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MULTIDIMENSIONAL SINGULAR λ-LEMMA

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Abstract. The well known λ-Lemma [3] states the following: Let f be a C^1-diffeomorphism of $\mathbb{R}^n$ with a hyperbolic fixed point at 0 and m- and p-dimensional stable and unstable manifolds W^S and W^U, respectively ($m+p=n$). Let D be a p-disk in W^U and w be another p-disk in W^U meeting W^S at some point A transversely. Then $\bigcup_{n\geq 0} f^n(w)$ contains p-disks arbitrarily C^1-close to D. In this paper we will show that the same assertion still holds outside of an arbitrarily small neighborhood of 0, even in the case of non-transverse homoclinic intersections with finite order of contact, if we assume that 0 is a low order non-resonant point.

1. Introduction

Let M be a smooth manifold without boundary and $f : M \to M$ be a C^1 map that has a hyperbolic fixed point at the origin. The well known λ-Lemma [3] gives an important description of chaotic dynamics. The basic assumption of this theorem is the presence of a transverse homoclinic point.

Theorem 1.1 (Palis)**.** *Let f be a C^1 diffeomorphism of $\mathbf{R^n}$ with a hyperbolic fixed point at* 0 *and m- and p-dimensional stable and unstable manifolds W^S and W^U $(m+p=n)$. Let D be a p-disk in W^U, and w be another p-disk in W^U meeting W^S at some point A* transversely*. Then $\bigcup_{n\geq 0} f^n(w)$ contains p-disks arbitrarily C^1-close to D.*

The assumption of transversality is not easy to verify for a concrete dynamical system. Obviously, the conclusion of the Theorem of Palis is not true for an arbitrary degenerate (non-transverse) crossing. Example by Newhouse illustrates this situation (See picture 1).

In this paper we prove an analog of the λ-Lemma for the non-transverse case in arbitrary dimension. Suppose W^S and W^U are sufficiently smooth and cross non-transversally at an isolated homoclinic point, i.e. they have a *singular homoclinic crossing*. In Section 2 we define the order of contact for this crossing (Definition 2.3) and show that it is preserved under a diffeomorphic transformation (Lemma 2.5).

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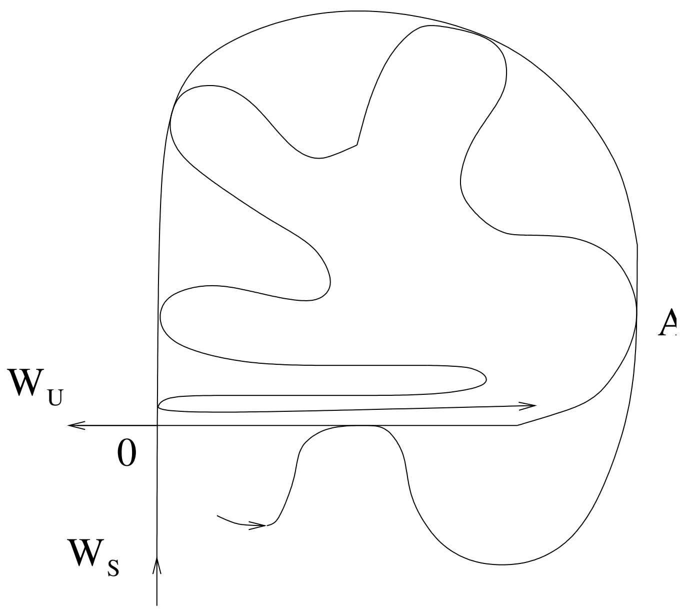

Figure 1. Newhouse example. Branches of W_U are not C^1-close near 0

We prove Singular λ-Lemma for the case of singular finite order homoclinic crossing of manifolds which have a graph portion (see Definition 2.6), under non-resonance restriction. See Lemma 3.1 in Section 3.

2. Definitions and Lemmas

In this section we are considering two immersed C^r manifolds in $\mathbb{R}^n$, $r > 1$. Suppose they meet at an isolated point A. We will discuss the structure of these manifolds in the neighborhood of the point A. First, assume that each manifold is a curve.

Hirsch in his work [2] describes the order of contact for two curves and formulates the following definition:

Definition 2.1. Let Λ_i $(i = 1, 2)$ denote two immersed C^r curves in $\mathbb{R}^2$, $r > 1$. Suppose the two curves meet at point A. Let $t \mapsto u_i(t)$ be a C^r parameterization of Λ_i, both defined for t in some interval I, with non-vanishing tangent vectors $u_i'(t)$. Suppose $0 \in I$ and $A = u_i(0)$. The order of contact of the two curves at A is the unique real number l in the range $1 \le l \le r$, if it exists, such that $u_1 - u_2$ has a root of order l at 0.

For our higher-dimensional proof we can reformulate this definition for two curves in $\mathbb{R}^n$:

Definition 2.2. Let Λ_i $(i = 1, 2)$ denote two immersed C^r curves in $\mathbb{R}^n$, $r > 1$. Suppose the two curves meet at point A. Let $t \mapsto u_i(t)$ be a C^r parameterization of Λ_i, both defined for t in some interval I, with non-vanishing tangent vectors $u_i'(t)$. Suppose $0 \in I$ and $A = u_i(0)$. The order of contact of the two curves at A is the unique real number l in the range $1 \le l \le r$, if it exists, such that $|u_1 - u_2|$ has a root of order l at 0.

Now we can define the order of contact for two manifolds of arbitrary dimensions.

Definition 2.3. Let W^S and W^U denote two immersed C^r manifolds in $\mathbb{R}^n$, $r > 1$. Suppose the two manifolds meet at an isolated point A. The order of contact α at A is the unique real number α in the range $1 \le \alpha \le r$, if it exists, such that

$$\begin{aligned}\alpha = \sup \big\{ l | C^r\text{-curve } \gamma_1 \in W^S \text{ has order of contact } l \text{ with another} \\ C^r\text{-curve } \gamma_2 \in W^U \text{ and } A \in \gamma_1 \cap \gamma_2 \big\}\end{aligned}$$

The order of contact is preserved under a diffeomorphism. This result is first proven for curves (Lemma 2.4).

Lemma 2.4. *Consider a C^∞ surface without boundary and a C^r diffeomorphism ϕ that maps a neighborhood N' of this surface onto some neighborhood $N \subset \mathbf{R^2}$. Assume that $u(t), v(t)$ are C^r curves, such that $u(0) = v(0)$. Then, ϕ preserves the order of contact of these curves.*

Proof. Without lost of generality, we assume that $u(0) = v(0) = 0$. We have curves

$$\phi \circ u(t), \quad \phi \circ v(t),$$

transformed by the diffeomorphism ϕ. There are positive constants m and M such that

$$m \le \frac{|u(t) - v(t)|}{|t|^l} \le M, \quad \text{as } t \to 0.$$

By the C^1 Mean Value Theorem,

$$\phi(x) - \phi(y) = \Big[\int_0^1 (D\phi)_{\sigma(s)} ds \Big] (x - y),$$

where $\sigma(s) = (1 - s)x + sy$. Then

$$(\phi \circ u)(t) - (\phi \circ v)(t) = \Big[\int_0^1 (D\phi)_{\sigma(s)} ds \Big] (u(t) - v(t)),$$

where $\sigma(s) = (1 - s)u(t) + sv(t)$. Therefore,

$$\frac{(\phi \circ u)(t) - (\phi \circ v)(t)}{t^l} = \Big[\int_0^1 (D\phi)_{\sigma(s)} ds \Big] \big(\frac{u(t) - v(t)}{t^l}\big)$$

As $t \to 0$, $\sigma(s) \to u(0)$ and the matrix $\int_0^1 (D\phi)_{\sigma(s)} ds$ tends to the invertible matrix $(D\phi)_{u(0)}$. The ratio $\frac{u(t)-v(t)}{t^l}$ is a vector whose norm is bounded by M and m, $0 < m \le M < \infty$. Hence

$$m \le \Big[\int_0^1 (D\phi)_{\sigma(s)} ds \Big] \big(\frac{u(t) - v(t)}{t^l}\big) \le M.$$

□

This lemma can easily be generalized for higher dimensions.

Lemma 2.5. *Consider a C^∞ surface without boundary and a C^r diffeomorphism ϕ that maps a neighborhood N' of this surface onto some neighborhood $N \subset \mathbf{R^n}$. Assume that $u(t), v(t)$ are C^r manifolds, such that $u(0) = v(0)$. Then, ϕ preserves the order of contact of these manifolds.*

This Lemma follows from Lemma 2.4 and Definition 2.3.

For the estimates in the proof of the Singular λ-Lemma we need the following definition of a graph portion.

Definition 2.6. Let f be a C^r diffeomorphism of $\mathbf{R^n}$ with a hyperbolic fixed point at the origin. Denote by W^S (resp., W^U) the associated stable (resp., unstable) manifold, and by m (resp., p) its dimension ($m + p = n$, $p < m$). Let A be a homoclinic point of W^S and W^U. Suppose that there exists a small p-disk in W^U around point A (call it $\mathcal{U}$), and there exists another small p-disk in W^U around the origin (call it $\mathcal{V}$). Define a local coordinate system E_1 at 0, which spans $\mathcal{V}$. Similarly, define a local coordinate system E_2 in some neighborhood of 0 (we can assume that A belongs to this neighborhood), centered at 0, which spans W^S in this neighborhood. Let $E = E_1 + E_2$. If $\mathcal{U}$ is a graph of a bijective (in E) function defined on $\mathcal{V}$, then $\mathcal{U}$ will be called a graph portion.

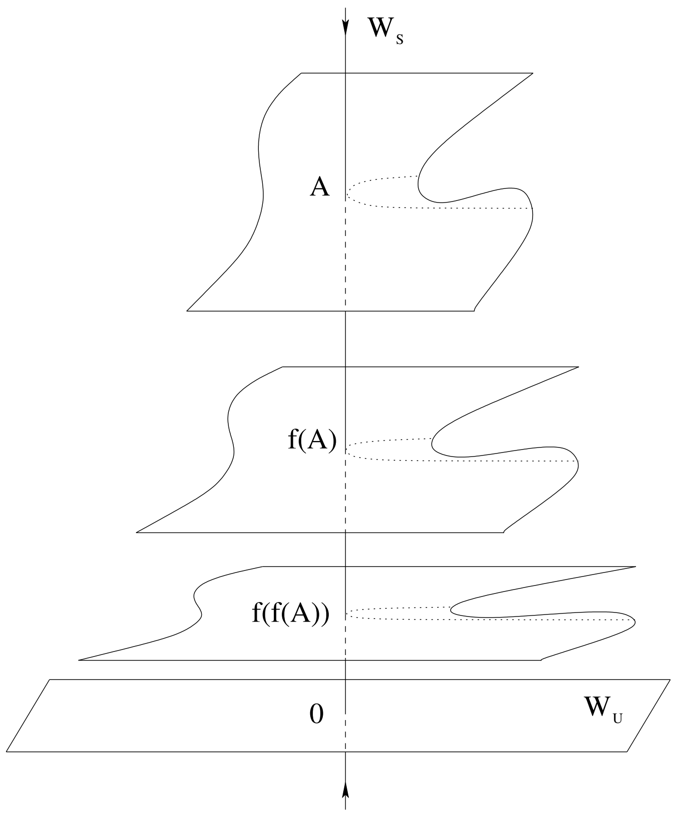

FIGURE 2. In this picture the iterated part of the W^U manifold is not a graph portion of the manifold W^U. It will not become C^1-close to the bottom part with the iterations.

There is another assumption that we have to make for the proof of our λ-Lemma. The assumption is stronger than the regular first order non-resonance condition, but weaker than the second order non-resonance. We will call our restriction one-and-a-half order resonance.

Definition 2.7. Let f be a C^2-diffeomorphism of $\mathbb{R}^n$ with a hyperbolic fixed point at 0 and m- and p-dimensional stable and unstable manifolds, and $f(x,y) : \mathbf{R^n} \to \mathbf{R^n}$ has the linear part $((\mathcal{A}x)_1, \dots, (\mathcal{A}x)_p, (\mathcal{B}y)_1, \dots, (\mathcal{B}y)_m)$. Then, the following condition will be called one-and-a-half order non-resonance condition:
If $a \in \operatorname{spec}\mathcal{A}$ and $b \in \operatorname{spec}\mathcal{B}$, then $ab \notin (spec\mathcal{A} \cup \operatorname{spec}\mathcal{B})$.

3. Singular λ-Lemma

Using the above definitions we formulate the following Singular λ-Lemma.

Lemma 3.1. *Let f be a C^r-diffeomorphism of $\mathbb{R}^n$ with a hyperbolic fixed point at 0 and m- and p-dimensional stable and unstable manifolds W^S and W^U ($p \le m$, $m+p=n$). Let $\mathcal{V}$ be a p-disk in W^U and Λ be a graph portion in W^U having a homoclinic crossing with W^S at some point A. Assume that Λ and W^S have order of contact r ($1 < r < \infty$) at A. Also assume that f is one-and-a-half order non-resonant. Then for any $\rho > 0$, for an arbitrarily small ϵ-neighborhood $\mathcal{U} \subset \mathbb{R}^n$ of the origin and for the graph portion Λ, $(\bigcup_{n\ge 0} f^n(\Lambda)) \setminus \mathcal{U}$ contains disks ρ-C^1 close to $\mathcal{V} \setminus \mathcal{U}$.*

Remark 3.2. There is no loss of generality to assume that $p \le m$, because we can always replace f with f^{-1}.

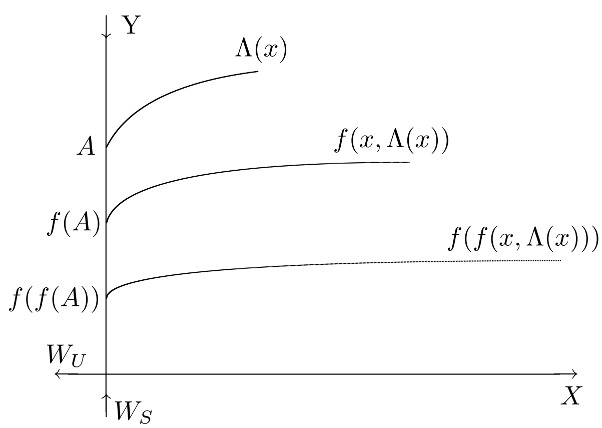

FIGURE 3. Iterations of the graph portion Λ with the diffeomorphism f

Proof of Lemma 3.1. Let $\alpha = 1/l$ ($0 < \alpha < 1$). Since Λ is a graph portion that has finite order of contact with W^S, we can assume that locally Λ is represented by the graph of the following form:

$$\Lambda(x) = A + r(x) : \mathbb{R}^p \to \mathbb{R}^m, \quad r(0) = 0,$$

and for any sufficiently small $\sigma > 0$

$$|r(x)| \le \text{const} \cdot |x|^\alpha \quad \text{and} \quad |\frac{\partial}{\partial x_i} r(x)| \le \text{const} \cdot |x|^{\alpha-1}$$

for all $|x| < \sigma$, $i = 1, \dots, p$. Let $x = (x_1, \dots, x_p) \in \mathbf{R^p}$, $y = (y_1, \dots, y_m) \in \mathbf{R^m}$ $(p + m = n)$ and $f(x, y) : \mathbf{R^n} \to \mathbf{R^n}$ has the linear part

$$((\mathcal{A}x)_1, \dots, (\mathcal{A}x)_p, (\mathcal{B}y)_1, \dots, (\mathcal{B}y)_m).$$

Assume that $\|\mathcal{A}^{-1}\|$, $\|\mathcal{B}\| < \lambda < 1$. Choose an arbitrarily small Δ. If there is a cross terms const $\cdot x_i y_j$ in the power expansion of this map around 0, then we assume one-and-a-half-order non-resonance condition. Then, by Flattening Theorem (See [4]) there exists smooth change of coordinates, such that locally f can be written in the form $f(x, y) = (S_1(x, y), S_2(x, y))$, where

$$\begin{aligned} S_1(x,y) =& \Big(\big((\mathcal{A}x)_1 + \phi_1(x) + \sum_{i=1,\dots,p; j=1,\dots,m} x_i y_j U^1_{ij}(x,y)\big), \dots, \\ & \big((\mathcal{A}x)_p + \phi_p(x) + \sum_{i=1,\dots,p; j=1,\dots,m} x_i y_j U^p_{ij}(x,y)\big)\Big) \end{aligned}$$

and

$$\begin{aligned} S_2(x,y) =& \Big(\big((\mathcal{B}y)_1 + \psi_1(y) + \sum_{i=1,\dots,p; j=1,\dots,m} x_i y_j V^1_{ij}(x,y)\big), \dots, \\ & \big((\mathcal{B}y)_m + \psi_m(y) + \sum_{i=1,\dots,p; j=1,\dots,m} x_i y_j V^m_{ij}(x,y)\big)\Big). \end{aligned}$$

Here $U(0) = V(0) = 0$, $\|\phi\|_{C^1}$, $\|\psi\|_{C^1}$, $\|U\|_{C^0}$, $\|V\|_{C^0} \le \Delta$, and $\|U\|_{C^1}$, $\|V\|_{C^1}$ are bounded.

Consider $f(x, \Lambda(x)) = (T^\Lambda_1(x), T^\Lambda_2(x))$. We will work with $(x, T^\Lambda_2 \circ (T^\Lambda_1)^{-1}(x))$ and deduce that $f^n(x, \Lambda(x))$ is C^1-small for n big enough and $\sigma > 0$ sufficiently small. First we will show that in C^1-topology $(T^\Lambda_1)^{-1}$ is Δ-close to $\mathcal{A}^{-1}$. For simplicity we will denote T^Λ_1 by T_1 and T^Λ_2 by T_2.

$$\begin{aligned} T_1(x) =& \Big((\mathcal{A}x)_1 + \phi_1(x) + \sum_{i=1,\dots,p; j=1,\dots,m} x_i \Lambda_j(x) U^1_{ij}(x, \Lambda(x)), \dots, \\ & (\mathcal{A}x)_p + \phi_p(x) + \sum_{i=1,\dots,p; j=1,\dots,m} x_i \Lambda_j(x) U^p_{ij}(x, \Lambda(x))\Big). \end{aligned}$$

Claim 3.3.
$$\| \sum_{i=1,\dots,p; j=1,\dots,m} x_i \Lambda_j(x) U^t_{ij}(x, \Lambda(x))\|_{C^1} < K \cdot \Delta$$
for $|x| < \sigma$ ($\sigma > 0$ sufficiently small, $K > 0$).

Proof. Fix some $l \in \{1, \dots, p\}$. Recall that $\Lambda(x) = A + r(x)$.

$$\begin{aligned} \big|\frac{\partial}{\partial x_l} x_i \Lambda_j(x)\big| &\le \delta_{il}|\Lambda(x)| + |x_i| \cdot |\frac{\partial}{\partial x_l}\Lambda_j(x)| \\ &\le \delta_{il}(|A| + |x|^\alpha) + |x| \cdot O(1)|x|^{\alpha-1} \\ &\le |A|\delta_{il} + (\delta_{il} + O(1))|x|^\alpha = O(1) \end{aligned}$$

Here

$$\delta_{il} = \begin{cases} 1 & \text{if } i = l, \\ 0 & \text{if } i \ne l. \end{cases}$$

Through the proof of this Theorem, $O(1)$ will be the set

$$\begin{aligned}O(1) =\big\{&\gamma(\zeta) : \mathbb{R} \mapsto \mathbb{R} \text{ such that there exists a positive constant } c \text{ with}\\ &|\gamma(\zeta)| \le c \text{ for all sufficiently small } \zeta\big\}\end{aligned}$$

Also,

$$\big|\frac{\partial}{\partial x_l} U^t_{ij}(x, \Lambda(x))\big| = \big|\frac{\partial}{\partial x_l} U^t_{ij}(x, y) + \sum_{k=1}^{m} \frac{\partial}{\partial y_k} U^t_{ij}(x, y) \cdot \frac{\partial}{\partial x_l} \Lambda_k(x)\big| = O(1)\,.$$

Therefore,

$$\begin{aligned}&\big\| \sum_{i=1,\dots,p;j=1,\dots,m} x_i \Lambda_j(x) U^t_{ij}(x, \Lambda(x))\big\|_{C^1}\\ &\le \sum_{i=1,\dots,p;j=1,\dots,m} \big|\sum_{l=1}^{p} \frac{\partial}{\partial x_l}(x_i \Lambda_j(x) U^t_{ij}(x, \Lambda(x)))\big|\\ &\le \sum_{i=1,\dots,p;j=1,\dots,m} \sum_{l=1}^{p} \big|\frac{\partial}{\partial x_l}(x_i \Lambda_j(x)) \cdot U^t_{ij}(x, \Lambda(x)) + x_i \Lambda_j(x) \cdot \frac{\partial}{\partial x_l} U^t_{ij}(x, \Lambda(x))\big|\\ &\le \Delta \cdot O(1),\end{aligned}$$

if σ is sufficiently small and $|x| < \sigma$ (Arbitrarily small Δ was chosen above). The estimate proves the claim. $\square$

Now, we continue the proof of Lemma 3.1. As it was noted earlier in the proof, $\|\phi\|_{C^1} \le \Delta$, by Flattening Theorem. This estimate and the assertion of the Claim imply that $\|\mathcal{A}-T_1\|_{C^1} = O(1)\cdot\Delta$. This obviously implies $\|\mathcal{A}^{-1}-T_1^{-1}\|_{C^1} = O(1)\cdot\Delta$.

Now we can do the main estimate, – the estimate for $\|T_2 \circ T_1^{-1}\|_{C^k}$ $(k = 0, 1)$.

$$\begin{aligned}T_2 \circ T_1^{-1} =\Big(&(\mathcal{B}\Lambda(T_1^{-1}))_1 + \psi_1(\Lambda(T_1^{-1}))\\ &+ \sum_{i=1,\dots,p;j=1,\dots,m} (T_1^{-1})_i(\Lambda(T_1^{-1}))_j V^1_{ij}(T_1^{-1}, \Lambda(T_1^{-1})), \dots,\\ &(\mathcal{B}\Lambda(T_1^{-1}))_m + \psi_m(\Lambda(T_1^{-1}))\\ &+ \sum_{i=1,\dots,p;j=1,\dots,m} (T_1^{-1})_i(\Lambda(T_1^{-1}))_j V^m_{ij}(T_1^{-1}, \Lambda(T_1^{-1}))\Big)\end{aligned}$$

We will begin by estimating each term of this vector.

$$\mathcal{B}\Lambda(T_1^{-1}) = \mathcal{B} \cdot A + \mathcal{B} \cdot r(T_1^{-1}(x)).$$

$$|\mathcal{B} \cdot r(T_1^{-1}(x))| = O(1) \cdot \|\mathcal{B}\||T_1^{-1}(x)|^{\alpha} = O(1) \cdot \|\mathcal{B}\|(\|\mathcal{A}^{-1}\| + \Delta)^{\alpha}|x|^{\alpha}.$$

By the chain rule,

$$\begin{aligned}&\big|\frac{\partial}{\partial x_l} \mathcal{B} \cdot r(T_1^{-1}(x))\big|\\ &= O(1) \cdot \|\mathcal{B}\|\|T_1^{-1}\|_{C^1}|T_1^{-1}(x)|^{\alpha-1}\\ &= O(1) \cdot \|\mathcal{B}\|(\|\mathcal{A}^{-1}\| + \Delta)(\|\mathcal{A}^{-1}\| + \Delta)^{\alpha-1}|x|^{\alpha-1}\\ &= O(1) \cdot \|\mathcal{B}\|(\|\mathcal{A}^{-1}\| + \Delta)^{\alpha}|x|^{\alpha-1}\\ &= O(1) \cdot \lambda|x|^{\alpha-1}\end{aligned}$$

with $\lambda < 1$. Moreover,

$$|\frac{\partial}{\partial x_l}\mathcal{B}^n \cdot r(T_1^{-n}(x))| = O(1)\cdot \|\mathcal{B}\|^n(\|\mathcal{A}^{-1}\|^n+\Delta)^\alpha |x|^{\alpha-1} = O(1)\cdot \lambda^n|x|^{\alpha-1}$$

This term can be made small if we perform enough iterations by the map f. I.e., $(\mathcal{B}^n\Lambda T_1^{-n})_m$ is C^1-small outside of a fixed neighborhood of 0, if n is big enough. For the estimates of the next term one can use the following expansion:

$$\psi_1(\Lambda(T_1^{-1}(x))) = \psi_1(A + r(T_1^{-1}(x))) = \psi_1(A) + D\psi_1(A)\cdot r(T_1^{-1}(x)) + R(T_1^{-1}(x)),$$

where $R(T_1^{-1}(x)) = o(|(T_1^{-1}(x))^\alpha)|$. Here the set $o(1)$ is the following set of functions:

$$\begin{aligned} o(1) = \big\{ &\gamma(\zeta) : \mathbb{R} \mapsto \mathbb{R} \text{ such that for any positive constant } c \\ &\text{and for all sufficiently small } \zeta < \sigma, |\gamma(\zeta)| < c \big\} \end{aligned}$$

Similar to the previous calculations $\psi_1(\Lambda(T_1^{-1}(x)))$ can be made small in C^1-norm if we perform enough iterations with the map f. Finally, we will note that the last term

$$\sum_{i=1,\dots,p;j=1,\dots,m} (T_1^{-1})_i(\Lambda(T_1^{-1}))_j V_{ij}^t(T_1^{-1},\Lambda(T_1^{-1}))$$

can be written as a composition $\Sigma^t \circ T_1^{-1}(x)$, where

$$\Sigma^t(x) = \sum_{i=1,\dots,p;j=1,\dots,m} x_i\Lambda_j(x)V_{ij}^t(x,\Lambda(x)).$$

Consider $\frac{\partial}{\partial x_l}\Sigma^t \circ T_1^{-1}(x)$.

$$\frac{\partial}{\partial x_l}\Sigma^t \circ T_1^{-1}(x) = \sum_{i=1}^{p}\frac{\partial}{\partial x_i}\Sigma^t \circ T_1^{-1}(x)\cdot \frac{\partial}{\partial x_l}(T_1^{-1}(x))_i.$$

We have already shown that

$$\Big\| \sum_{i=1,\dots,p;j=1,\dots,m} x_i\Lambda_j(x)U_{ij}^t(x,\Lambda(x))\Big\|_{C^1} = O(1)\cdot\Delta.$$

Similar, one can show that

$$\|\Sigma^t\|_{C^1} = \Big\| \sum_{i=1,\dots,p;j=1,\dots,m} x_i\Lambda_j(x)V_{ij}^t(x,\Lambda(x))\Big\|_{C^1} = O(1)\cdot\Delta.$$

Also,

$$\|T_1^{-1}\|_{C^1} \le \|\mathcal{A}^{-1}\|_{C^1} + \|T_1^{-1}-\mathcal{A}^{-1}\|_{C^1} \le \|\mathcal{A}^{-1}\|_{C^1} + O(1)\cdot\Delta.$$

The estimates on $\|\Sigma^t\|_{C^1}$ and $\|T_1^{-1}\|_{C^1}$, together with the fact that $T(0)=0$, imply that

$$\|\Sigma^t\circ T_1^{-1}\|_{C^1} = O(1)\cdot\Delta.$$

Thus, for any small positive number ρ and for any small (but bigger than a fixed ϵ) $|x|$ one can find n such that $(x,(T_2^\Lambda)^n \circ (T_1^\Lambda)^{-n}(x))$ is ρ-C^1-close to $\mathcal{V}$. This implies that for any $\rho>0$ and for an arbitrarily small ϵ-neighborhood $\mathcal{U}\subset\mathbb{R}^n$ of the origin, $(\bigcup_{n\ge0} f^n(\Lambda))\setminus\mathcal{U}$ contains p-disks ρ-C^1-close to $\mathcal{V}\setminus\mathcal{U}$. $\square$

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