Electronic Journal of Differential Equations, Vol. 2003(2003), No. 33, pp. 1–25.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

ANALYTIC SOLUTION TO A CLASS OF INTEGRO-DIFFERENTIAL EQUATIONS

XUMING XIE

Abstract. In this paper, we consider the integro-differential equation

$$\epsilon^2 y''(x) + L(x)\mathcal{H}(y) = N(\epsilon, x, y, \mathcal{H}(y)),$$

where $\mathcal{H}(y)[x] = \frac{1}{\pi}(P)\int_{-\infty}^{\infty} \frac{y(t)}{t-x}dt$ is the Hilbert transform. The existence and uniqueness of analytic solution in appropriately chosen space is proved. Our method consists of extending the equation to an appropriately chosen region in the complex plane, then use the Contraction Mapping Theorem.

1. Introduction

The second order ordinary differential equations with singular perturbation have been discussed in works such as [11, 18]. While singular integral equations have also been studied systematically in [9]. Many physical problems can be modelled by singular integro-differential equation

$$\epsilon^2 y''(x) + Q(x)y(x) + L(x)\mathcal{H}(y) = N(\epsilon^2, x, y, y', \mathcal{H}(y), \mathcal{H}(y')), \tag{1.1}$$

for $x \in (-\infty, +\infty)$, where $\mathcal{H}(y)[x] = \frac{1}{\pi}(P)\int_{-\infty}^{\infty} \frac{y(t)}{t-x}dt$ is the Hilbert transform.

Saffman and Taylor [14] studied the displacement of a viscous fluid by a less viscous fluid in a Hele-Shaw cell. It was noted that a single finger of the less viscous fluid is eventually formed and propagates at constant velocity keeping a steady shape. In the absence of surface tension ($\epsilon = 0$), Saffman and Taylor obtained a family of exact solutions. When the surface tension is non-zero, by conformal mapping, Maclean and Saffman [8] have reduced the determination of the finger to the solution of two coupled nonlinear integrodifferential equations. In Maclean-Saffman equations, the integral term is a Cauchy type singular integral over the interval [0,1]. By a transformation of the independent variable, Combescot et al [5] and Chapman [4] cast the finger problem with small surface tension as an integrodifferntial equation on the whole real line. By using Saffman-Taylor exact solutions, the integrodifferntial equations of Combescot et al and Chapman can be reduced to equation (1.1) with $Q(x) \neq 0$ and $L(x) \neq 0$. The works mentioned above include numerical and asymptotic studies. Recently, Xie and Tanveer [20, 17] reformulated the Saffman-Taylor finger problem as solving the integro-differential equation

$$\epsilon^2 y''(x) + Q(x)y'(x) = N(\epsilon, x, y, y', \mathcal{H}(\mathcal{G}[y])) \tag{1.2}$$

2000 *Mathematics Subject Classification.* 34A20, 45E05.
Key words and phrases. Analytic solution, singular integro-differential equation.

Submitted August 13, 2002. Published March 28, 2003.

where $\mathcal{G}[y]$ is an operator. Existence of steadily translating finger solutions (i.e analytic solution of (1.2)) were proved rigorously if relative finger width is in a set of infinite discrete values and surface tension is sufficiently small.

Dendritic crystal growth has long been a subject of interest to both physicists and Mathematicians. The simplest example of dentrite growth is the growth of needle crysyal in solidification from a pure undercooled melt. The growth of a steadily moving interface between solid and liquid is the ultimate evolution of the Mullins-Sekerka instability. When surface tension is neglected, Ivanstov [6] found an infinite continuous family of parabolic crystal interfaces. When surface tension is taken into account, in the limit of small Peclet number, Pelce and Pomeau [12, 1] reduced the Nash-Gicksman [10] equation to a simpler set of integrodifferential equations in which the singular integral terms are no longer of Cauchy type. In a recent work, Xie [19] reduced the one-sided needle crystal growth problem to solving an integrodifferential equation of form (1.2); symmetric analytic solutions are obtained if the surface tension is small and the crystalline anisotropy is in a set of infinite discrete values.

In this paper, we consider (1.1) with $L(x) \neq 0$, $Q(x) = 0$, and N does not depend on y'. i.e., equations of form

$$\epsilon^2 y''(x) + L(x)\mathcal{H}(y) = N(\epsilon, x, y, \mathcal{H}(y))\,. \tag{1.3}$$

We believe that the method developed in this paper will be useful for other type of integro-differential equations such as Nash-Glicksman equations [10, 12, 15] for the two-sided steady needle crystal growth problem.

Although equation (1.3) is given on the real x axis, we will extend the equation to some domains in the complex plane as in the viscous fingering case ([20, 17]). The main reason to go to complex plane is that it is possible to control the nonlocal integral terms in (1.3) and estimate the decay rate of the derivatives.

Notation and Main result. We define regions in complex z-plane:

Definition 1.1. Let $\mathcal{R}_{\alpha,\varphi}$ be an open connected region on the complex plane bounded by the lines

$$r_u = r_{u,1} \cup r_{u,2} \cup r_{u,3}, \quad r_l = r_{l,1} \cup r_{l,2} \cup r_{l,3}$$

where

$$\begin{gathered}
r_{u,1} = \{z : z = \alpha i - R + re^{(\pi-\varphi)i}, 0 \leq r \leq \infty\},\\
r_{u,2} = \{z : z = \alpha i + r, -R \leq r \leq R\},\\
r_{u,3} = \{z : z = \alpha i - R + re^{\varphi i}, 0 \leq r \leq \infty\},\\
r_{l,1} = \{z : z = -\frac{\alpha}{2} i - R + re^{(\pi+\frac{\varphi i}{2})}, 0 \leq r \leq \infty\},\\
r_{l,2} = \{z : z = -\frac{\alpha}{2} i + r, -R \leq r \leq R\},\\
r_{l,3} = \{z : z = -\frac{\alpha}{2} i - R + re^{-\frac{\varphi i}{2}}, 0 \leq r \leq \infty\},
\end{gathered}$$

where $1 > \alpha > 0$, $0 < \varphi < \pi/2$ and $R > 0$.

Denoting by $*$ the complex conjugate, we define

$$\tilde{\mathcal{R}}_{\alpha,\varphi} = \mathcal{R}^*_{\alpha,\varphi} \equiv \{z^* : z \in \mathcal{R}\} \tag{1.4}$$

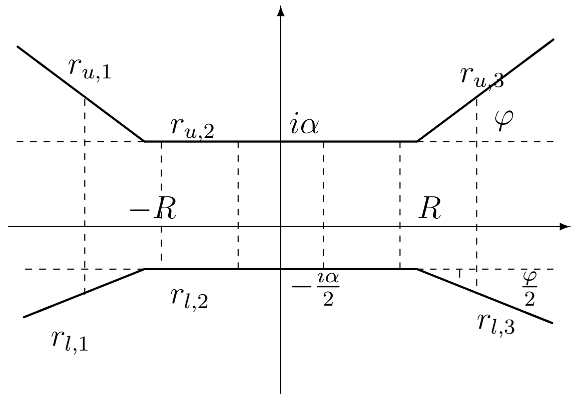

FIGURE 1. Region $\mathcal{R}_{\alpha,\varphi}$ in the complex plane

Definition 1.2. We define function

$$P(z) = \int_0^z \sqrt{iL(t)}dt, \quad \tilde{P}(z) = \int_0^z \sqrt{-iL(t)}dt \tag{1.5}$$

We assume that there exist numbers $1 > \alpha_0 > 0$, $R > 0$ and $\frac{\pi}{2} > \varphi_0 > 0$, so that $L(z), P(z)$, and $\tilde{P}(z)$ satisfy the following properties:
Property 1. $L(z)$ is analytic in $\mathcal{R}_{\alpha_0,\varphi_0} \cup \tilde{\mathcal{R}}_{\alpha_0,\varphi_0}$, and $L(z) \neq 0$ in $\mathcal{R}_{\alpha_0,\varphi_0} \cup \tilde{\mathcal{R}}_{\alpha_0,\varphi_0}$ and

$$|L(z)| = C|z|^\gamma(1 + o(1)), |L'(z)| = C|z - 2i|^{\gamma-1}(1 + o(1)), \tag{1.6}$$

$$|L''(z)| = C|z - 2i|^{\gamma-2}(1 + o(1)) \text{ as } |z| \to \infty, z \in \mathcal{R}_{\alpha_0,\varphi_0} \cup \tilde{\mathcal{R}}_{\alpha_0,\varphi_0} \tag{1.7}$$

where $\gamma > -2$ and C independent of ϵ.
Property 2. A branch of $\sqrt{iL}$ and $\sqrt{-iL}$ in (1.5) can be chosen so that $P(z)$ and $\tilde{P}(z)$ are analytic in $\mathcal{R}_{\alpha_0,\varphi_0} \cup \tilde{\mathcal{R}}_{\alpha_0,\varphi_0}$ and $\operatorname{Re} P(-\infty) = -\infty$, $\operatorname{Re} P(\infty) = \infty$,$\operatorname{Re} \tilde{P}(-\infty) = -\infty$, $\operatorname{Re} \tilde{P}(\infty) = \infty$.
Property 3. For $\operatorname{Re} z \geq R$, $0 < \alpha \leq \alpha_0$, $0 < \varphi \leq \varphi_0$, $\operatorname{Re} P(t)$ (resp. $\operatorname{Re} \tilde{P}(t)$) is increasing with increasing s along any ray $r = \{t : t = z + se^{i\theta}, 0 < s < \infty, -\varphi < \theta < \varphi\}$ in $\mathcal{R}_{\alpha,\varphi}$ (resp. in $\tilde{\mathcal{R}}_{\alpha,\varphi}$) from z to $z+\infty e^{i\theta}$ and $C_1|t-2i|^{\gamma/2} \leq \frac{d}{ds}$ Re $P(t(s))$ (*resp.* $C_1|t - 2i|^{\gamma/2} \leq \frac{d}{ds}$ Re $\tilde{P}(t(s))$) where C_1 is a constant which can be made independent of ϵ and $C_1 > 0$.
Property 4. For $\operatorname{Re} z \leq -R$, $0 < \alpha \leq \alpha_0$, $0 < \varphi \leq \varphi_0$, $\operatorname{Re} P(t)$ (resp. $\operatorname{Re} \tilde{P}(t)$) is decreasing with increasing s along ray $r = \{t : t = z + se^{i(\pi-\theta)}, 0 < s < \infty, -\varphi < \theta < \varphi\}$ in $\mathcal{R}_{\alpha,\varphi}$ (resp. $\tilde{\mathcal{R}}_{\alpha,\varphi}$) from z to $z+\infty e^{i(\pi-\varphi)}$ and $\frac{d}{ds}$ Re $P(t(s)) \leq -C_2|t - 2i|^{\gamma/2}$,(resp. $\frac{d}{ds}$ Re $\tilde{P}(t(s))| \leq -C_2|t - 2i|^{\gamma/2}$) where C_2 is constant which can be made independent of ϵ and $C_2 > 0$.
Property 5. For any $z \in \mathcal{R}_{\alpha,\varphi}$ (resp. $z \in \tilde{\mathcal{R}}_{\alpha,\varphi}$), $0 < \alpha \leq \alpha_0$, $0 < \varphi \leq \varphi_0$, there is a path $\mathcal{P}(z, \infty)$ in $\mathcal{R}_{\alpha,\varphi}$ (resp. $\tilde{\mathcal{P}}(z, \infty)$ in $\tilde{\mathcal{R}}_{\alpha,\varphi}$) which is $\mathbf{C}^1$ curve connecting z to ∞ so that $\frac{d}{ds}$ $[Re\ P(t(s))] \geq C|t - 2i|^{\gamma/2} > 0$ for $t(s) \in \mathcal{P}(z, \infty)$
(*resp.* $\frac{d}{ds}$ $[Re\ \tilde{P}(t(s))] \geq C|t - 2i|^{\gamma/2} > 0$ for $t(s) \in \tilde{\mathcal{P}}(z, \infty)$), s being an arc length of $\mathcal{P}(z, \infty)$ (resp. $\tilde{\mathcal{P}}(z, \infty)$), which increases toward $t = \infty$.

Property 6. For any $z \in \mathcal{R}_{\alpha,\varphi}$,(resp. $z \in \tilde{\mathcal{R}}_{\alpha,\varphi}$), $0 < \alpha \leq \alpha_0,\ 0 < \varphi \leq \varphi_0$, there is a path $\mathcal{P}(z, -\infty)$ in $\mathcal{R}_{\alpha,\varphi}$ (resp. $\tilde{\mathcal{P}}(z, -\infty)$ in $\tilde{\mathcal{R}}_{\alpha,\varphi}$) which is $\mathbf{C}^1$ curve connecting z to $-\infty$ so that $\frac{d}{ds}\ [Re\ P(t(s))] \leq -C|t-2i|^{\gamma/2} < 0$ for $t(s) \in \mathcal{P}(z, \infty)$ (*resp.* $\frac{d}{ds}[\mathrm{Re}\,\tilde{P}(t(s))] \leq -C|t-2i|^{\gamma/2} < 0$ for $t(s) \in \tilde{\mathcal{P}}(z,\infty)$), s being an arc length of $\mathcal{P}(z, -\infty)$ (resp. $\tilde{\mathcal{P}}(z, -\infty)$), which increases toward $t = -\infty$.

Remark 1.3. In section 4, we are going to give two explicit functions of $L(z)$ and region $\mathcal{R}_{\alpha_0,\varphi_0}$ so that Property 1-6 hold. Note that Property 1-6 are crucial to prove Lemmas 2.12 and 2.13.

We assume that $N(\epsilon, z, u, v)$ can be written as

$$N(\epsilon, z, u, v) = \sum_{k=2}^{n} p_k(z)T_k(u,v) + \epsilon^2 \sum_{k=0}^{l} f_k(z)Q_k(u,v) \tag{1.8}$$

where $Q_k(u,v), T_k(u,v)$ is analytic in $\{(u,v) : |u| < \frac{1}{\rho}, |v| < \frac{1}{\rho}\}$ and

$$\begin{gathered} Q_k(u,v) = \sum_{\alpha_k+\beta_k \geq k} q_{\alpha_k,\beta_k} u^{\alpha_k} v^{\beta_k}, \quad T_k(u,v) = \sum_{\alpha_k+\beta_k \geq k} t_{\alpha_k,\beta_k} u^{\alpha_k} v^{\beta_k}, \\ |q_{\alpha_k,\beta_k}| \leq A\rho^{\alpha_k+\beta_k}, \quad |t_{\alpha_k,\beta_k}| \leq A\rho^{\alpha_k+\beta_k}, \end{gathered} \tag{1.9}$$

where A and ρ are some positive constants. Then $f_k(z)$ and $p_k(z)$ are analytic in $\mathcal{R}_{\alpha_0,\varphi_0} \cup \tilde{\mathcal{R}}_{\alpha_0,\varphi_0}$ and for $z \in \mathcal{R}_{\alpha_0,\varphi_0} \cup \tilde{\mathcal{R}}_{\alpha_0,\varphi_0}$,

$$|f_k(z)| \leq C|z-2i|^{-\tau+\gamma+k\tau}, \quad |p_k(z)| \leq C|z-2i|^{-\tau+\gamma+k\tau} \tag{1.10}$$

Let $0 < \tau < 1$ be fixed and independent of ϵ. Let $\mathcal{D}$ be any connected open set in complex z-plane. We introduce the following function spaces:

$$\begin{aligned} \mathbf{A}_k(\mathcal{D}) = &\{y(z) : y(z) \text{ is analytic in } \mathcal{D} \text{ and continuous in } \overline{\mathcal{D}}, \\ &\text{with } \sup_{z\in\overline{\mathcal{D}}} |(z-2i)^{k+\tau} y(z)| < \infty\} \end{aligned}$$

for $k = 0, 1$, with $\|y\|_{k,\mathcal{D}} := \sup_{z\in\overline{\mathcal{D}}} |(z-2i)^{k+\tau} y(z)|$

Clearly, $\mathbf{A}_k(\mathcal{D})$ are Banach spaces, and $\mathbf{A}_2(\mathcal{D}) \subset \mathbf{A}_1(\mathcal{D}) \subset \mathbf{A}_0(\mathcal{D}) \equiv \mathbf{A}(\mathcal{D})$.

$$\begin{aligned} \mathbf{A}_k &\equiv \mathbf{A}_k(\mathcal{R}_{\alpha_0,\varphi_0}), \quad \|y\|_k = \|\cdot\|_{k,\mathcal{R}_{\alpha_0,\varphi_0}} \text{ for } y \in \mathbf{A}_k \\ \tilde{\mathbf{A}}_k &\equiv \mathbf{A}_k(\tilde{\mathcal{R}}_{\alpha_0,\varphi_0}), \quad \|\tilde{y}\|_k = \|\cdot\|_{k,\tilde{\mathcal{R}}_{\alpha_0,\varphi_0}} \text{ for } \tilde{y} \in \tilde{\mathbf{A}}_k \end{aligned}$$

Let δ be a constant such that $0 < \delta < 1$, We define

$$\mathbf{A}_{0,\delta} = \{y : y \in \mathbf{A}_0, \|y\|_0 \leq \delta\}, \quad \tilde{\mathbf{A}}_{0,\delta} = \{\tilde{y} : \tilde{y} \in \tilde{\mathbf{A}}_0, \|\tilde{y}\|_0 \leq \delta\} \tag{1.11}$$

We will prove the following result.

Theorem 1.4. *For sufficiently small ϵ and δ, there exists a unique solution $y \in \mathbf{A}_{0,\delta}(\mathcal{R}_{\alpha_0,\varphi_0} \cup \tilde{\mathcal{R}}_{\alpha_0,\varphi_0})$ to equation (1.3). Furthermore $\|y\|_{0,\mathcal{R}_{\alpha_0,\varphi_0}\cup\tilde{\mathcal{R}}_{\alpha_0,\varphi_0}} \sim O(\epsilon^2)$.*

The proof of this Theorem will be given at the end of §3, after some preliminary results. The solution strategy followed in this paper is as follows: In §2, we first derive two integro-differential equations. One is the extension of equation (1.3) to the upper part of $\mathcal{R}$ and the other is the extension of equation (1.3) to lower part of $\tilde{\mathcal{R}}$. Since the integral terms $I_\pm$ derived from the Hilbert transform $\mathcal{H}$ (see Definition 2.4 in the sequel) are not contraction terms (or small terms), the classical contraction argument does not work for both equations. By integration by parts and Hilbert Inverse transform, we can change the integral term into a small term. However in doing so, we get derivatives of $I_\pm$, the classical contraction argument still fails. To circumvent this difficulty, we formulate a coupled system of integral equations using the equations in $\{Im\ z > 0\} \cap \mathcal{R}$ and in $\{Im\ z < 0\} \cap \tilde{\mathcal{R}}$ at the same time. In §3, we use contraction argument to show the existence and uniqueness of solution to the coupled system of integral equations. Then, we further show the solution to the coupled system is actually the solution to (1.3). In §4, in order to demonstrate the relevance of the method, we give two explicit functions for $L(x)$ in (1.3) and show that there are constants α_0 and φ_0 so that property 1-6 hold in $\mathcal{R}_{\alpha_0,\varphi_0}$. Therefore, Theorem 1.4 can be applied to these two examples. These simple model problems are derived from more complex and physically sound problem [8, 15, 2].

2. Formulation of equivalent integral equations in complex regions

In this section, for simple notation, we use $\mathcal{R}$ to denote $\mathcal{R}_{\alpha_0,\varphi_0}$ and $\tilde{\mathcal{R}}$ to denote $\tilde{\mathcal{R}}_{\alpha_0,\varphi_0}$ respectively. We will use C (and sometimes C_1, C_2) as generic constant, whose value is allowed to differ from Lemma to Lemma and from line to line. However C does not depend on ϵ.

Lemma 2.1. *Let $\Gamma = \{t, t = \xi_0 + \rho e^{i\varphi}, 0 \le \rho < \infty\}$ be a ray, with $2i$ not in Γ. $\mathcal{D}$ is a region with $\operatorname{dist}(2i, \mathcal{D}) > 0$ and*

$$\operatorname{dist}(\xi, \Gamma) \ge m|\xi - \xi_0| > 0; \ \textit{for } \xi \in \mathcal{D}; \tag{2.1}$$

$$\operatorname{dist}(t, \mathcal{D}) \ge m|t - \xi_0|; \ \textit{for } t \in \Gamma; \tag{2.2}$$

$$\operatorname{dist}(t, 2i) \ge m|t - \xi_0|; \ \textit{for } t \in \Gamma; \tag{2.3}$$

for some constant $m > 0$ independent of ϵ. Assume $g(\xi)$ to be a continuous function on Γ with $\|g\|_{0,\Gamma} < \infty$, then for $k = 0, 1, 2$,

$$\sup_{\mathcal{D}} \big|(\xi - 2i)^{k+\tau} \int_\Gamma \frac{g(t)}{(t-\xi)^{k+1}} dt\big| \le C\|g\|_{0,\Gamma}; \tag{2.4}$$

where constant C that depends on φ and m only .

Proof. This lemma was proved in [20], we give the proof here for completeness.

$$\big|(\xi - 2i)^{k+\tau} \int_\Gamma \frac{g(t)}{(t-\xi)^{k+1}} dt\big| \le \|g\|_{0,\gamma} |\xi - 2i|^{k+\tau} \int_\Gamma \frac{|dt|}{|(t-\xi)|^{k+1}|t-2i|^\tau}; \tag{2.5}$$

On Γ,$t - \xi_0 = \rho e^{i\varphi}$, $|dt| = d\rho$. Breaking up the integral in (2.5) into two parts:

$$\begin{aligned} &\int_\Gamma \frac{|dt|}{|(t-\xi)|^{k+1}|t-2i|^\tau} \\ &= \int_0^{|\xi-\xi_0|} \frac{d\rho}{|(t-\xi)|^{k+1}|t-2i|^\tau} + \int_{|\xi-\xi_0|}^\infty \frac{d\rho}{|(t-\xi)|^{k+1}|t-2i|^\tau}; \end{aligned} \tag{2.6}$$

for the first integral in (2.6),we use (2.1) and (2.3) and for the second, we use (2.2) and (2.3) to obtain:(on scaling ρ by $|\xi - \xi_0|$)

$$\int_\Gamma \frac{|dt|}{|(t-\xi)|^{k+1}|t-2i|^\tau} = \frac{C}{|\xi - \xi_0|^{k+\tau}} (\int_0^1 \frac{d\rho}{\rho^\tau} + \int_1^\infty \frac{d\rho}{\rho^{k+\tau+1}});$$

$\square$

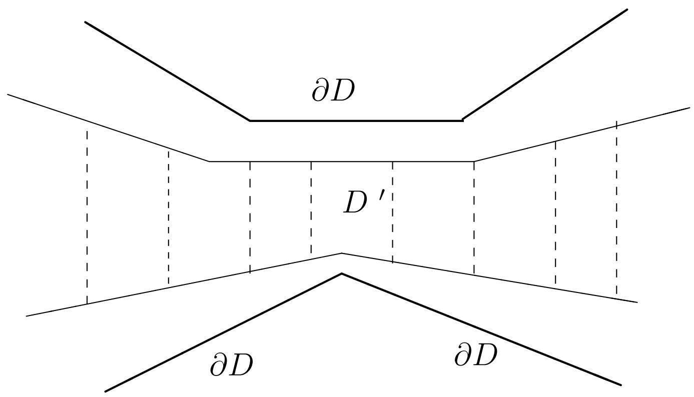

Figure 2. Angular subset $\mathcal{D}'$ of $\mathcal{D}$

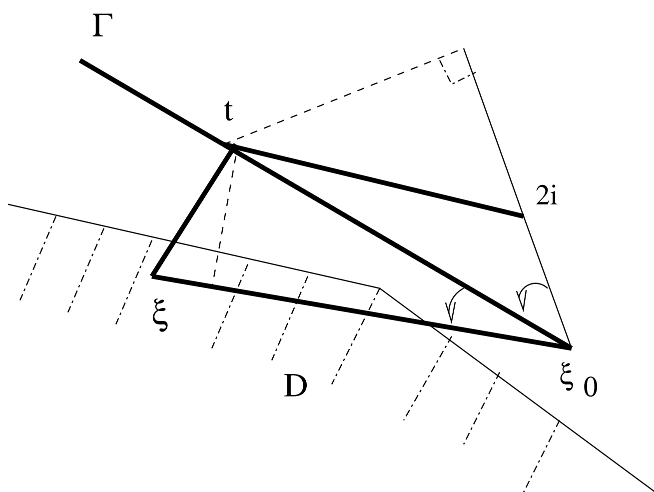

Figure 3. Relevant to Lemma 2.1 and Remark 2.3

Definition 2.2. Let $\mathcal{D}$ be an open connected set in complex plane with one or more straight line boundaries. $\mathcal{D}'$ is defined as an angular subset of $\mathcal{D}$ if $\mathcal{D}' \subset \mathcal{D}$, $dist(\mathcal{D}', \partial\mathcal{D}) > 0$ and $\mathcal{D}'$ has straight line boundaries that make a nonzero angle with respect to $\partial\mathcal{D}$ asymptotically at large distances from the origin (see Figure 2). This means that if $z' \in \mathcal{D}'$ and $z \in \partial\mathcal{D}$,then $dist(z, \partial\mathcal{D}') \geq C|z|\sin\theta_0$, as $|z| \to \infty$; $\mathrm{dist}(z', \partial\mathcal{D}) \geq C|z'|\sin\theta_0$, as $|z'| \to \infty$,where C is some positive constant and $0 < \theta_0 \leq \pi/2$.

Remark 2.3. Note if Γ is in $\mathcal{D}'_c$,an angular subset of $\mathcal{D}_c$ (complement of $\mathcal{D}$),then (2.1) and (2.2) hold (see Figure 2). Also note (2.3) is valid for any Γ in $\mathcal{R}$.

Definition 2.4. Let y be continuous on $(-\infty,\infty)$ and and $\|y\|_{0,(-\infty,\infty)}<\infty$, we define $I_+(y)(z)$ and $I_-(y)(z)$ as

$$I_+(y)(z)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y(t)}{t-z}dt \text{ for } \operatorname{Im} z>0; \tag{2.7}$$

$$I_-(y)(z)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y(t)}{t-z}dt \text{ for } \operatorname{Im} z<0; \tag{2.8}$$

Lemma 2.5. *If $y\in\mathbf{A}_0$ and $\tilde y\in\tilde{\mathbf{A}}_0$, then*
(1) $I_+(y)\in\mathbf{A}_0, \|I_+(y)\|_0\le C\|y\|_0$.
(2) $\frac{d^k}{dz^k}I_+(\tilde y)\in\mathbf{A}_k, \|\frac{d^k}{dz^k}I_+(\tilde y)\|_k\le C\|\tilde y\|_0$ for any integer $k\ge 1$

Proof. (1) For $z\in\{\operatorname{Im} z\ge 0\}$, By Cauchy integral formula, we have

$$I_+(y)(z)=\frac{1}{\pi}\int_{r_l}\frac{y(t)}{t-z}dt \tag{2.9}$$

Using Lemma 2.1 with $\mathcal{D}=\{\operatorname{Im} z>0\}$,we have $\sup_{\mathcal{D}}|(z-2i|^{\tau}|I_+(y)|\le C\|y\|_0$.

For $z\in\{\operatorname{Im} z<0\}\cap\mathcal{R}$, by Plemelj formula and Cauchy integral formula, $I_+(y)$ can be analytically continued to low part of $\mathcal{R}$,

$$I_+(y)(z)=I_-(y)(z)+2iy(z)=\frac{1}{\pi}\int_{r_u}\frac{y(t)}{t-z}dt+2iy(z); \tag{2.10}$$

Using Lemma 2.1 with $\mathcal{D}=\{\operatorname{Im} z<0\}\cap\mathcal{R}$, we have $\sup_{\mathcal{D}}|(z-2i|^{\tau}|I_-(y)(z)|\le C\|y\|_0$.

Using (2.10) and $y\in\mathbf{A}_0$, we get the lemma.

(2) For $z\in\{\operatorname{Im} z\ge 0\}$,By Cauchy integral formula,we have

$$\frac{d^k}{dz^k}I_+(\tilde y)=\frac{1}{k!\pi}\int_{\tilde r_u}\frac{\tilde y(t)}{(t-z)^{(k+1)}}dt \tag{2.11}$$

where $\tilde r_u=[r_u]^*$. Using Lemma 2.1 with $\mathcal{D}=\{\operatorname{Im} z>0\}$, we have $\sup_{\mathcal{D}}|z-2i|^{k+\tau}|\frac{d^k}{dz^k}I_+(\tilde y)|\le C\|\tilde y\|_0$. For $z\in\{\operatorname{Im} z<0\}\cap\mathcal{R}$, by (2.8), Cauchy integral formula and Plemelj formula, $I_+(\tilde y)$ can be analytically continued to low part of $\tilde{\mathcal{R}}$,

$$I_+(\tilde y)=I_-(\tilde y)+2i\tilde y=\frac{1}{\pi}\int_{\tilde r_l}\frac{\tilde y(t)}{t-z}dt+2i\tilde y; \tag{2.12}$$

So

$$\frac{d^k}{dz^k}I_+(\tilde y)=\frac{d^k}{dz^k}I_-(\tilde y)+2i\frac{d^k}{dz^k}\tilde y=\frac{1}{k!\pi}\int_{\tilde r_l}\frac{\tilde y(t)}{(t-z)^k}dt+2i\frac{d^k}{dz^k}\tilde y; \tag{2.13}$$

Using Lemma 2.1 with $\mathcal{D}=\{\operatorname{Im} z<0\}\cap\mathcal{R}$, we have $\sup_{\mathcal{D}}|z-2i|^{k+\tau}|\frac{d^k}{dz^k}I_-(\tilde y)|\le C\|\tilde y\|_0$. Now by the Cauchy integral formula: for $z\in\{\operatorname{Im} z<0\}\cap\mathcal{R}$

$$\tilde y(z)=\frac{1}{2\pi i}\int_{\tilde r_u}-\int_{\tilde r_l}\frac{\tilde y}{(t-z)}dt; \tag{2.14}$$

where $\tilde r_l=[r_l]^*, \tilde r_u=[r_u]^*$. So

$$\frac{d^k}{dz^k}\tilde y(z)=\frac{1}{2k!\pi i}\int_{\tilde r_u}-\int_{\tilde y_l}\frac{\tilde y}{(t-z)^{k+1}}dt; \tag{2.15}$$

Using Lemma 2.1 with $\mathcal{D}=\{\operatorname{Im} z<0\}\cap\mathcal{R}$ which is an angular subset of $\tilde{\mathcal{R}}$, we have $\sup_{\mathcal{D}}|(z-2i|^{k+\tau}|\frac{d^k}{dz^k}(\tilde y)|\le C\|\tilde y\|_0$. $\square$

Remark 2.6. In general, from Cauchy integral formula and Lemma 2.1, the following statements are true:

(1) If $y \in \mathbf{A}_0(\mathcal{D})$ and $\mathcal{D}'$ is an angular subset of $\mathcal{D}$, then $\frac{d^k y}{dz^k} \in \mathbf{A}_k(\mathcal{D}')$ and $\|\frac{d^k y}{dz^k}\|_{k,\mathcal{D}'} \le C\|y\|_{0,\mathcal{D}}$.
(2) If $y \in \mathbf{A}_0(\mathcal{D})$, $(-\infty,\infty) \subset \mathcal{D}$, then $I_\pm(y) \in \mathbf{A}_0(\mathcal{D})$ and $\|I_\pm(y)\|_{0,\mathcal{D}} \le C\|y\|_{0,\mathcal{D}}$.
(3) If $y \in \mathbf{A}_0(\mathcal{D})$, $\mathcal{D}' \cap \{\operatorname{Im} z < 0\}$ is an angular subset of $\mathcal{D}$, and $(-\infty,\infty) \subset \mathcal{D}$, then for any integer $k \ge 1$, $\frac{d^k I_+(y)}{dz^k} \in \mathbf{A}_k(\mathcal{D}')$ and $\|\frac{d^k I_+(y)}{dz^k}\|_{k,\mathcal{D}'} \le C\|y\|_{0,\mathcal{D}}$.
(4) If $y \in \mathbf{A}_0(\mathcal{D})$, $\mathcal{D}' \cap \{\operatorname{Im} z > 0\}$ is an angular subset of $\mathcal{D}$, and $(-\infty,\infty) \subset \mathcal{D}$, then for any integer $k \ge 1$, $\frac{d^k I_-(y)}{dz^k} \in \mathbf{A}_k(\mathcal{D}')$ and $\|\frac{d^k I_-(y)}{dz^k}\|_{k,\mathcal{D}'} \le C\|y\|_{0,\mathcal{D}}$.

Lemma 2.7. *If $y \in \mathbf{A}_0$ and $\tilde{y} \in \tilde{\mathbf{A}}$, then*

(1) $I_-(\tilde{y}) \in \tilde{\mathbf{A}}_0, \|I_-(\tilde{y})\|_0 \le C\|\tilde{y}\|_0$
(2) $\frac{d^k}{dz^k} I_-(y) \in \tilde{\mathbf{A}}_k, \|\frac{d^k}{dz^k} I_-(y)\|_k \le C\|y\|_0$ *for any integer* $k \ge 1$

The proof of this lemma is parallel to that of Lemma 2.5.

Lemma 2.8. *If $y \in \mathbf{A}_0$ and $\tilde{y} \in \tilde{\mathbf{A}}_0$, then*

(1) *The Hilbert transform $\mathcal{H}(y)(x) = (P)\int_{-\infty}^{\infty} \frac{y(t)}{t-x} dt$ can be extended to region $\mathcal{R}$ and $\mathcal{H}(y)(z) \in \mathbf{A}_0$,$\|\mathcal{H}(y)\|_0 \le C\|y\|_0$.*
(2) *The Hilbert transform $\mathcal{H}(\tilde{y})(x) = (P)\int_{-\infty}^{\infty} \frac{\tilde{y}(t)}{t-x} dt$ can be extended to region $\tilde{\mathcal{R}}$ and $\mathcal{H}(\tilde{y})(z) \in \tilde{\mathbf{A}}_0$,$\|\mathcal{H}(\tilde{y})\|_0 \le C\|\tilde{y}\|_0$.*

Proof. We prove only (1). The proof of (2) is similar. Using Plemelj formula, we have

$$\begin{aligned} \mathcal{H}(y)(z) &= I_+(y)(z) - iy(z); \text{ for } z \in \{\operatorname{Im} z > 0\} \cap \mathcal{R} \\ \mathcal{H}(y)(z) &= I_-(y)(z) + iy(z); \text{ for } z \in \{\operatorname{Im} z < 0\} \cap \mathcal{R} \end{aligned} \tag{2.16}$$

the lemma follows from Lemma 2.5. □

Formulation of Equivalent integral equations.

Lemma 2.9. *Let $y(z) \in \mathbf{A}_0$, then $y(z)$ is a solution of (1.3) on real axis, if and only if $y(z)$ satisfies*

$$\epsilon^2 y''(z) - iL(z)y(z) = -L(z)I_+(y)(z) + N(\epsilon, z, y, I_+(y) - iy), \tag{2.17}$$

for $z \in \mathcal{R} \cap \{\operatorname{Im} z > 0\}$.

Proof. If $y \in \mathbf{A}_0$ satisfies (1.3), extending (1.3) to upper half complex plane Im $z > 0$ and using Plemelj formulae and Lemma 2.8, we get the equation (2.17). Conversely, in (2.17) , let z go to to real axis from above, using Plemelj formulae, we get (1.3). □

Lemma 2.10. *If $\tilde{y}(z) \in \tilde{\mathbf{A}}_0$ is a solution of (1.3) on real axis, then for $z \in \tilde{\mathcal{R}} \cap \{\operatorname{Im} z < 0\}$, $\tilde{y}(z)$ satisfies*

$$\epsilon^2 \tilde{y}''(z) + iL(z)\tilde{y}(z) = -L(z)I_-(\tilde{y})(z) + N(\epsilon, z, \tilde{y}, I_-(\tilde{y}) + i\tilde{y}), \tag{2.18}$$

Proof. If $y \in \mathbf{A}_0$ satisfies (1.3), extending (1.3) to lower half complex plane Im $z < 0$ and using Plemelj formulae and Lemma 2.8, we get the equation (2.18). Conversely, in (2.18) , let z go to to real axis from below, using Plemelj formulae, we get (1.3). □

Note that the equation $\epsilon^2\phi'' - iL\phi = 0$ has the following two Wentzel-Kramers-Brillouin solutions (i.e WKB solutions):

$$Y_1(z) = L^{-1/4}(z)\exp\{\frac{1}{\epsilon}P(z)\} \tag{2.19}$$

$$Y_2(z) = L^{-1/4}(z)\exp\{-\frac{1}{\epsilon}P(z)\} \tag{2.20}$$

The Wronskian of these two solutions is

$$W = -\frac{\sqrt{2}(1+i)}{\epsilon}, \tag{2.21}$$

While two WKB solutions to $\epsilon^2\tilde{\phi}'' + iL\tilde{\phi} = 0$ are

$$\tilde{Y}_1(z) = L^{-1/4}(z)\exp\{\frac{1}{\epsilon}\tilde{P}(z)\} \tag{2.22}$$

$$\tilde{Y}_2(z) = L^{-1/4}(z)\exp\{-\frac{1}{\epsilon}\tilde{P}(z)\} \tag{2.23}$$

The Wronskian of $\tilde{Y}_1(z)$ and $\tilde{Y}_2(z)$ is

$$\widetilde{W} = -\frac{\sqrt{2}(1-i)}{\epsilon}, \tag{2.24}$$

and $Y_1(z), Y_2(z)$ satisfy the equation

$$\epsilon^2\phi''(z) - iL(z)\phi(z) + \epsilon^2 L_1(z)\phi(z) = 0 \tag{2.25}$$

where

$$L_1(z) = \frac{L''(z)}{4L(z)} - \frac{5(L'(z))^2}{16L^2(z)} \tag{2.26}$$

Remark 2.11. From Property 1, $L_1(z)$ is analytic in $\mathcal{R} \cup \tilde{\mathcal{R}}$, and $L_1(z) \sim O(|z-2i|^{-2})$.

The functions $\tilde{Y}_1(z), \tilde{Y}_2(z)$ satisfy the equation

$$\epsilon^2\tilde{\phi}''(z) + iL(z)\tilde{\phi}(z) + \epsilon^2 L_1(z)\tilde{\phi}(z) = 0 \tag{2.27}$$

so (2.17) can be written as

$$\epsilon^2\phi'' - iL(z)\phi(z) + L_1(z)\epsilon^2\phi(z) = -L(z)I_+(\phi)(z) + N_1(\epsilon,\phi)(z) \tag{2.28}$$

where N_1 is an operator

$$N_1(\epsilon,\phi)(z) = \epsilon^2 L_1(z)\phi(z) + N(\epsilon, z, \phi(z), I_+(\phi)(z) - i\phi(z)), \tag{2.29}$$

while (2.18) can be written as

$$\epsilon^2\tilde{\phi}'' + iL(z)\tilde{\phi}(z) + L_1(z)\epsilon^2\tilde{\phi}(z) = -L(z)I_-(\tilde{\phi}) + \tilde{N}_1(\epsilon,\tilde{\phi})(z), \tag{2.30}$$

where $\tilde{N}_1$ is an operator defined by

$$\tilde{N}_1(\epsilon,\tilde{\phi}) = \epsilon^2 L_1(z)\tilde{\phi}(z) + N(\epsilon, z, \tilde{\phi}(z), I_-(\tilde{\phi})(z) + i\tilde{\phi}(z)), \tag{2.31}$$

Lemma 2.12. *If* $\sup_{\mathcal{R}} |y(z)(z-2i)^m| < \infty$, *for* $m \geq 0$ *,then*

$$|Y_1(z)\int_\infty^z y(t)Y_2(t)dt| \leq \epsilon\frac{C\sup|(z-2i)^m y(z)|}{|z-2i|^{\gamma+m}} \quad \textit{for } z \in \mathcal{R},$$

where C is a constant independent of ϵ and $y(z)$.

Proof. **Case 1**: when $\operatorname{Re} z > 2R$, on path $\mathcal{P} = \{t : t = z + s, 0 < s < \infty\}$, $\operatorname{Re}(P(t) - P(z))$ goes from 0 to ∞ as $s \to \infty$.

$$\begin{aligned}
&|Y_1(z) \int_\infty^z y(t)Y_2(t)dt| \\
&= |L^{-1/4} \int_z^\infty y(t)L(t)^{-1/4} e^{-\frac{1}{\sqrt{2\epsilon}}(P(t)-P(z))} dt| \\
&\le \sup |(z-2i)^m y(z)|\, |L(z)|^{-1/4} \int_z^\infty |L(t)|^{-1/4} |t-2i|^{-m} e^{-\frac{1}{\epsilon}\operatorname{Re}(P(t)-P(z))} dt \\
&\le \epsilon \sup |(z-2i)^m y(z)|\, |L(z)|^{-1/4} \int_0^\infty \frac{|L(t(s))|^{-1/4} |t(s)-2i|^{-m}}{\operatorname{Re} P'(t(s))} d[e^{-\frac{1}{\epsilon}\operatorname{Re}(P(t)-P(z))}]
\end{aligned}$$

Note that $|L^{-1/4}(z)| \sim C|z-2i|^{-\gamma/4}$, $|z-2i| \le |t-2i|$ for t on the integral range, and we have $\operatorname{Re} P'(t(s))| \ge C|t-2i|^{\gamma/2}$, so we have

$$\big|Y_1(z) \int_\infty^z y(t)Y_2(t)dt\big| \le C\epsilon \sup |(z-2i)^m y(z)||z-2i|^{-m-\gamma}$$

Case 2: when $|\operatorname{Re} z| \le 2R$, by property 5, there is a path $\mathcal{P}(z,\infty)$ on which $\operatorname{Re} P(t)$ increases as t goes from z to ∞. Using the same steps as in Case 1, we can get the estimate.

Case 3: when $\operatorname{Re} z \le -2R$, we choose paths connecting z to ∞ , $\mathcal{P} = \mathcal{P}_1 \cup [\operatorname{Re} z/2, \infty)$, where

$$\mathcal{P}_1 = \{t : t = \operatorname{Re} z/2 + \rho e^{i \arg(z - \frac{\operatorname{Re} z}{2})}, 0 \le \rho \le |z - \frac{\operatorname{Re} z}{2}|\}$$

Note on $\mathcal{P}_1$,by Property 4, we have

$$\operatorname{Re}(P(z) - P(t)) \le -C_1 \int_{|\operatorname{Re} t|}^{|\operatorname{Re} z|} r^{\gamma/2} dr \le -C_1(|\operatorname{Re} z|^{1+\gamma/2} - |\operatorname{Re} t|^{1+\gamma/2})$$

So

$$\begin{aligned}
&\big|Y_1(z) \int_{\mathcal{P}_1} y(t)Y_2(t)dt\big| \\
&= \big|L^{-1/4} \int_{\mathcal{P}_1} y(t)L(t)^{-1/4} e^{-\frac{1}{\epsilon}(P(t)-P(z))} dt\big| \\
&\le \sup |(z+i)^m y(z)||L(z)^{-1/4}| \int_{|\operatorname{Re} z/2|}^{|\operatorname{Re} z|} |L(t)|^{-1/4} |t-2i|^{-m} e^{-\frac{1}{\epsilon}\operatorname{Re}(P(t)-P(z))} dt \\
&\le C \sup |(z+i)^m y(z)||z|^{-m-\gamma/2} \int_{|\operatorname{Re} z|/2}^{|\operatorname{Re} z|} e^{-C_1 \frac{1}{\epsilon}(|\operatorname{Re} z|^{1+\gamma/2} - t^{1+\gamma/2})} dt \\
&\le C\epsilon \sup |(z+i)^m y(z)||z|^{-m-\gamma}
\end{aligned}$$

and

$$
\begin{aligned}
&\big|Y_1(z)\int_{\operatorname{Re} z/2}^{\infty} y(t)Y_2(t)dt\big| \\
&= \big|L^{-1/4}\int_{\operatorname{Re} z/2}^{\infty} y(t)L(t)^{-1/4}e^{-\frac{1}{\epsilon}(P(t)-P(z))}dt\big| \\
&\le \sup|(z-2i)^m y(z)||L(z)|^{-1/4}\int_{\operatorname{Re} z/2}^{\infty}|L(t)|^{-1/4}|t-2i|^{-m}e^{-\frac{1}{\epsilon}\operatorname{Re}(P(t)-P(z))}dt \\
&\le \epsilon\sup|(z-2i)^m y(z)||L(z)^{-1/4}| \\
&\quad\times\int_{\operatorname{Re} z/2}^{\infty}\frac{|L(t(s))|^{-1/4}|t(s)-2i|^{-m}}{\operatorname{Re} P'(t(s))}d\big[e^{-\frac{1}{\epsilon}\operatorname{Re}(P(t)-P(z))}\big] \\
&\le C\epsilon\sup|(z-2i)^m y(z)||L|^{-1/4}e^{-\frac{1}{\epsilon}\operatorname{Re}(P(\operatorname{Re} z/2)-P(z))} \\
&\le C\epsilon\sup|(z-2i)^m y(z)||z-2i|^{-m-\gamma}
\end{aligned}
$$

Note that

$$
e^{-\frac{1}{\epsilon}\operatorname{Re}(P(\operatorname{Re} z/2)-P(z))}\le e^{-\frac{C_1}{\epsilon}|\operatorname{Re} z|^{1+\gamma}}\le C|z-2i|^{-l}
$$

for any integer $l>0$, which completes the proof. □

Lemma 2.13. *If $\sup_{\mathbf{R}}|y(z)|z-2i|^m|<\infty$, for $m\ge 0$, then $z\in\mathcal{R}$,*

$$
\big|Y_2(z)\int_{-\infty}^{z} y(t)Y_1(t)dt\big|\le\epsilon\frac{C\sup|(z+i)^m y(z)|}{|z-2i|^{\gamma+m}}\,,
$$

where C is a constant independent of ϵ and $y(z)$.

The proof is similar to the proof of Lemma 2.12.

Lemma 2.14. *If $\sup_{\tilde{\mathcal{R}}}|\tilde{y}(z)|z-2i|^m|<\infty$, for $m\ge 0$, then*

$$
\begin{gathered}
\big|\tilde{Y}_1(z)\int_{\infty}^{z}\tilde{y}(t)\tilde{Y}_2(t)dt\big|\le\epsilon\frac{C\max|(z-2i)^m\tilde{y}(z)|}{|z-i|^{\gamma+m}} \\
\big|\tilde{Y}_2(z)\int_{-\infty}^{z}\tilde{y}(t)\tilde{Y}_1(t)dt\big|\le\epsilon\frac{C\max|(z-2i)^m\tilde{y}(z)|}{|z-i|^{\gamma+m}}
\end{gathered}
$$

where C is a constant independent of ϵ and $\tilde{y}(z)$.

The proof is similar to the proof of Lemmas 2.12 and 2.13.

Lemma 2.15. *Let $u,v\in\mathbf{A}_{0,\delta}$, then for δ sufficiently small and $z\in\mathcal{R}$*

$$
|N(\epsilon,z,u,v)|\le C|z-2i|^{-\tau+\gamma}(\epsilon^2+\delta(\|u\|_0+\|v\|_0)) \tag{2.32}
$$

Proof. By (1.8), (1.9) and (1.10):

$$
\begin{aligned}
&|N(\epsilon,z,u,v)|\\
&\le \sum_{k=2}^{n}|p_k(z)||T_k(u,v)|+\epsilon^2\sum_{k=0}^{l}|f_k(z)||Q_k(u,v)|\\
&\le C\sum_{k=2}^{n}|z-2i|^{-\tau+\gamma+k\tau}\sum_{\alpha_k+\beta_k\ge k}|t_{\alpha_k,\beta_k}||u|^{\alpha_k}|v|^{\beta_k}\\
&\quad+\epsilon^2 C\sum_{k=0}^{l}|z-2i|^{-\tau+\gamma+k\tau}\sum_{\alpha_k+\beta_k\ge k}|q_{\alpha_k,\beta_k}||u|^{\alpha_k}|v|^{\beta_k}\\
&\le C\sum_{k=2}^{n}|z-2i|^{-\tau+\gamma+k\tau}\sum_{\alpha_k+\beta_k\ge k}A\rho^{\alpha_k+\beta_k}(|z-2i|^{-\tau}\|u\|)^{\alpha_k}(|z-2i|^{-\tau}\|v\|)^{\beta_k}\\
&+\epsilon^2 C\sum_{k=0}^{l}|z-2i|^{-\tau+\gamma+k\tau}\sum_{\alpha_k+\beta_k\ge k}A\rho^{\alpha_k+\beta_k}(|z-2i|^{-\tau}\|u\|_0)^{\alpha_k}(|z-2i|^{\tau}\|v\|_0)^{\beta_k}\\
&\le C|z-2i|^{-\tau+\gamma}\sum_{k=2}^{n}\sum_{\alpha_k+\beta_k\ge k}A(\rho(\|u\|_0+\|v\|_0))^{\alpha_k+\beta_k}\\
&\quad+C\epsilon^2|z-2i|^{-\tau+\gamma}\sum_{k=0}^{l}\sum_{\alpha_k+\beta_k\ge k}A(\rho\delta)^{\alpha_k+\beta_k}
\end{aligned}
\tag{2.33}
$$

If δ is less than $\frac{1}{4\rho}$, then

$$
\sum_{k=2}^{n}\sum_{\alpha_k+\beta_k\ge k}A(\rho(\|u\|+\|v\|))^{\alpha_k+\beta_k}\le C\delta(\|u\|_0+\|v\|_0),
$$

$$
\sum_{k=0}^{l}\sum_{\alpha_k+\beta_k\ge k}A(\rho\delta)^{\alpha_k+\beta_k}\le C
$$

hence, the Lemma follows from (2.33). □

Lemma 2.16. *Let $\tilde u,\tilde v\in\tilde{\mathbf{A}}_{0,\delta}$, then for δ sufficiently enough and $z\in\tilde{\mathcal{R}}$*

$$
|N(\epsilon,z,\tilde u,\tilde v)|\le C|z-2i|^{-\tau+\gamma}(\epsilon^2+\delta(\|\tilde u\|_0+\|\tilde v\|_0)) \tag{2.34}
$$

The proof is similar to that of Lemma 2.15.

We want to convert (2.17) and (2.18) into integral equations by using variation of parameters.

Definition 2.17. Define operator $\mathcal{U}$ so that for function $N(z)$ satisfying

$$
\sup_{z\in\mathcal{R}}|z-2i|^{m}|N(z)|<\infty, m+\gamma>0
$$

$$
\mathcal{U}(N)[z]:=\frac{Y_1(z)}{\sqrt{2}(1+i)\epsilon}\int_{\infty}^{z}N(t)Y_2(t)dt-\frac{Y_2(z)}{\sqrt{2}(1+i)\epsilon}\int_{-\infty}^{z}N(t)Y_1(t)dt \tag{2.35}
$$

Remark 2.18. In light of Lemma 2.12 and Lemma 2.13, we have

$$
\sup_{z\in\mathcal{R}}|z-2i|^{m+\gamma}|\mathcal{U}N[z]|<C\sup_{z\in\mathcal{R}}|z-2i|^{m}|N(z)|
$$

Definition 2.19. Let $\phi \in \mathbf{A}_{0,\delta}$, we define operator $G(\epsilon, \phi)$ so that

$$G(\epsilon,\phi)[z] := \mathcal{U}(N_1(\epsilon,\phi))[z] \tag{2.36}$$

Lemma 2.20. *Let $\phi(z) \in \mathbf{A}_{0,\delta}$, then for sufficiently small δ, $G(\epsilon,\phi) \in \mathbf{A}_0$ and*

$$\|G(\epsilon,\phi)\|_0 \le C(\delta\|\phi\|_0 + \epsilon^2) \tag{2.37}$$

Proof. Since $\phi(z) \in \mathbf{A}_{0,\delta}$, by lemma 2.5, $I_+(\phi) \in \mathbf{A}_0$, $\|I_+(\phi)\| \le C\|\phi\|_0$. By (2.29) and Lemma 2.15, we have

$$|N_1(\epsilon,t,\phi)[z]| \le C|z-2i|^{-\tau+\gamma}(\epsilon^2 + \delta\|\phi\|_0)$$

Then the lemma follows from Remark 2.18. □

Lemma 2.21. *Let $\phi(z) \in \mathbf{A}_{0,\delta}$, ϕ is a solution to (2.17) if and only if it is a solution to the following integral equation:*

$$\phi(z) = \mathcal{U}(-LI_+(\phi))[z] + G(\epsilon,\phi)[z] \tag{2.38}$$

Proof. Using variation of parameters, the equation (2.17) is equivalent to the integral equation

$$\phi(z) = C_1Y_1(z) + C_2Y_2(z) + \mathcal{U}(-LI_+(\phi))[z] + G(\epsilon,\phi)[z]$$

since $\phi(z) \in \mathbf{A}_{0,\delta}$,by (1.6) and Lemma 2.5, we have $|L(z)I_+(\phi)(z)| \le C|z-2i|^{-\tau+\gamma}$. By Remark 2.18, $\mathcal{U}(LI_+(\phi)) \in \mathbf{A}_0$. Note $Y_1(z) \to \infty$ as $z \to -\infty$ and $Y_2(z) \to \infty$ as $z \to \infty$, we must have $C_1 = C_2 = 0$. □

By the same method, we convert (2.18) into integral equations.

Definition 2.22. Define operator $\tilde{\mathcal{U}}$ so that for function $\tilde{N}(z)$ satisfying

$$\sup_{z\in\tilde{\mathcal{R}}} |z-2i|^m|\tilde{N}(z)| < \infty, m+\gamma > 0$$

$$\tilde{\mathcal{U}}(\tilde{N})[z] := \frac{\tilde{Y}_1(z)}{\sqrt{2}(1-i)\epsilon}\int_\infty^z \tilde{N}(t)\tilde{Y}_2(t)dt - \frac{\tilde{Y}_2(z)}{\sqrt{2}(1-i)\epsilon}\int_{-\infty}^z \tilde{N}(t)\tilde{Y}_1(t)dt \tag{2.39}$$

Remark 2.23. In light of Lemma 2.14, we have

$$\sup_{z\in\tilde{\mathcal{R}}} |z-2i|^{m+\gamma}\tilde{\mathcal{U}}(\tilde{N})[z] < \sup_{z\in\tilde{\mathcal{R}}} |z-2i|^m|\tilde{N}(z)|$$

Definition 2.24. Let $\tilde{\phi}(z) \in \tilde{\mathbf{A}}_{0,\delta}$, we define operator $\tilde{G}(\epsilon,\tilde{\phi})$ so that

$$\tilde{G}(\epsilon,\tilde{\phi})[z] := \mathcal{U}(\tilde{N}_1(\epsilon,\tilde{\phi}))[z] \tag{2.40}$$

Lemma 2.25. *Let $\tilde{\phi}(z) \in \tilde{\mathbf{A}}_{0,\delta}$, then for sufficiently small δ, $G(\epsilon,\tilde{\phi}) \in \tilde{\mathbf{A}}_0$ and*

$$\|\tilde{G}(\epsilon,\tilde{\phi})\|_0 \le C(\delta\|\tilde{\phi}\|_0 + \epsilon^2) \tag{2.41}$$

The proof is similar to that of Lemma 2.20.

Lemma 2.26. *Let $\tilde{\phi}(z) \in \tilde{\mathbf{A}}_{0,\delta}$, $\tilde{\phi}$ is a solution to (2.18) if and only if it is a solution to the following integral equation:*

$$\tilde{\phi}(z) = \tilde{\mathcal{U}}(-LI_-(\tilde{\phi}))[z] + \tilde{G}(\epsilon,\tilde{\phi})[z] \tag{2.42}$$

The proof is similar to the proof of Lemma 2.21.

We would not be able to use contraction argument in integral equation (2.38) or (2.42) to get existence of solution, since the linear part of the left hand side of (2.38) or (2.42), $\mathcal{U}(-LI_\pm(\phi))$, is not a contraction. In the following lemma, we are going to deal with this linear term.

Lemma 2.27. *Let $\phi(z) \in \mathbf{A}_{0,\delta}$, ϕ is a solution to (2.38) if and only if it is a solution to the following integral equation:*

$$\phi(z) = iI_+(G(\epsilon,\phi))(z) + i\epsilon I_+(G_1(\phi,\phi))(z) + G(\epsilon,\phi)[z] + \epsilon G_1(\phi,\phi)[z] \tag{2.43}$$

where G_1 is an operator acting on two functions u and v

$$\begin{aligned} G_1(u,v)[z] &= \frac{iY_1(z)}{2\epsilon}\int_\infty^z \Big(\frac{1}{4}L^{-1/2}(t)L'(t)I_+(u)(t) + L^{1/2}(t)\frac{d}{dt}\{I_+(v)(t)\}\Big)Y_2(t)dt \\ &+ \frac{iY_2(z)}{2\epsilon}\int_{-\infty}^z \Big(\frac{1}{4}L^{-1/2}(t)L'(t)I_+(u) + L^{1/2}(t)\frac{d}{dt}\{I_+(v)(t)\}\Big)Y_1(t)dt \end{aligned} \tag{2.44}$$

Proof. Integrating by parts,

$$\mathcal{U}(-LI_+(\phi)) = -iI_+(\phi) + \epsilon G_1(\phi,\phi) \tag{2.45}$$

From (2.38),(2.45):

$$\phi(z) = -iI_+(\phi)(z) + G(\epsilon,\phi)[z] + \epsilon G_1(\phi.\phi))[z] \tag{2.46}$$

In above equation,Let $\operatorname{Im} z \to 0^+$, i.e z goes to $x = \operatorname{Re} z$ on the real axis, using Plemelj formulae:

$$\mathcal{H}(\phi)(x) = -iG(\epsilon,\phi)[x] - i\epsilon G_1(\phi,\phi)[x] \tag{2.47}$$

Using Hilbert inverse formulae:

$$\phi(x) = i\mathcal{H}(G(\epsilon,\phi))(x) + i\epsilon\mathcal{H}(G_1(\phi,\phi))(x) \tag{2.48}$$

Extending (2.48) to upper half z-plane, we get (2.43).
Conversely,since the above steps are reversible,it can be seen that a solution $\phi \in \mathbf{A}$ to (2.43) satisfies (2.38). □

Lemma 2.28. *Let $\tilde\phi(z) \in \tilde{\mathbf{A}}_{0,\delta}$, $\tilde\phi$ is a solution to (2.42) if and only if it is a solution to the following integral equation:*

$$\tilde\phi = -iI_-(\tilde G(\epsilon,\tilde\phi)) - i\epsilon I_-(\tilde G_1(\tilde\phi,\tilde\phi)) + \tilde G(\epsilon,\tilde\phi) + \epsilon\tilde G_1(\tilde\phi,\tilde\phi) \tag{2.49}$$

where $\tilde G_1$ is an operator acting on $\tilde u$ and $\tilde v$

$$\begin{aligned} \tilde G_1(\tilde u,\tilde v) &= -\frac{i\tilde Y_1(z)}{2\epsilon}\int_\infty^z \Big(\frac{1}{4}L^{-1/2}(t)L'(t)I_-(\tilde u) + L^{1/2}(t)\frac{d}{dt}\{I_-(\tilde v)(t)\}\Big)\tilde Y_2(t)dt \\ &- \frac{i\tilde Y_2(z)}{2\epsilon}\int_{-\infty}^z \Big(\frac{1}{4}L^{-1/2}(t)L'(t)I_-(\tilde u) + L^{1/2}(t)\frac{d}{dt}\{I_-(\tilde v)(t)\}\Big)\tilde Y_1(t)dt \end{aligned} \tag{2.50}$$

The proof is parallel to that of Lemma 2.27.

To get the small ϵ factor in front of $G_1(\phi,\phi)$ terms in equation (2.43), we paid a price. Instead we get the derivative term $\frac{d}{dt}I_+(\phi)$ in the expression of $G_1(\phi,\phi)$. Since $\frac{d}{dt}I_+(\phi)$ can not be estimated in terms of $\|\phi\|_0$ (see Remark 2.6),we are not be able to use contraction argument to prove existence in (2.43). Same situation is true for (2.49). To circumvent this difficulty, we replace the derivative term $\frac{d}{dt}I_+(\phi)$ in $G_1(\phi,\phi)$ with $\frac{d}{dt}I_+(\tilde\phi)$ to get $G_1(\phi,\tilde\phi)$, and $\frac{d}{dt}I_-(\tilde\phi)$ in $\tilde G_1(\tilde\phi,\tilde\phi)$ with $\frac{d}{dt}I_-(\phi)$ to get $\tilde G_1(\tilde\phi,\phi)$. That is, we consider a **coupled system** of integral equations:

$$\phi(z) = iI_+(G(\epsilon\phi)) + i\epsilon I_+(G_1(\phi,\tilde\phi)) + G(\epsilon,\phi) + \epsilon G_1(\phi,\tilde\phi) \tag{2.51}$$

$$\tilde\phi(z) = -iI_-(\tilde G(\epsilon,\tilde\phi)) - i\epsilon I_-(\tilde G_1(\tilde\phi,\phi)) + \tilde G(\epsilon,\tilde\phi) + \epsilon\tilde G_1(\tilde\phi,\phi) \tag{2.52}$$

Remark 2.29. If $\phi \in \mathbf{A}_{0,\delta}, \tilde{\phi} \in \tilde{\mathbf{A}}_{0,\delta}$ are solutions to the coupled system (2.51) and (2.52), and $\phi(x) = \tilde{\phi}(x)$ for real x,then (2.51) is equivalent to (2.43) in $\mathcal{R} \cup \{\operatorname{Im} z > 0\}$,(2.52) is equivalent to (2.49) in $\tilde{\mathcal{R}} \cup \{\operatorname{Im} z < 0\}$. Therefore $\phi \equiv \tilde{\phi}$ in $\mathcal{R} \cup \tilde{\mathcal{R}}$, and by lemma 2.9, Lemma 2.21 and Lemma 2.27, either ϕ or $\tilde{\phi}$ is a solution to (1.3).

3. Existence and Uniqueness of Analytic Solutions

In this section, we use a contraction argument to prove the existence and uniqueness of solutions of a coupled system. Then we argue that solutions to the coupled system are solutions of (1.3).

Lemma 3.1. *If $y \in \mathbf{A}_{0,\delta}, \tilde{y}(z) \in \tilde{\mathbf{A}}_{0,\delta}$, then $G_1(y, \tilde{y}) \in \mathbf{A}_0$ and*

$$\|G_1(y, \tilde{y})\|_0 \le C(\|y\|_0 + \|\tilde{y}\|_0) \tag{3.1}$$

where C is independent of ϵ and $y, \tilde{y}$.

Proof. By Lemma 2.5, $\frac{d}{dz} I_+(\tilde{y}) \in \mathbf{A}_1$, and $\|\frac{d}{dz} I_+(\tilde{y})\|_1 \le C\|\tilde{y}\|_0$. Since $|L^{1/2}| \sim O(|z - 2i|^{\gamma/2})$, we have

$$\sup_{z\in\mathcal{R}} |z - 2i|^{1+\tau-\gamma/2} |L^{1/2}(z) \frac{d}{dz} I_+(\tilde{y})| \le C\|\tilde{y}\|_0 \tag{3.2}$$

By Lemma 2.5, $I_+(y) \in \mathbf{A}_0$, $\|I_+(y)\|_0 \le C\|y\|_0$.Using (1.6), we have

$$\sup_{z\in\mathcal{R}} |z - 2i|^{1+\tau-\gamma/2} |L^{-1/2}(z) L'(z) I_+(y)| \le C\|y\|_0 \tag{3.3}$$

The Lemma follows from Lemma 2.12, Lemma 2.13 and (2.44). □

Lemma 3.2. *If $y \in \mathbf{A}_{0,\delta}, \tilde{y}(z) \in \tilde{\mathbf{A}}_{0,\delta}$, then $\tilde{G}_1(\tilde{y}, y) \in \tilde{\mathbf{A}}_0$ and*

$$\|\tilde{G}_1(\tilde{y}, y)\|_0 \le C(\|y\|_0 + \|\tilde{y}\|_0). \tag{3.4}$$

where C is independent of ϵ and $y, \tilde{y}$.

The proof is similar to the proof of Lemma 3.1.

Lemma 3.3. *If $y_k \in \mathbf{A}_{0,\delta}, \tilde{y}_k(z) \in \tilde{\mathbf{A}}_{0,\delta}$, then*

$$\|G_1(y_1, \tilde{y}_1) - G_1(y_1, \tilde{y}_2)\|_0 \le C(\|y_1 - y_2\|_0 + \|\tilde{y}_1 - \tilde{y}_2\|_0) \tag{3.5}$$

where C is independent of ϵ, y_k, and $\tilde{y}_k$.

Proof. By (2.44), $G_1(y, \tilde{y})$ is linear in y and $\tilde{y}$, the lemma follows from lemma 3.1. □

Lemma 3.4. *If $y_k \in \mathbf{A}_{0,\delta}, \tilde{y}_k(z) \in \tilde{\mathbf{A}}_{0,\delta}$, then*

$$\|\tilde{G}_1(\tilde{y}_1, y_1) - \tilde{G}_1(\tilde{y}_2, y_2)\|_0 \le C(\|y_1 - y_2\|_0 + \|\tilde{y}_1 - \tilde{y}_2\|_0) \tag{3.6}$$

where C is independent of ϵ, y_k, and $\tilde{y}_k$.

The proof is similar to the proof of Lemma 3.3.

Lemma 3.5. *Let $u_k \in \mathbf{A}_{0,\delta}, v_k(z) \in \mathbf{A}_{0,\delta}$, then for sufficiently small δ,*

$$|N(\epsilon, z, u_1, v_1) - N(\epsilon, z, u_2, v_2)| \le C|z-2i|^{-\tau+\gamma}(\epsilon^2+\delta)(\|u_1-u_2\|_0+\|v_1-v_2\|_0) \tag{3.7}$$

where C is independent of ϵ and u_k, v_k.

Proof. For $\alpha_k \geq 1, \beta_k \geq 1$,

$$\begin{aligned} &|u_1^{\alpha_k} v_1^{\beta_k} - u_2^{\alpha_k} v_2^{\beta_k}| \\ &\leq |(u_1^{\alpha_k} - u_2^{\alpha_k})||v_1|^{\beta_k} + |u_2|^{\alpha_k} |v_1^{\beta_k} - v_2^{\beta_k}| \\ &\leq \alpha_k (|u_1|^{\alpha_k - 1} + |u_2|^{\alpha_k - 1})|u_1 - u_2||v_1|^{\beta_k} + \beta_k (|v_1|^{\beta_k - 1} + |v_2|^{\beta_k - 1})|v_1 - v_2||u_2|^{\alpha_k} \\ &\leq |z - 2i|^{-(\alpha_k + \beta_k)\tau} \big\{ 2\alpha_k \delta^{\alpha_k + \beta_k - 1} \|u_1 - u_2\|_0 + 2\beta_k \delta^{\alpha_k + \beta_k - 1} \|v_1 - v_2\|_0 \big\} \\ &\leq |z - 2i|^{-(\alpha_k + \beta_k)\tau} \big\{ 2(\alpha_k + \beta_k) \delta^{\alpha_k + \beta_k - 1} (\|u_1 - u_2\|_0 + \|v_1 - v_2\|_0) \big\} \end{aligned} \tag{3.8}$$

By (1.8) and (1.9):

$$\begin{aligned} &|N(\epsilon, z, u_1, v_1) - N(\epsilon, z, u_2, v_2)| \\ &\leq \sum_{k=2}^{n} |p_k(z)| \sum_{\alpha_k + \beta_k \geq k} |t_{\alpha_k, \beta_k}||u_1^{\alpha_k} v_1^{\beta_k} - u_2^{\alpha_k} v_2^{\beta_k}| \\ &\quad + \epsilon^2 \sum_{k=0}^{l} |f_k(z)| \sum_{\alpha_k + \beta_k \geq 1} |q_{\alpha_k, \beta_k}||u_1^{\alpha_k} v_1^{\beta_k} - u_2^{\alpha_k} v_2^{\beta_k}| \\ &\leq C|z - 2i|^{-\tau + \gamma} (\|u_1 - u_2\|_0 + \|v_1 - v_2\|_0) \\ &\quad \times \sum_{k=2}^{n} \sum_{\alpha_k + \beta_k \geq k} A\rho^{\alpha_k + \beta_k} \{2(\alpha_k + \beta_k)\delta^{\alpha_k + \beta_k - 1}\} \\ &\quad + C\epsilon^2 |z - 2i|^{-\tau + \gamma} (\|u_1 - u_2\|_0 + \|v_1 - v_2\|_0) \\ &\quad \times \sum_{k=1}^{l} \sum_{\alpha_k + \beta_k \geq k} A\rho^{\alpha_k + \beta_k} \{2(\alpha_k + \beta_k)\delta^{\alpha_k + \beta_k - 1}\} \end{aligned} \tag{3.9}$$

For $\delta \leq \frac{1}{4\rho}$,

$$\begin{gathered} \sum_{\alpha_k + \beta_k \geq 2} A\rho^{\alpha_k + \beta_k} \{2(\alpha_k + \beta_k)\delta^{\alpha_k + \beta_k - 1}\} \leq C\delta \\ \sum_{\alpha_k + \beta_k \geq 1} A\rho^{\alpha_k + \beta_k} \{2(\alpha_k + \beta_k)\delta^{\alpha_k + \beta_k - 1}\} \leq C \end{gathered}$$

Equation (3.7) follows from (3.9) and the inequalities above. □

Lemma 3.6. *If $y_k \in \mathbf{A}_{0,\delta}$, then*

$$\|G(\epsilon, y_1) - G(\epsilon, y_2)\|_0 \leq C(\epsilon^2 + \delta)\|y_1 - y_2\|_0 \tag{3.10}$$

where C is independent of ϵ .

Proof. By lemma 2.5, we have $I_+(y_k) \in \mathbf{A}_0, \|I_+(y_k)\| \leq C\|y_k\|_0$. Replacing $u_k = y_k, v_k = I_+(y_k) - iy_k$ in Lemma 3.5, we have

$$|N(\epsilon, z, u_1, v_1) - N(\epsilon, z, u_2, v_2)| \leq C|z - 2i|^{-\tau + \gamma} (\epsilon^2 + \delta)\|y_1 - y_2\|_0 \tag{3.11}$$

By Remark 2.11,

$$|\epsilon^2 L_1(z)(y_1 - y_2)| \leq \epsilon^2 |z - 2i|^{-\tau - 2} \|y_1 - y_2\|_0 \,. \tag{3.12}$$

Then the lemma follows from (2.29), Definition 2.19 and Remark 2.18. □

Lemma 3.7. *If $\tilde{y}_k \in \tilde{\mathbf{A}}_{0,\delta}, k = 1, 2$, then*

$$\|\tilde{G}(\epsilon, \tilde{y}_1) - \tilde{G}(\epsilon, \tilde{y}_2)\|_0 \le C(\epsilon^2 + \delta)\|\tilde{y}_1 - \tilde{y}_2\|_0 \tag{3.13}$$

where C is independent of ϵ.

The proof is simimlar to the proof of Lemma 3.6.

Lemma 3.8. *If $\phi \in \mathbf{A}_{0,\delta}, \tilde{\phi} \in \mathbf{A}_{0,\delta}$ is a solution of equations (2.51) and (2.52) for sufficiently small δ but independent of ϵ, then $\|\phi\|_0 \le K_0\epsilon^2, \|\tilde{\phi}\|_0 \le K_0\epsilon^2$. where K_0 is some constant independent of ϵ.*

Proof. By (2.51) and Lemmas 2.5, 2.20, 3.1:

$$\begin{aligned} \|\phi\|_0 &\le \|I_+(G(\epsilon, \phi))\|_0 + \epsilon\|I_+(G_1(\phi, \tilde{\phi}))\| + \|G(\epsilon, \phi)\|_0 + \epsilon\|G_1(\phi, \tilde{\phi})\|_0 \\ &\le C_1\|G(\epsilon, \phi)\|_0 + C_1\epsilon\|G_1(\phi, \tilde{\phi})\|_0 \\ &\le C_2(\epsilon^2 + \delta\|\phi\|_0) + C_2\epsilon(\|\phi\|_0 + \|\tilde{\phi}\|_0) \end{aligned} \tag{3.14}$$

By (2.52), Lemmas 2.7, 2.25, 3.2:

$$\begin{aligned} \|\tilde{\phi}\|_0 &\le \|I_-(\tilde{G}(\epsilon, \tilde{\phi}))\|_0 + \epsilon\|I_-(\tilde{G}_1(\tilde{\phi}, \phi))\| + \|\tilde{G}(\epsilon, \tilde{\phi})\|_0 + \epsilon\|\tilde{G}_1(\tilde{\phi}, \phi)\|_0 \\ &\le C_1\|\tilde{G}(\epsilon, \tilde{\phi})\|_0 + C_1\epsilon\|\tilde{G}_1(\tilde{\phi}, \phi)\|_0 \\ &\le C_2(\epsilon^2 + \delta\|\tilde{\phi}\|_0) + C_2\epsilon(\|\phi\|_0 + \|\tilde{\phi}\|_0)\,. \end{aligned} \tag{3.15}$$

Adding (3.15) to (3.14),

$$\|\phi\|_0 + \|\tilde{\phi}\|_0 \le C_3(\epsilon + \delta)(\|\phi\|_0 + \|\tilde{\phi}\|_0) + C_3\epsilon^2 \tag{3.16}$$

Choosing δ small enough so that $C_3(\epsilon + \delta) \le \frac{1}{2}$, the lemma follows from (3.16). □

Definition 3.9.

$$\mathbf{E} = \mathbf{A}_{0,\delta} \oplus \tilde{\mathbf{A}}_{0,\delta} \tag{3.17}$$

$$\|(y, \tilde{y})\|_{\mathbf{E}} = \|y\|_{\mathbf{A}_0} + \|\tilde{y}\|_{\tilde{\mathbf{A}}_0} \tag{3.18}$$

It is clear that $\mathbf{E}$ is a Banach space.

Definition 3.10. Let

$$\mathbf{E}_\epsilon = \{(y, \tilde{y}) \in \mathbf{E} \mid \|y\|_0 \le K\epsilon^2, \|\tilde{y}\|_0 \le K\epsilon^2\}, \tag{3.19}$$

Definition 3.11. For $\mathbf{e} = (\phi, \tilde{\phi}) \in \mathbf{E}$, we define operator $O(\mathbf{e})$ as follows:

$$O(\mathbf{e}) = (O_1(\mathbf{e}), O_2(\mathbf{e})) \tag{3.20}$$

where

$$O_1(\mathbf{e}) = iI_+(G(\epsilon, \phi)) + i\epsilon I_+(G_1(\phi, \tilde{\phi})) + G(\epsilon, \phi) + \epsilon G_1(\phi, \tilde{\phi}) \tag{3.21}$$

$$O_2(\mathbf{e}) = -iI_-(\tilde{G}(\epsilon, \tilde{\phi})) - i\epsilon I_-(\tilde{G}_1(\tilde{\phi}, \phi))) + \tilde{G}(\epsilon, \tilde{\phi}) + \epsilon\tilde{G}_1(\tilde{\phi}, \phi) \tag{3.22}$$

Theorem 3.12. *For sufficiently small ϵ and properly chosen K,the operator $O(\mathbf{e})$ is a contraction mapping from $\mathbf{E}_\epsilon$ to $\mathbf{E}_\epsilon$; therefore there exists a unique solution $(\phi, \tilde{\phi}) \in \mathbf{E}_\epsilon$ to (2.51),(2.52).*

Proof. By (3.21) and replacing $\delta = K\epsilon^2$ in (3.14), we have $\|O_1(\mathbf{e})\|_0 \le C\epsilon^2 + CK\epsilon^3 + CK\epsilon^4$.By (3.22) and replacing $\delta = K\epsilon^2$ in (3.15), we have $\|O_2(\mathbf{e})\|_0 \le C\epsilon^2 + CK\epsilon^3 + CK\epsilon^4$. Hence for K suitably chosen, we have $O(\mathbf{E}_\epsilon) \subset \mathbf{E}_\epsilon$.
Let $(\phi_1, \tilde{\phi}_1) \in \mathbf{E}_\epsilon, (\phi_2, \tilde{\phi}_2) \in \mathbf{E}_\epsilon$. By (3.21), Lemmas 2.5, 3.3, 3.5:

$$\begin{aligned}
&\| O_1(\phi_1, \tilde{\phi}_1) - O_1(\phi_2, \tilde{\phi}_2)\|_0\\
&\le \|I_+(G(\epsilon, \phi_1) - G(\epsilon, \phi_2))\| + \epsilon\|I_+(G_1(\phi_1, \tilde{\phi}_1) - G_1(\phi_2, \tilde{\phi}_2))\|\\
&quad + \|(G(\epsilon, \phi_1) - G(\epsilon, \phi_2))\| + \epsilon\|(G_1(\phi_1, \tilde{\phi}_1) - G_1(\phi_2, \tilde{\phi}_2))\|\\
&\le C\|(G(\epsilon, \phi_1) - G(\epsilon, \phi_2))\| + \epsilon C\|(G_1(\phi_1, \tilde{\phi}_1) - G_1(\phi_2, \tilde{\phi}_2))\|\\
&\le \epsilon\|(y_1 - y_2, \tilde{y}_1 - \tilde{y}_2)\|_{\mathbf{E}}\,.
\end{aligned}$$

By (3.22), Lemmas 2.7, 3.4, 3.6, we have

$$\begin{aligned}
&\| O_2(\phi_1, \tilde{\phi}_1) - O_1(\phi_2, \tilde{\phi}_2)\|_0\\
&\le \|I_-(\tilde{G}(\epsilon, \tilde{\phi}_1) - G(\epsilon, \tilde{\phi}_2))\| + \epsilon\|I_-(\tilde{G}_1(\tilde{\phi}_1, \phi_1) - \tilde{G}_1(\tilde{\phi}_2, \phi_2))\|\\
&\quad + \|(\tilde{G}(\epsilon, \tilde{\phi}_1) - \tilde{G}(\epsilon, \tilde{\phi}_2))\| + \epsilon\|(\tilde{G}_1(\tilde{\phi}_1, \phi_1) - \tilde{G}_1(\tilde{\phi}_2, \phi_2))\|\\
&\le C\|(\tilde{G}(\epsilon, \tilde{\phi}_1) - \tilde{G}(\epsilon, \phi_2))\| + \epsilon C\|(\tilde{G}_1(\tilde{\phi}_1, \phi_1) - \tilde{G}_1(\tilde{\phi}_2, \phi_2))\|\\
&\le C\epsilon\|(\phi_1 - \phi_2, \tilde{\phi}_1 - \tilde{\phi}_2)\|_{\mathbf{E}}
\end{aligned}$$

□

Next, we show that $\phi \equiv \tilde{\phi}$ in the above Theorem. First, we show that $u = \phi - \tilde{\phi}$ satisfies a homogeneous differential equation on real axis. Then we use the a priori estimates obtained in §2 and §3 to prove $u \equiv 0$.

Definition 3.13. We define the differential operators

$$\mathcal{V}\phi = \epsilon^2\phi'' + (-iL + \epsilon^2 L_1)\phi\,, \tag{3.23}$$

$$\tilde{\mathcal{V}}\tilde{\phi} = \epsilon^2\tilde{\phi}'' + (iL + \epsilon^2 L_1)\tilde{\phi}\,. \tag{3.24}$$

Remark 3.14. By (2.35) and (2.39), we have $\mathcal{V}\mathcal{U}N = N$ and $\tilde{\mathcal{V}}\tilde{\mathcal{U}}\tilde{N} = \tilde{N}$.

Lemma 3.15. *Let $(\phi, \tilde{\phi})$ be as in Theorem 3.12,then $(\phi, \tilde{\phi})$ satisfies the following equation on real axis:*

$$\epsilon^2\tilde{\phi}'' + L\mathcal{H}(\phi) = \epsilon^2 i(\mathcal{H}(\tilde{\phi} - \phi))'' + \epsilon^2 L_1[(\phi - \tilde{\phi}) + i\mathcal{H}(\tilde{\phi} - \phi)] + N(\epsilon, x, \phi, \mathcal{H}(\phi)) \tag{3.25}$$

Proof. In equation (2.51), let $\operatorname{Im} z \to 0^+$. Using Plemelj formula,

$$\phi = i\mathcal{H}(G(\epsilon, \phi)) + i\epsilon\mathcal{H}(G_1(\phi, \tilde{\phi})), \tag{3.26}$$

On the real axis. Applying Hilbert inverse Transform:

$$\mathcal{H}(\phi) = -iG(\epsilon, \phi) - i\epsilon G_1(\phi, \tilde{\phi}) \tag{3.27}$$

on real axis. Extending above to $\operatorname{Im} z > 0$, and using Plemelj formula,

$$\phi = -iI_+(\phi) + G(\epsilon, \phi) + \epsilon G_1(\phi, \tilde{\phi}). \tag{3.28}$$

Using (2.45), we have

$$-iI_+(\phi) = \mathcal{U}(-LI_+(\phi)) - \epsilon G_1(\phi, \phi). \tag{3.29}$$

Substituting (3.29) in (3.28), we have

$$\phi = G(\epsilon, \phi) + \epsilon G_1(\phi, \tilde{\phi} - \phi) + \mathcal{U}(-LI_+(\phi))\,. \tag{3.30}$$

Using (2.45), we have

$$\epsilon G_1(\phi, \tilde{\phi} - \phi) = iI_+(\tilde{\phi} - \phi) + \mathcal{U}(-LI_+(\tilde{\phi} - \phi)). \tag{3.31}$$

Substituting (3.31) in (3.30), we have

$$\phi = G(\epsilon, \phi) + \mathcal{U}(-LI_+(\tilde{\phi})) + iI_+(\tilde{\phi} - \phi). \tag{3.32}$$

Applying $\mathcal{V}$ to (3.30) and Using Remark 3.14,

$$\mathcal{V}\phi = R_1(\epsilon, \phi) - LI_+(\tilde{\phi}) + \mathcal{V}(iI_+(\tilde{\phi} - \phi)) \tag{3.33}$$

Let $\operatorname{Im} z \to 0^+$ in (2.33) and using Plemelj formula, we get the lemma. □

Lemma 3.16. *Let $(\phi, \tilde{\phi})$ be as in Theorem 3.12, then $(\phi, \tilde{\phi})$ satisfies the following equation on real axis:*

$$\epsilon^2\phi'' + L\mathcal{H}(\tilde{\phi}) = \epsilon^2 i(\mathcal{H}(\tilde{\phi} - \phi))'' + \epsilon^2 L_1[(\tilde{\phi} - \phi) + i\mathcal{H}(\tilde{\phi} - \phi)] + N(\epsilon, x, \tilde{\phi}, \mathcal{H}(\tilde{\phi})) \tag{3.34}$$

Proof. Starting with (2.52), using the same steps as in the proof of Lemma 3.15, we get the lemma. □

From now on in this section, we are going to work with different domains. We use notation $\mathbf{A}(\mathcal{D})$ to indicate the dependence of function space on domain $\mathcal{D}$. Let $\mathcal{R}_1 = \mathcal{R}_{\alpha_0,\varphi_0} \cap \tilde{\mathcal{R}}_{\alpha_0,\varphi_0}$. By Definition 1.1,$\mathcal{R}_1 = \mathcal{R}_{\alpha_0/2,\varphi_0/2} \cup \tilde{\mathcal{R}}_{\alpha_0/2,\varphi_0/2}$.

Lemma 3.17. *Let $(\phi, \tilde{\phi})$ be as in Theorem 3.12,then*

$$N(\epsilon, x, \tilde{\phi}, \mathcal{H}(\tilde{\phi})) - N(\epsilon, x, \phi, \mathcal{H}(\phi)) = B_1(\epsilon, x)(\tilde{\phi} - \phi) + B_2(\epsilon, x)\mathcal{H}(\tilde{\phi} - \phi) \tag{3.35}$$

where $B_1(\epsilon, x)$ and $B_2(\epsilon, x)$ can be extended to $\mathcal{R}_1$ and

$$\sup_{z\in\mathcal{R}_1} |z - 2i|^{\tau-\gamma}|B_j(\epsilon, z)| \leq C\epsilon^2, \quad j = 1, 2 \tag{3.36}$$

Proof. By Theorem 3.12, $\phi \in \mathbf{A}(\mathcal{R}_1), \tilde{\phi} \in \mathbf{A}(\mathcal{R}_1)$ and

$$\|\phi\|_{0,\mathcal{R}_1} \leq K\epsilon^2, \|\tilde{\phi}\|_{0,\mathcal{R}_1} \leq K\epsilon^2 \tag{3.37}$$

Let $u_1 = \tilde{\phi}, v_1 = \mathcal{H}(\tilde{\phi})$ and $u_2 = \phi, v_2 = \mathcal{H}(\phi)$. By Lemma 2.5 and Lemma 2.7:

$$\|u_j\|_{0,\mathcal{R}_1} \leq K\epsilon^2, \quad \|v_j\|_{0,\mathcal{R}_1} \leq K\epsilon^2, \quad j = 1, 2 \tag{3.38}$$

For any integer $m \geq 1$, define

$$g_m(x) := \frac{[u_1(x)]^m - [u_2(x)]^m}{u_1(x) - u_2(x)} = \sum_{k=0}^{m-1} u_1^k u_2^{m-1-k} \tag{3.39}$$

$$h_m(x) := \frac{[v_1(x)]^m - [v_2(x)]^m}{v_1(x) - v_2(x)} = \sum_{k=0}^{m-1} v_1^k v_2^{m-1-k} \tag{3.40}$$

Then g_m and h_m can be extended to $\mathcal{R}_1$ and for $z \in \mathcal{R}_1$,

$$\begin{aligned} |g_m(z)| &\leq \sum_{j+k=m-1} |u_1^j||u_2^k| \\ &\leq \sum_{j+k=m-1} \{|z - 2i|^{-\tau}\|u_1\|_0\}^j\{|z - 2i|^{-\tau}\|u_2\|_0\}^k \\ &\leq Km\epsilon^{2(m-1)}|z - 2i|^{-\tau(m-1)} \end{aligned} \tag{3.41}$$

Similarly

$$|h_m(z)| \leq Km\epsilon^{2(m-1)}|z - 2i|^{-\tau(m-1)}\,. \tag{3.42}$$

For $\alpha_k+\beta_k\ge 1$,

$$\begin{aligned} u_1^{\alpha_k}v_1^{\beta_k}-u_2^{\alpha_k}v_2^{\beta_k} &= (u_1^{\alpha_k}-u_2^{\alpha_k})v_1^{\beta_k}+u_2^{\alpha_k}(v_1^{\beta_k}-v_2^{\beta_k}) \\ &= (u_1-u_2)g_{\alpha_k}v_1^{\beta_k}+(v_1-v_2)u_2^{\alpha_k}h_{\beta_k} \end{aligned} \tag{3.43}$$

By (1.8) and (1.9), we have

$$\begin{aligned} N(\epsilon,x,u_1,v_1)-N(\epsilon,x,u_2,v_2) &= \sum_{k=2}^{n} p_k(x)\sum_{\alpha_k+\beta_k\ge k} t_{\alpha_k,\beta_k}[u_1^{\alpha_k}v_1^{\beta_k}-u_2^{\alpha_k}v_2^{\beta_k}] \\ &\quad + \epsilon^2\sum_{k=1}^{n} f_k(x)\sum_{\alpha_k+\beta_k\ge k} q_{\alpha_k,\beta_k}[u_1^{\alpha_k}v_1^{\beta_k}-u_2^{\alpha_k}v_2^{\beta_k}] \end{aligned} \tag{3.44}$$

Substituting (3.43) in (3.44), we have (3.35) with B_1 and B_2 given by:

$$B_1(\epsilon,x)=\sum_{k=2}^{n} p_k(x)\sum_{\alpha_k+\beta_k\ge k} t_{\alpha_k,\beta_k}g_{\alpha_k}(x)v_1^{\beta_k}+\epsilon^2\sum_{k=1}^{n} f_k(x)\sum_{\alpha_k+\beta_k\ge k} q_{\alpha_k,\beta_k}g_{\alpha_k}(x)v_1^{\beta_k} \tag{3.45}$$

$$B_2(\epsilon,x)=\sum_{k=2}^{n} p_k(x)\sum_{\alpha_k+\beta_k\ge k} t_{\alpha_k,\beta_k}h_{\alpha_k}(x)v_1^{\beta_k}+\epsilon^2\sum_{k=1}^{n} f_k(x)\sum_{\alpha_k+\beta_k\ge k} q_{\alpha_k,\beta_k}h_{\alpha_k}(x)v_1^{\beta_k} \tag{3.46}$$

To obtain (3.36) and to show the convergence of the series in B_1 and B_2, we consider

$$\begin{aligned} &\sum_{k=2}^{n}|p_k(x)|\sum_{\alpha_k+\beta_k\ge k}|t_{\alpha_k,\beta_k}||g_{\alpha_k}(x)||v_1|^{\beta_k} \\ &\le \sum_{k=2}^{n}|z-2i|^{-\tau+\gamma+k\tau} \\ &\quad\times\sum_{\alpha_k+\beta_k\ge k} A\rho^{\alpha_k+\beta_k}(\alpha_k(K\epsilon^2)^{\alpha_k-1}|z-2i|^{-\tau(\alpha_k-1)})(|z-2i|^{-\tau}\|v_1\|)^{\beta_k} \\ &\le |z-2i|^{\gamma-\tau}\sum_{k=2}^{n}\sum_{\alpha_k+\beta_k\ge k} A\rho\alpha_k(K\epsilon^2\rho)^{\alpha_k+\beta_k-1} \\ &\le C\epsilon^2|z-2i|^{\gamma-\tau} \end{aligned} \tag{3.47}$$

The other series can be estimated similarly. Then the proof is complete. □

Lemma 3.18. *Let $u=\phi-\tilde\phi$, then u satisfies the following homogeneous equation on real axis:*

$$\epsilon^2u''-L\mathcal{H}(u)=-2\epsilon^2L_1u+B_1(\epsilon,x)u+B_2(\epsilon,x)\mathcal{H}(u) \tag{3.48}$$

This lemma follows from Lemmas 3.16 and 3.17.

Lemma 3.19. *The function u satisfies the homogeneous equations:*

$$\epsilon^2u''+(iL+\epsilon^2L_1)u=LI_+(u)-\epsilon^2L_1u+B_1(\epsilon,z)u+B_2(\epsilon,z)(I_+(u)-iu) \tag{3.49}$$

for $z\in\mathcal{R}_{\alpha_0/2,\varphi_0/2}$, and

$$\epsilon^2u''+(-iL+\epsilon^2L_1)u=LI_-(u)-\epsilon^2L_1u+B_1(\epsilon,z)u+B_2(\epsilon,z)(I_-(u)+iu) \tag{3.50}$$

for $z\in\tilde{\mathcal{R}}_{\alpha_0/2,\varphi_0/2}$.

Proof. Extending (3.48) to $\{\operatorname{Im} z > 0\}$, using Plemelj formula, we obtain (3.49). While extending (3.48) to $\{\operatorname{Im} z < 0\}$, using Plemelj formula, we obtain (3.50). □

Definition 3.20. We define operator

$$\tilde{G}_2(\epsilon, u)[z] := \tilde{\mathcal{U}}(-\epsilon^2 L_1 u + B_1(\epsilon, z)u + B_2(\epsilon, z)(I_+(u) - iu))[z] \tag{3.51}$$

where $\tilde{\mathcal{U}}$ is given by (2.39).

Lemma 3.21. *If $u \in \mathbf{A}(\mathcal{R}_{\alpha_0/2,\varphi_0/2})$, then $\tilde{G}_2(\epsilon, u) \in \mathbf{A}(\mathcal{R}_{\alpha_0/2,\varphi_0/2})$ and*

$$\|\tilde{G}_2(\epsilon, u)\|_{0,\mathcal{R}_{\alpha_0/2,\varphi_0/2}} \le C\epsilon^2 \|u\|_{0,\mathcal{R}_{\alpha_0/2,\varphi_0/2}} \tag{3.52}$$

Proof. By Remark 2.6, $\|I_+(u)\|_{0,\mathcal{R}_{\alpha_0/2,\varphi_0/2}} \le C\|u\|_{0,\mathcal{R}_{\alpha_0/2,\varphi_0/2}}$. Using (3.36), we have

$$\left|-\epsilon^2 L_1 u + B_1(\epsilon, z)u + B_2(\epsilon, z)(I_+(u) - iu)\right| \le C\epsilon^2 |z - 2i|^{\gamma-\tau} \|u\|_{0,\mathcal{R}_{\alpha_0/2,\varphi_0/2}} \tag{3.53}$$

We note that Lemma 2.14 still hold if we replace $\tilde{\mathcal{R}}$ by $\mathcal{R}_{\alpha_0/2,\varphi_0/2}$, the lemma follows from Lemma 2.14 , equation (3.51) and (3.53). □

Definition 3.22. We define operator $\tilde{G}_3(u)$ by

$$\begin{aligned}\tilde{G}_3(u)[z] := \frac{i\tilde{Y}_1(z)}{2} \int_\infty^z [L^{1/2}(t) \frac{d}{dt}\{I_+(u)(t)\} + \frac{1}{4} L^{-1/2}(t) L'(t) I_+(u)(t)] \tilde{Y}_2(t) dt \\ + \frac{i\tilde{Y}_2(z)}{2} \int_{-\infty}^z [L^{1/2}(t) \frac{d}{dt}\{I_+(u)(t)\} + \frac{1}{4} L^{-1/2}(t) L'(t) I_+(u)(t)] \tilde{Y}_1(t) dt\end{aligned} \tag{3.54}$$

Lemma 3.23. *If $u \in \mathbf{A}(\mathcal{R}_1)$, then $\tilde{G}_3(u) \in \mathbf{A}(\mathcal{R}_{\alpha_0/2,\varphi_0/2})$ and*

$$\|\tilde{G}_3(u)\|_{0,\mathcal{R}_{\alpha_0/2,\varphi_0/2}} \le C\|u\|_{0,\mathcal{R}_1} \tag{3.55}$$

Proof. Since $\mathcal{R}_{\alpha_0/2,\varphi_0/2} \cap \{\operatorname{Im} z < 0\}$ is an angular subset of $\mathcal{R}_1$,from Remark 2.6, $\|\frac{d}{dz} I_+(u)\|_{1,\mathcal{R}_{\alpha_0/2,\varphi_0/2}} \le C\|u\|_{0,\mathcal{R}_1}$.
Since $|L^{1/2}| \sim O(|z - 2i|^{\gamma/2})$, we have

$$\sup_{z \in \mathcal{R}_{\alpha_0/2,\varphi_0/2}} |z - 2i|^{1+\tau-\gamma/2} |L^{1/2}(z) \frac{d}{dz} I_+(u)| \le C\|u\|_{0,\mathcal{R}_1} \tag{3.56}$$

By Lemma 2.5, $I_+(u) \in \mathbf{A}_{0,\mathcal{R}_1}$, $\|I_+(u)\|_{0,\mathcal{R}_1} \le C\|u\|_{0,\mathcal{R}_1}$.Using (1.6), we have

$$\sup_{z \in \mathcal{R}_{\alpha_0/2,\varphi_0/2}} |z - 2i|^{1+\tau-\gamma/2} |L^{-1/2}(z) L'(z) I_+(u)| \le C\|u\|_{0,\mathcal{R}_1} \tag{3.57}$$

The proof follows from Lemma 2.14, (3.54), (3.56) and (3.57). □

Lemma 3.24. *The function u satisfies the following integral equation for $z \in \mathcal{R}_{\alpha_0/2,\varphi_0/2}$:*

$$u = i[I_+(\tilde{G}_2(\epsilon, u)) - i\tilde{G}_2(\epsilon, u)] + i\epsilon[I_+(\tilde{G}_3(u)) - i\tilde{G}_3(u)] \tag{3.58}$$

Proof. Using variation of parameters in (3.49):

$$u = \tilde{\mathcal{U}}(LI_+(u)) + \tilde{G}_2(\epsilon, u) \tag{3.59}$$

Integration by parts,

$$\tilde{\mathcal{U}}(LI_+(u)) = -iI_+(u) + \epsilon\tilde{G}_3(u) \tag{3.60}$$

From (3.59) and (3.60),

$$u = -iI_+(u) + \epsilon\tilde{G}_3(u) + \tilde{G}_2\,. \tag{3.61}$$

In the above equation, let $\operatorname{Im} z \to 0^+$, using Plemelj formula:

$$\mathcal{H}(u) = -i\epsilon\tilde{G}_3(u) - i\tilde{G}_2 \tag{3.62}$$

Taking inverse Hilbert Transforms,

$$u = i\epsilon\mathcal{H}(\tilde{G}_3(u)) + i\mathcal{H}(\tilde{G}_2). \tag{3.63}$$

Extending (3.63) to $\operatorname{Im} z > 0$, we complete the proof. □

Definition 3.25. We define operator

$$G_2(\epsilon, u)[z] := \mathcal{U}(-\epsilon^2 L_1 u + B_1(\epsilon, z)u + B_2(\epsilon, z)(I_-(u) + iu)) \tag{3.64}$$

where $\mathcal{U}$ is given by (2.35).

Lemma 3.26. *If $u \in \mathbf{A}(\tilde{\mathcal{R}}_{\alpha_0/2,\varphi_0/2})$, then $G_2(\epsilon, u) \in \mathbf{A}(\tilde{\mathcal{R}}_{\alpha_0/2,\varphi_0/2})$ and*

$$\|G_2(\epsilon, u)\|_{0,\tilde{\mathcal{R}}_{\alpha_0/2,\varphi_0/2}} \le C\epsilon^2\|u\|_{0,\tilde{\mathcal{R}}_{\alpha_0/2,\varphi_0/2}} \tag{3.65}$$

The proof is similar to the proof of Lemma 3.21.

Definition 3.27. We define operator

$$\begin{aligned} G_3(u)[z] := &-\frac{iY_1(z)}{2}\int_\infty^z [L^{1/2}(t)\frac{d}{dt}\{I_-(u)(t)\} + \frac{1}{4}L^{-1/2}(t)L'(t)I_-(u)(t)]Y_2(t)dt \\ &- \frac{iY_2(z)}{2}\int_{-\infty}^z [L^{1/2}(t)\frac{d}{dt}\{I_+(u)(t)\} + \frac{1}{4}L^{-1/2}(t)L'(t)I_-(u)(t)]Y_1(t)dt \end{aligned} \tag{3.66}$$

Lemma 3.28. *If $u \in \mathbf{A}(\mathcal{R}_1)$, then $G_3(u) \in \mathbf{A}(\tilde{\mathcal{R}}_{\alpha_0,\varphi_0/2})$ and*

$$\|G_3(u)\|_{0,\tilde{\mathcal{R}}_{\alpha_0,\varphi_0/2}} \le C\|u\|_{0,\mathcal{R}_1} \tag{3.67}$$

The proof is similar to the proof of Lemma 3.23.

Lemma 3.29. *the function u satisfies the following integral equations for $z \in \tilde{\mathcal{R}}_{\alpha_0,\varphi_0/2}$,*

$$u = -i[I_-(G_2(\epsilon, u) + iG_2(\epsilon, u))] - i\epsilon[I_-(G_3(u)) + iG_3(u)] \tag{3.68}$$

The proof is parallel to that of Lemma 3.24.

Lemma 3.30. *Let $(\phi, \tilde{\phi})$ be as in Theorem 3.12, then $\phi \equiv \tilde{\phi}$ in $\mathcal{R}_1$*

Proof. By (3.58), Lemmas 2.5, 3.21, 3.23, we have

$$\begin{aligned} \|u\|_{0,\mathcal{R}_{\alpha_0,\varphi_0/2}} &\le \|I_+(\tilde{G}_2(\epsilon, u))\|_{0,\mathcal{R}_{\alpha_0,\varphi_0/2}} + \epsilon\|I_+(\tilde{G}_3(u))\|_{0,\mathcal{R}_{\alpha_0,\varphi_0/2}} \\ &\quad + \|\tilde{G}_2(\epsilon, \phi)\|_{0,\mathcal{R}_{\alpha_0,\varphi_0/2}} + \epsilon\|\tilde{G}_3(u)\|_{0,\mathcal{R}_{\alpha_0,\varphi_0/2}} \\ &\le C_1\|\tilde{G}_2(\epsilon, u)\|_{0,\mathcal{R}_{\alpha_0,\varphi_0/2}} + C\epsilon\|\tilde{G}_3(u)\|_{0,\mathcal{R}_{\alpha_0,\varphi_0/2}} \\ &\le C(\epsilon + \epsilon^2)\|u\|_{0,\mathcal{R}_1}\,. \end{aligned} \tag{3.69}$$

By (3.68), Lemmas 2.7, 3.26, 3.28, we have

$$\begin{aligned} \|u\|_{0,\tilde{\mathcal{R}}_{\alpha_0,\varphi_0/2}} &\le \|I_-(G_2(\epsilon, u))\|_{0,\tilde{\mathcal{R}}_{\alpha_0,\varphi_0/2}} + \epsilon\|I_-(G_3(u))\|_{0,\tilde{\mathcal{R}}_{\alpha_0,\varphi_0/2}} \\ &\quad + \|G_2(\epsilon, \phi)\|_{0,\tilde{\mathcal{R}}_{\alpha_0,\varphi_0/2}} + \epsilon\|G_3(u)\|_{0,\tilde{\mathcal{R}}_{\alpha_0,\varphi_0/2}} \\ &\le C_1\|G_2(\epsilon, u)\|_{0,\tilde{\mathcal{R}}_{\alpha_0,\varphi_0/2}} + C\epsilon\|G_3(u)\|_{0,\tilde{\mathcal{R}}_{\alpha_0,\varphi_0/2}} \\ &\le C(\epsilon + \epsilon^2)\|u\|_{0,\mathcal{R}_1}\,. \end{aligned} \tag{3.70}$$

Adding (3.69) to (3.70),

$$\|u\|_{0,\mathcal{R}_1} \le \|u\|_{0,\tilde{\mathcal{R}}_{\alpha_0,\varphi_0/2}} + \|u\|_{0,\mathcal{R}_{\alpha_0,\varphi_0/2}} \le C(\epsilon+\epsilon^2)\|u\|_{0,\mathcal{R}_1}\,. \tag{3.71}$$

For sufficiently small ϵ, (3.71) implies $\|u\|_{0,\mathcal{R}_1} = 0$. So $u \equiv 0$ in $\mathcal{R}_1$. □

The proof of Theorem 1.4 follows from Remark 2.29, Lemma 3.8, Lemma 3.12 and Lemma 3.30.

4. Some Explicit Examples

In this section we give explicit functions of $L(x)$ so that Properties 1–6 hold.

Example 1 Let $L(x)$ be constant. Without loss of generality, we assume $L(x) \equiv 1$, then $\gamma = 0$, $P(z) = e^{i\frac{\pi}{4}}z$, $\tilde{P}(z) = e^{-i\frac{\pi}{4}}z$. Let $r = \{t : z + se^{i\theta}, 0 \le s < \infty\}$ be a ray, and z be a complex number. Then along ray r,

$$\operatorname{Re} P(t(s)) = s\cos(\frac{\pi}{4}+\theta) + \operatorname{Re}(e^{i\frac{\pi}{4}}z), \quad \operatorname{Re}\tilde{P}(z) = t\cos(-\frac{\pi}{4}+\theta) + \operatorname{Re}(e^{-i\frac{\pi}{4}}z).$$

Therefore,

$$\begin{gathered}
\frac{d\operatorname{Re} P(t(s))}{ds} = \cos(\frac{\pi}{4}+\theta) > 0, \text{ for } -\frac{\pi}{4} < \theta < \frac{\pi}{4}, \\
\frac{d\operatorname{Re} P(t(s))}{ds} = \cos(\frac{\pi}{4}+\theta) < 0, \text{ for } \frac{3\pi}{4} < \theta < \frac{5\pi}{4}, \\
\frac{d\operatorname{Re} \tilde{P}(t(s))}{ds} = \cos(\frac{\pi}{4}+\theta) > 0, \text{ for } -\frac{\pi}{4} < \theta < \frac{\pi}{4}, \\
\frac{d\operatorname{Re} \tilde{P}(t(s))}{ds} = \cos(\frac{\pi}{4}+\theta) < 0, \text{ for } \frac{3\pi}{4} < \theta < \frac{5\pi}{4},
\end{gathered}$$

Also for $R > 0$, on the line $\{\operatorname{Im} t = \text{constant}, -R \le s = \operatorname{Re} t \le R\}$,

$$\operatorname{Re} P(t(s)) = s\cos\frac{\pi}{4} + \text{costant}, \quad \operatorname{Re}\tilde{P}(t(s)) = s\cos\frac{\pi}{4} + \text{costant},$$

So

$$\frac{d\operatorname{Re} P(t(s))}{ds} = \cos\frac{\pi}{4} > 0, \quad \frac{d\operatorname{Re} \tilde{P}(t(s))}{ds} = \cos\frac{\pi}{4} > 0,$$

Therefore, if we choose α_0 and φ_0 so that $0 < \alpha_0 < 1$, $0 < \varphi_0 < \frac{\pi}{4}$, $R > 0$ can be chosen arbitrarily, then Property 1-6 hold from above equations.

Example 2 Let $L(x) = x^2 + a^2$ with $a \ge 1$. Without loss of generality, we assume $L(x) = x^2 + 1$, then $\gamma = 2$, $P(z) = e^{i\frac{\pi}{4}}\int_0^z \sqrt{1+t^2}dt$, $\tilde{P}(z) = e^{-i\frac{\pi}{4}}\int_0^z \sqrt{1+t^2}dt$, where we use lines $\{ti : t \ge 1\}$ and $\{ti : t \le -1\}$ as the branch cut of $\sqrt{t^2+1}$ and

$$\sqrt{t^2+1} = \sqrt{|t^2+1|}e^{1/2[\arg(t+i)+\arg(t-i)]}$$

where $-\frac{3\pi}{2} \le \arg(t-i) \le \frac{\pi}{2}, -\frac{\pi}{2} \le \arg(t+i) \le \frac{3\pi}{2}$.

Lemma 4.1. *For $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, then as $|z| \to \infty$ and $t \in \{t(s) : t(s) = z + se^{i\theta}, 0 < s < \infty\}$,*

$$\begin{gathered}
\operatorname{Re} P(t(s)) = |t|^2\cos(\frac{\pi}{4}+2\theta)(1+o(1)), \\
\frac{d\operatorname{Re} P(t(s))}{ds} = 2|t|\cos(\frac{\pi}{4}+2\theta)(1+o(1)),
\end{gathered} \tag{4.1}$$

Proof. Note that

$$P(t(s)) = e^{i\frac{\pi}{4}} \int_0^z \sqrt{1+\xi^2}d\xi + e^{i\frac{\pi}{4}} \int_z^t (s)\sqrt{1+\xi^2}d\xi\,. \tag{4.2}$$

The first integral of (4.2) is independent of s, and can be treated as a constant. We choose the path of integration in the second integral of (4.2) to be the ray segment $\{\xi : \xi = z + \rho e^{i\theta}, 0 < \rho < s\}$, then

$$\begin{aligned} P(t(s)) &= e^{i\frac{\pi}{4}} \int_0^s \sqrt{(z+\rho e^{i\theta})+i}\sqrt{(z+\rho e^{i\theta})-i}e^{i\theta}d\rho + \text{constant} \\ &= se^{i(\theta+\frac{\pi}{4})} \int_0^1 \sqrt{(z+s\omega e^{i\theta})+i}\sqrt{(z+s\omega e^{i\theta})-i}d\omega \end{aligned} \tag{4.3}$$

The proof follows from the above equation and the fact that $\arg(z+s\omega e^{i\theta}+i) \to \theta$, $\arg(z+s\omega e^{i\theta}-i) \to \theta$, as $|z| \to \infty$. □

Lemma 4.2. *For $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, if $|z| \to \infty$ and $t \in \{t(s) : t = z + se^{i\theta}, 0 < s < \infty\}$, then*

$$\begin{aligned} \operatorname{Re}\tilde{P}(t(s)) &= |t|^2 \cos(-\frac{\pi}{4} + 2\theta)(1+o(1)), \\ \frac{d\operatorname{Re}\tilde{P}(t(s))}{ds} &= 2|t|\cos(-\frac{\pi}{4}+2\theta)(1+o(1)), \end{aligned} \tag{4.4}$$

The proof is similar to the proof of Lemma 4.1.

Lemma 4.3. *For $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$, if $|z| \to \infty$ and $t \in \{t(s) : t = z + se^{i\theta}, 0 < s < \infty\}$, then*

$$\begin{aligned} \operatorname{Re}P(t) &= |t|^2 \cos(-\frac{3\pi}{4} + 2\theta)(1+o(1)), \\ \frac{d\operatorname{Re}P(t(s))}{ds} &= 2|t|\cos(-\frac{3\pi}{4}+2\theta)(1+o(1)), \end{aligned} \tag{4.5}$$

Proof. The proof follows from (4.3) and the fact that $\arg(z + s\omega e^{i\theta} + i) \to \theta$, $\arg(z+s\omega e^{i\theta}-i) \to \theta - 2\pi$, as $|z| \to \infty$. □

Lemma 4.4. *For $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$, if $|z| \to \infty$ and $t \in \{t(s) : t = z + se^{i\theta}, 0 < s < \infty\}$, then*

$$\begin{aligned} \operatorname{Re}\tilde{P}(t(s)) &= |t|^2 \cos(-\frac{5\pi}{4} + 2\theta)(1+o(1)), \\ \frac{d\operatorname{Re}\tilde{P}(t(s))}{ds} &= 2|t|\cos(-\frac{5\pi}{4}+2\theta)(1+o(1)), \end{aligned} \tag{4.6}$$

Lemma 4.5. *Let $R > 0$ be any fixed number, there exists a number $0 < \alpha_0 < 1$ so that $\frac{d\operatorname{Re}P(t(s))}{ds} > 1$ on line segment $\{z : z = s + id, -R < s < R\}$ where $-\alpha_0 \le d \le \alpha_0$.*

Proof. When $t(s)$ is on the real axis, i.e., $d = 0$, we have

$$\operatorname{Re}P(t(s)) = \cos(\pi/4)\int_0^s \sqrt{1+t^2}dt$$

so $\frac{d\operatorname{Re}P(z)}{ds} = 1 + s^2 > 1$. Since $\frac{d\operatorname{Re}P(t)}{ds}$ is continuous with respect to d, we get the lemma. □

Now by Lemmas 4.1–4.4, there exist R large enough and $\varphi_0 = \pi/8$ so that

$$\frac{d\,\mathrm{Re}\,P(t)}{ds} > C|t| > 0, \frac{d\,\mathrm{Re}\,\tilde{P}(t)}{ds} > C|t| > 0 \tag{4.7}$$

for $z > R$, $t \in \{t(s) = z + se^{i\theta},\ 0 < s < \infty\}$, $-\varphi_0 < \theta < \varphi_0$. Also

$$\frac{d\,\mathrm{Re}\,P(t)}{ds} < -C|t| < 0, \frac{d\,\mathrm{Re}\,\tilde{P}(t)}{ds} < -C|t| < 0 \tag{4.8}$$

for $z < -R$, $t \in \{t(s) = z + se^{i(\pi-\theta)},\ 0 < s < \infty\}$, $-\varphi_0 < \theta < \varphi_0$.

Choose α_0 so that Lemma 4.5 holds. It can be checked easily that Properties 1–6 hold in $\mathcal{R}_{\alpha_0,\varphi_0}$, by (4.7), (4.8) and Lemma 4.5.

Acknowlegment. The author wants to thank Professor Saleh Tanveer for suggesting this problem and for having helpful discussions.

References

[1] Ben Amar, M.; Pomeau,Y., Europhysics letters, **2** (1986), pp 307.
[2] Ben Amar, M. Theory of needle crystal. Physica D 31 (1988), pp409.
[3] G. F. Carrier, M. Krook & C. E. Pearson, *Functions of a complex variable,* McGraw-Hill, 1966.
[4] S. J. Chapman, On the role of Stokes lines in the selection of Saffman-Taylor fingers with small surface tension, Eur. J. Appl. Math. 10, 513-534 (1999)
[5] R.Combescot, V.Hakim, T.Dombre, Y.Pomeau, A.Pumir, Analytic theory of Saffman-Taylor fingers, *Physical Review A* vol.37, number 4(1988), pp1270-1283.
[6] G. P. Ivantsov, Dokl. Acad. Nauk SSSR 58, 569 (1949).
[7] M. D. Kunka, M. R. Foster, S. Tanveer, Dendrtic crystal growth for weak undercooling, Phys. Rev. E 56, 3068-3100.
[8] J. W. Mclean, P. G. Saffman, T he effect of surface tension on the shape of fingers in a Hele Shaw cell, *J.Fluid Mech.* vol.102(1981), pp 455-469.
[9] N. I. Muskhelishvili, Singular Integral Equations, Noordhoff International Publishing, 1977.
[10] G. E. Nash, Naval Research Lab., Report Nos. 7670 and 7680. (unpublished).
[11] F. W. J. Olver, Asymtotics and Special Functions, Academic Press, New York, 1974.
[12] Pelce, P. and Y. Pomeau, Stud. Appl. Math. 74, 245 (1986).
[13] P. G. Saffman, Viscous Fingering in a Hele-shaw cell. *J.Fluid Mech.*, **173**, 73 (1986).
[14] P. G. Saffman, G. I. Taylor, The penetration of a fluid into a porous medium of Hele-Shaw cell containing a more viscous fluid, *Proc. R. Soc. London Ser. A.* 145(1958), 312.
[15] S. Tanveer, Analytic theory for the selection of a two-dimensional needle crystal at arbitrary Peclet number, Phys.Rev. A 40(8) (1989), pp4756.
[16] S. Tanveer, Surprises in Viscous fingering, J. Fluid Mech., vol. 409(2000), pp.273-308.
[17] Tanveer, S.; Xie, X. Analyticity and nonexistence of classical steady Hele-Shaw fingers. *Communications on pure and applied mathematics* vol. 56 (3), 353-402 (2003).
[18] W. Wasow, *Asymtotic Expansion for Odinary Diffential Equations*, Interscience Publishers, New York,1968.
[19] Xie, X. Rigorous results in selection of steady needle crystals, submitted, available at http://www.math.udel.edu/˜ xie
[20] Xie, X. & Tanveer, S. Rigorous results in steady finger selection in viscous fingering. *Archive for rational mechanics and analysis* vol. 166 (2003) 3, 219-286.

Xuming Xie
Department of Mathematical Sciences, University of Delaware,
501 Ewing Hall, Newark, DE 19716, USA
E-mail address: xie@math.udel.edu