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MAGNETIC BARRIERS OF COMPACT SUPPORT AND EIGENVALUES IN SPECTRAL GAPS

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Abstract. We consider Schrödinger operators $H = -\Delta + V$ in $L_2(\mathbb{R}^2)$ with a spectral gap, perturbed by a strong magnetic field $\mathcal{B}$ of compact support. We assume here that the support of $\mathcal{B}$ is connected and has a connected complement; the total magnetic flux may be zero or non-zero. For a fixed point E in the gap, we show that (for a sequence of couplings tending to ∞) the signed spectral flow across E for the magnetic perturbation is equal to the flow of eigenvalues produced by a high potential barrier on the support of the magnetic field. This allows us to use various estimates that are available for the high barrier case.

1. Introduction

We study the discrete eigenvalues that may appear in a spectral gap of a Schrödinger operator acting in $L_2(\mathbb{R}^2)$,

$$H = -\Delta + V, \tag{1.1}$$

under perturbation by a strong magnetic field $\mathcal{B}$ of compact support. In the present paper we restrict our attention to the case where the support of $\mathcal{B}$ as well as its complement are connected sets.

For a bounded and measurable potential $V = V(x)$ and a magnetic vector potential $\vec{a} \in C^1(\mathbb{R}^2;\mathbb{R}^2)$ with $\operatorname{curl}\vec{a} = \mathcal{B}$, the associated Schrödinger operator is defined as

$$H(\lambda\vec{a}) = (-\mathrm{i}\nabla - \lambda\vec{a}(x))^2 + V(x), \quad \lambda \geq 0. \tag{1.2}$$

Our main interest is then in the (signed) flow of the eigenvalues of $H(\lambda\vec{a})$ across a fixed observation point E in a spectral gap of H, as the coupling λ tends to infinity. For periodic H, the eigenvalues of $H(\lambda\vec{a})$ can be interpreted as electronic bound states in a thin wafer of solid matter (semi-conductor or insulator) which is locally penetrated by a strong magnetic field.

For a sequence of couplings $\lambda_k \to \infty$ we will establish a direct link between the magnetic problem and the classical case where H is perturbed by $\mu\chi_\Omega$, for $\mu \to \infty$; here χ_Ω denotes the characteristic function of the bounded, open set

$$\Omega = \{x \in \mathbb{R}^2 : \mathcal{B}(x) \neq 0\}. \tag{1.3}$$

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To describe the relevant properties of the high barrier problem, let

$$M = \mathbb{R}^2 \setminus \Omega \tag{1.4}$$

and let H_M denote the operator $-\Delta + V$, acting in M with Dirichlet boundary conditions on ∂M. It is well known that $H + \mu\chi_\Omega$ converges to H_M in the norm resolvent sense, as $\mu \to \infty$, and a finite number of eigenvalues of $H + \mu\chi_\Omega$ may cross a fixed point E inside a gap of H; we denote this number (counting multiplicities) as $\mathcal{N}(\Omega, E)$, cf. (3.2), (3.4). All eigenvalues of $H + \mu\chi_\Omega$ are increasing functions of the coupling μ. Estimates for $\mathcal{N}(\Omega, E)$ will be discussed in Section 3. Compactness implies that $\sigma_{\mathrm{ess}}(H_M) = \sigma_{\mathrm{ess}}(H)$ so that the spectrum of H_M is discrete in $\mathbb{R} \setminus \sigma_{\mathrm{ess}}(H)$.

The eigenvalues of $H + \mu\chi_\Omega$ cross E according to the repulsion produced by the potential barrier $\mu\chi_\Omega$. By the Avron-Herbst-Simon-bound [2] and Dirichlet decoupling [14], a similar repulsive effect is to be expected in the case of a magnetic perturbation. The magnetic case is more delicate, however, because eigenvalues will not depend monotonically on the coupling and, in general, we have to deal with vector potentials that do not vanish identically in M. Because of the lack of monotonicity, we will rather count the signed flow of spectral multiplicity instead of the number of eigenvalues that cross E. Defining the *flux of* $\mathcal{B}$,

$$\Phi = \int \mathcal{B}(x)\,\mathrm{d}x, \tag{1.5}$$

we must distinguish, as usual, between the cases $\Phi = 0$ and $\Phi \neq 0$. For $\Phi = 0$, we may pass to an equivalent vector potential that vanishes in the interior G of M (assuming some regularity of ∂G) and much simpler proofs are available (cf. Remark5.6); the situation is similar in dimensions 3 and higher. Hence the most difficult case is the one where we are in $\mathbb{R}^2$ and Φ is non-zero; in addition, this is the case that is closest to the experimental situation in physics. Note that we may assume $\lambda \geq 0$ and $\Phi \geq 0$ without loss of generality since the operators $H(\lambda\vec{a})$ and $H(-\lambda\vec{a})$ are anti-unitarily equivalent under complex conjugation.

We will work with the following assumptions:

Assumption 1.1. The sets $\overline{\Omega} = \operatorname{supp}\mathcal{B}$ and $G = (\operatorname{supp}\mathcal{B})^C$ are both connected, with $\operatorname{supp}\mathcal{B} \neq \emptyset$.

Loosely speaking, Assumption 1.1 means that $\operatorname{supp}\mathcal{B}$ is connected and has no holes. Assumption 1.1 implies that the fundamental group of G is isomorphic to $\mathbb{Z}$.

Assumption 1.2. The Dirichlet Laplacian $-\Delta_M$ of the closed set M has form core $C_c^\infty(G)$.

The precise definition of $-\Delta_M$ is given in Section 2. It is sufficient for Assumption 1.2 to hold that $\partial\Omega$ has measure zero and that G satisfies the segment condition; cf. Section 2 for details and a more general criterion.

Following Safronov [29, 30], we next define a function that counts the signed spectral multiplicity of $H(\lambda\vec{a})$ crossing E: we let $\mathcal{M}(\lambda; \mathcal{B}, E)$ denote the number of eigenvalues (counting multiplicities) of $H(\mu\vec{a})$ that cross E in upwards direction minus the number of eigenvalues crossing downwards, at couplings $\mu \in (0, \lambda)$; the eigenvalues that change direction at E are not counted. The precise definition of $\mathcal{M}(\lambda; \mathcal{B}, E)$ is given in Equation (2.7). Our main result reads as follows.

Theorem 1.3. *Let H and H_M as above and let $\mathcal{B}:\mathbb{R}^2\to\mathbb{R}$ a continuous function of compact support such that Assumptions 1.1 and 1.2 are satisfied. Let $E\in\mathbb{R}$, $E\notin\sigma(H)\cup\sigma(H_M)$. Then there exists $\Lambda_0\geq 0$ such that*

$$\mathcal{M}(\lambda;\mathcal{B},E)=\mathcal{N}(\Omega,E),\quad \lambda\geq\Lambda_0,\quad \lambda\Phi\in 2\pi\mathbb{Z}. \tag{1.6}$$

We therefore see that the repulsion produced by a strong magnetic field of compact support corresponds to the repulsion from a high potential barrier, supported on the set where $\mathcal{B}$ is non-zero, for a sequence of couplings going to infinity.

Remark 1.4. (i) For flux $\Phi=0$, the only restriction on λ is $\lambda\geq\Lambda_0$. For $\Phi>0$, we obtain a sequence $(\lambda_k)_{k\in\mathbb{N}}\subset(0,\infty)$, with $\lambda_k=2\pi k/\Phi$, such that (1.6) holds for $\lambda_k\geq\Lambda_0$.
(ii) For $E<\inf\sigma(H)$, we have $\mathcal{N}(\Omega,E)=0$ by monotonicity and $\mathcal{M}(\lambda;\mathcal{B},E)=0$ by the diamagnetic inequality.
(iii) In between two successive λ_k's a certain number of eigenvalues of $H(\lambda\vec{a})$ may cross and cross back. While the results of Herbst and Nakamura [18] imply that the eigenvalues in the gap will approach periodic functions, as $\lambda\to\infty$, it appears to be rather difficult to find conditions that would guarantee these perodic limiting functions to be non-constant. At the same time, it is rather unlikely that these periodic functions are constant.

Under the assumptions of Theorem 1.3 there exists a constant $c(E,\vec{a})\geq 0$ such that

$$|\mathcal{M}(\lambda;\mathcal{B},E)-\mathcal{N}(\Omega,E)|\leq c(E,\vec{a}),\quad \lambda\geq\Lambda_1, \tag{1.7}$$

for some $\Lambda_1\geq 0$; cf. Remark 5.5 for a proof. More precisely, it is shown in Remark 5.5 that the constant $c(E,\vec{a})$ in (1.7) can be estimated in terms of a (non-magnetic) eigenvalue problem for which the Birman-Schwinger principle is applicable.
(iv) In a subsequent paper we will discuss magnetic perturbations where the sets $G=M^{\mathrm{int}}$ and $\operatorname{supp}\mathcal{B}$ may have more than one component. Here a theorem of Dirichlet in number theory can be used in the construction of a suitable gauge leading to a "smallest possible" vector potential.
(v) Theorem 1.3 assumes $E\notin\sigma(H)\cup\sigma(H_M)$. If $E\notin\sigma(H)$ belongs to $\sigma(H_M)$, monotonicity with respect to E and Theorem 1.3 yield

$$\mathcal{N}(\Omega,E+\epsilon)\leq\liminf_{\lambda\to\infty,\,\lambda\Phi\in2\pi\mathbb{Z}}\mathcal{M}(\lambda;\mathcal{B},E)\leq\limsup_{\lambda\to\infty,\,\lambda\Phi\in2\pi\mathbb{Z}}\mathcal{M}(\lambda;\mathcal{B},E)\leq\mathcal{N}(\Omega,E), \tag{1.8}$$

for any sufficiently small $\epsilon>0$.

In the following corollaries, Theorem 1.3 is combined with simple upper and lower estimates on $\mathcal{N}(\Omega,E)$ of phase space volume type, taken from [11, 12]; cf. Section 3. The first corollary provides an upper bound:

Corollary 1.5. *Suppose that the assumptions of Theorem 1.3 are satisfied. Then there exist constant C_1, C_2, independent of $\mathcal{B}$, such that*

$$\lim_{\lambda\to\infty,\,\lambda\Phi\in2\pi\mathbb{Z}}\mathcal{M}(\lambda;\mathcal{B},E)\leq C_1R^2+C_2, \tag{1.9}$$

provided $\Omega=\{x;\mathcal{B}(x)\neq 0\}$ is contained in some disk of radius R.

The corresponding lower bound requires an additional assumption on the spectrum of H below E:

Assumption 1.6. Let $\dim_E(H_{B_R})$ denote the number of eigenvalues of H_{B_R} below E, counting multiplicities, where $H_{B_R} = -\Delta + V$ on $B_R := \{x \in \mathbb{R}^2 : |x| < R\}$, with Dirichlet boundary conditions. Let $E \in \mathbb{R} \setminus \sigma(H)$. We assume that there exist constants $c > 0$ and $R_0 \geq 0$ such that

$$\dim_E(H_{B_R}) \geq cR^2, \quad R \geq R_0. \tag{1.10}$$

For this assumption to hold it is sufficient that the i.d.s. for H exists (cf. [28, 21]) and has a positive value at E. If (1.10) holds, H must have some essential spectrum below E.

Corollary 1.7. *In addition to the conditions of Theorem 1.3, suppose that Assumption 1.6 is satisfied. Then there exist constants $c_1 > 0$, $c_2 \geq 0$, independent of $\mathcal{B}$, such that*

$$\lim_{\lambda\to\infty,\, \lambda\Phi\in 2\pi\mathbb{Z}} \mathcal{M}(\lambda; \mathcal{B}, E) \geq c_1 R^2 - c_2, \tag{1.11}$$

provided $\operatorname{supp}\mathcal{B}$ *contains a disk of radius R.*

The paper is organized as follows. In Section 2, we provide basic definitions and describe the fundamental approximation procedure by eigenvalue problems on large disks B_n. In particular, we construct a family of approximating operators $\tilde{H}_n(\lambda\vec{a})$, for $\lambda \geq 0$, acting in $L_2(B_n)$ with Dirichlet boundary conditions.

In Section 3 we recall basic properties of the discrete eigenvalues of $H + \mu\chi_U$, for $0 \leq \mu \to \infty$, where $U \subset \mathbb{R}^m$ is open and bounded.

Section 4 contains the key estimate for the number of eigenvalues below E for the approximating operators $\tilde{H}_n(\lambda\vec{a})$ on large disks B_n. Here we consider the counting functions

$$\mathcal{M}_n(\lambda; \mathcal{B}, E) = \dim_E(\tilde{H}_n(0)) - \dim_E(\tilde{H}_n(\lambda\vec{a})), \tag{1.12}$$

which correspond to the loss of eigenvalues below E due to the magnetic perturbation for the approximating problems on B_n. In a first step, we decouple the problem by a natural Dirichlet boundary condition along ∂G, for λ large. Then, for $\lambda\Phi \in 2\pi\mathbb{Z}$, it is shown that the operator on the "annulus" $B_n \cap G$, with Dirichlet boundary conditions, is unitarily equivalent to an operator without magnetic terms. For the latter the number of eigenvalues below E is easy to estimate. We also obtain a preliminary result on the existence of $\lim_{n\to\infty} \mathcal{M}_n(\lambda; \mathcal{B}, E)$, for λ large, $\lambda\Phi \in 2\pi\mathbb{Z}$, and we show that this limit is equal to $\mathcal{N}(\Omega, E)$.

In Section 5, we finally show that

$$\lim_{n\to\infty} \mathcal{M}_n(\lambda; \mathcal{B}, E) = \mathcal{M}(\lambda; \mathcal{B}, E), \tag{1.13}$$

provided λ is such that $E \notin \sigma(H(\lambda\vec{a}))$. We then prove Theorem 1.3 as well as Corollaries 1.5 and 1.7.

There are three appendices (Sections 6—8) containing some of the more technical arguments.

We conclude the introduction with a few remarks on related work in the literature. General information on magnetic Schrödinger operators can be found in Avron, Herbst and Simon [2], Mohamed and Raikov [24]. Scattering for decreasing magnetic fields has been studied by Loss and Thaller [22]. The problem studied in the present paper is somewhat related to the Bohm-Aharonov effect (cf. Helffer [9], Weder [33]), but with a (periodic) electric background potential.

A situation which has attracted some attention is that of a constant magnetic background, combined with localized electric perturbations (cf., e.g., Birman and

Raikov [5], Hempel and Levendorski [16], Levendorskii [23], Raikov [26]). The existence of eigenvalues in spectral gaps for a perturbation by a magnetic field of compact support has been studied previously in Hempel and Laitenberger [15] under the assumption of zero flux, in a rather special case. In [16] there is a result for the case where both the magnetic field and the vector potential decay exponentially at infinity; again, the flux is zero. The paper [13] constructs examples of Schrödinger operators $H(\lambda\vec{a})$ having an eigenvalue in the gap that asymptotically approaches a periodic or quasi-periodic function of the coupling λ; here the magnetic field has compact support consisting of a disk enclosed by an annulus.

It is interesting to note that the situation can be strikingly different for the Pauli operator in $\mathbb{R}^2$. Here some (purely magnetic) cases can be analyzed in depth by using the property of supersymmetry (cf. [6]). It turns out that, under suitable assumptions, eigenvalues will in fact move downwards where one might at first expect an upwards movement (cf. [3, 4]). We take as the "unperturbed" operator the Pauli Hamiltonian of a constant magnetic field $\mathcal{B}_0 > 0$, and add a perturbation by a magnetic field $\mathcal{B} \le 0$ of compact support. We therefore consider the pair of operators

$$H_\pm(\lambda) = (-\mathrm{i}\nabla - \vec{a}_0 - \lambda\vec{a})^2 \mp \mathcal{B}_0 \mp \lambda\mathcal{B}, \tag{1.14}$$

where we refer to [6] for the construction of the Pauli Hamiltonian in $\mathbb{R}^2$. Looking at $H_+(\lambda)$, one might expect that the repulsive effect of the potential barrier $-\lambda\mathcal{B} \ge 0$, combined with some repulsion coming from the magnetic term $\lambda\vec{a}$, should shift a finite number of eigenvalues from below through each spectral gap of H_+. But this cannot be true for the first gap $(0, 2\mathcal{B}_0)$: by supersymmetry, each non-zero point in the spectrum of $H_+(\lambda)$ must also appear in the spectrum of $H_-(\lambda)$, yet $H_-(0)$ has no spectrum below $2\mathcal{B}_0$. On the other hand, it can be shown that there exist eigenvalues that move downwards through the first gap, as λ increases. A more detailed discussion of the related phenomena can be found in [3, 4].

2. Notation and Preliminaries

General Notation. We write $x = (x_1, x_2) \in \mathbb{R}^2$ and $B_s = \{x \in \mathbb{R}^2 : |x| < s\}$; χ_s denotes the characteristic function of the ball B_s, for $s > 0$. The function spaces $C^k(U)$, $L_2(U)$, for $U \subset \mathbb{R}^d$ open, are defined as usual; the norm of $u \in \mathcal{L}_2(U)$ is given by $\|u\|^2 = \int_U |u|^2 \,\mathrm{d}x$. Scalar products in Hilbert space are written $\langle .,. \rangle$. The Sobolev space $\mathcal{H}^1(\mathbb{R}^2)$ consists of those functions $u \in L_2(\mathbb{R}^2)$ that have weak first derivatives $\partial_j u \in L_2(\mathbb{R}^2)$. $\mathcal{H}^1(\mathbb{R}^2)$ is a Hilbert space with norm $\|u\|_1^2 = \|u\|^2 + \sum_j \|\partial_j u\|^2$.

For a self-adjoint operator T acting in a Hilbert space $\mathcal{H}$ we denote the spectrum by $\sigma(T)$, the essential spectrum by $\sigma_{\mathrm{ess}}(T)$, and the spectral projections by $P_I(T)$, for any interval I of the real line. If T has only discrete spectrum in the interval I, then $\dim \operatorname{ran} P_I(T) = \operatorname{trace} P_I(T)$ is the number of eigenvalues of T in I, counting multiplicities. For $\alpha \le \beta \in \mathbb{R}$ and $\eta \in \mathbb{R}$, we will write

$$\dim_{(\alpha,\beta)}(T) = \dim \operatorname{ran} P_{(\alpha,\beta)}(T), \quad \dim_\eta(T) = \dim \operatorname{ran} P_{(-\infty,\eta)}(T). \tag{2.1}$$

The Dirichlet Laplacian of a Closed Set. We will mostly be concerned with self-adjoint operators defined via quadratic forms, with form domain given by a suitable subspace of the Sobolev space $\mathcal{H}^1(\mathbb{R}^2)$. For any closed set $K \subset \mathbb{R}^2$, a natural subspace of $\mathcal{H}^1(\mathbb{R}^2)$ [8, 19] is

$$\tilde{\mathcal{H}}_0^1(K) := \{u \in \mathcal{H}^1(\mathbb{R}^2); u(x) = 0 \;\text{ a.e. in } K^C\}, \tag{2.2}$$

where K^C denotes the complement of K. The self-adjoint operator associated with the quadratic form $\int |\nabla u|^2 \, dx$ on $\tilde{\mathcal{H}}_0^1(K)$ via the usual representation theorem ([20, Thm. VI-2.1]) will be called the *Dirichlet Laplacian on* K, denoted as $-\Delta_K$. Operators of this type occur naturally; for instance, if $U = K^C$, then $-\Delta + \mu\chi_U$ converges to $-\Delta_K$ in the strong resolvent sense, as $\mu \to \infty$.

If D is an open set, there is also the standard Sobolev space $\mathcal{H}_0^1(D)$, obtained by taking the closure of $C_c^\infty(D)$ in the $\|\cdot\|_1$-norm. For K closed, an alternative choice of a form domain for a Dirichlet Laplacian is $\mathcal{H}_0^1(K^{\rm int}) \subset \tilde{\mathcal{H}}_0^1(K)$; note that $\mathcal{H}_0^1(K^{\rm int})$ may be considerably smaller than $\tilde{\mathcal{H}}_0^1(K)$. We say that the Dirichlet Laplacian associated with K is *unique*, if $\mathcal{H}_0^1(K^{\rm int}) = \tilde{\mathcal{H}}_0^1(K)$.

Returning to the sets $M = \overline{M}$, $\Omega = M^C$ and $G = M^{\rm int}$ of Section 1, Assumption 1.2 is equivalent to the uniqueness of the Dirichlet Laplacian on $M = \Omega^C$. It follows directly from results in [19] that the Laplacian of $M = \overline{M}$ is unique if ∂G satisfies the segment condition and if $M \setminus \overline{G} = \overline{\Omega}^{\rm int} \setminus \Omega$ has measure zero.

While the above discussion of Dirichlet Laplacians applies also to $\mathbb{R}^d$, we note as an aside that there is a stronger criterion [17] that is specific to $\mathbb{R}^2$: in fact, the Dirichlet Laplacian of a closed set $M \subset \mathbb{R}^2$ will be unique if for each point $x \in \partial M$ there exists a continuous function $f : [0,1] \to \mathbb{R}^2$ such that $f(0) = x$ and $f(s) \notin M$, for all $s \in (0,1]$. This criterion follows from [19, Thms. 2.1, 2.5], combined with the fact that Brownian paths in $\mathbb{R}^2$ will immediately "spiral" around their starting point with probability 1 (cf. [25, Ch. 2, Section 7]; we warn the reader to be careful about the definition of τ_M in [19] and of τ_B in [25]).

Magnetic Schrödinger operators. We next turn to magnetic Schrödinger operators. As above, we assume that $\vec{a} \in C^1(\mathbb{R}^2; \mathbb{R}^2)$ is bounded and such that $\mathcal{B} = \operatorname{curl} \vec{a} = \partial_2 a_1 - \partial_1 a_2$ has compact support. For an electric potential $V \in L_\infty(\mathbb{R}^2, \mathbb{R})$ the magnetic Schrödinger operators $H(\vec{a}) = (-i\nabla - \vec{a}(x))^2 + V(x)$ can be easily realized as self-adjoint operators with form domain $\mathcal{H}^1(\mathbb{R}^2)$; cf., e.g., [6]. Magnetic operators defined on a closed set $K \subset \mathbb{R}^2$ with Dirichlet boundary conditions are obtained as for the Dirichlet Laplacian. We will also use the following simple commutator identities: for $\phi \in C_c^\infty$, we have $[-i\nabla - \vec{a}, \phi] = -i\nabla\phi$ and $[(-i\nabla - \vec{a})^2, \phi] = -2i\nabla\phi \cdot (-i\nabla - \vec{a}) - \Delta\phi$.

The vector potential $\vec{a}$ is not uniquely determined by the field $\mathcal{B}$; in fact, if we take any function $f \in C^2(\mathbb{R}^2)$, then $\vec{a} + \nabla f$ produces the same field as $\vec{a}$. Passing from $\vec{a}$ to $\vec{a} + \nabla f$ is called a "gauge transformation"; any two C^1 vector potentials on $\mathbb{R}^2$ associated with the same field are connected via a gauge transformation. It is a well-known and simple fact (cf. [6]) that $H(\vec{a})$ and $H(\vec{b})$ are unitarily equivalent if $\vec{b} = \vec{a} + \nabla f$; this is also true for unbounded vector potentials. In particular, the spectrum and also the usual parts of the spectrum are gauge-independent.

If the field $\mathcal{B}$ has compact support contained in B_R, and if the flux Φ is zero, then we can easily find a vector potential $\vec{a}$ such that $\operatorname{curl} \vec{a} = \mathcal{B}$ and $\vec{a}(x) = 0$ for $|x| \geq R$. For $\Phi > 0$, we have the following lemma.

Lemma 2.1. *Let $\mathcal{B} : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function of compact support and suppose $\Phi = \int \mathcal{B} > 0$. Let $R > 0$ be such that* $\operatorname{supp} \mathcal{B} \subset B_R$. *Then there exists a vector potential $\vec{a}$ of class C^1 such that* $\operatorname{curl} \vec{a} = \mathcal{B}$ *in* $\mathbb{R}^2$, $\operatorname{div} \vec{a}(x) = 0$ *for* $|x| > R$, *and*

$$|\vec{a}(x)| \leq \frac{\Phi}{2\pi} \frac{1}{|x|}, \quad |x| \geq R. \tag{2.3}$$

Proof. Writing $r = (x_1^2 + x_2^2)^{1/2}$, we let $g : [0,\infty) \to \mathbb{R}$ denote a C^1-function satisfying $g'(0) = 0$ and

$$g(r) = \frac{\Phi}{2\pi} r^{-2}, \quad r \geq R. \tag{2.4}$$

Define $\vec{a}_{\rm rad}(x) = (-x_2 g(r), x_1 g(r))$ and let $\mathcal{B}_{\rm rad} = \operatorname{curl} \vec{a}_{\rm rad}$. Then $\operatorname{div} \vec{a}_{\rm rad}$ and $\mathcal{B}_{\rm rad}$ vanish in the exterior of B_R, while $\int \mathcal{B}_{\rm rad}(x)\,\mathrm{d}x = \int \mathcal{B}(x)\,\mathrm{d}x = \Phi$. Since the flux of $\mathcal{B} - \mathcal{B}_{\rm rad}$ is zero, there exists a vector potential $\vec{b}$ of class C^1, vanishing outside of B_R, such that $\operatorname{curl} \vec{b} = \mathcal{B} - \mathcal{B}_{\rm rad}$, and we see that $\vec{a} = \vec{a}_{\rm rad} + \vec{b}$ has the required properties. $\square$

A direct consequence of Lemma 2.1 is that

$$\sigma_{\rm ess}(H(\vec{a})) = \sigma_{\rm ess}(H), \tag{2.5}$$

provided $\vec{a} \in C^1$ with $\operatorname{curl} \vec{a}$ of compact support. In fact, since gauge transformations are unitary, we may assume that $\vec{a}$ is as in Lemma 2.1. It is then easy to see that the resolvent difference $H^{-1} - H(\vec{a})^{-1}$ is compact.

While $\sigma_{\rm ess}(H(\lambda\vec{a})) = \sigma_{\rm ess}(H)$, for $\lambda \in \mathbb{R}$, the discrete spectrum of $H(\lambda\vec{a})$ will depend on λ, in general. For $\vec{a}$ as in Lemma 2.1, the operators $H(\lambda\vec{a})$ form a holomorphic self-adjoint family of type (A) in the sense of Kato [20], and it follows that the discrete eigenvalues of $H(\lambda\vec{a})$ are described by an (at most countable) family of analytic functions of λ. This family is locally finite in the sense that each compact subset of $\mathbb{R} \times (\mathbb{R} \setminus \sigma_{\rm ess}(H))$ is intersected by only finitely many of these functions. It follows that, for $E \notin \sigma(H)$, the set

$$M(\mathcal{B}, E) = \{\mu \in \mathbb{R} : E \in \sigma(H(\mu\vec{a}))\}, \tag{2.6}$$

is a discrete subset of the real line. In order to define the counting function $\mathcal{M}(\lambda; \mathcal{B}, E)$ for $\lambda > 0$, write $M(\mathcal{B}, E) \cap (0, \lambda) = \{\mu_1, \dots, \mu_\ell\}$, for a suitable $\ell = \ell(\lambda) \in \mathbb{N}$, where $\mu_1 < \dots < \mu_\ell$. For any $j = 1, \dots, \ell$, there is an open set $U_j \subset \mathbb{R} \times (\varrho(H) \cap \mathbb{R})$, with $(\mu_j, E) \in U_j$, such that the spectrum of the family $(H(\mu\vec{a}); \mu \geq 0)$ in U_j is described by the union of the graphs of a finite set of (pairwise distinct) analytic functions ϕ_{js}, $s = 1, \dots, r_j$, where $\phi_{js}(\mu_j) = E$. If ϕ_{js} carries multiplicity $m_{js} \in \mathbb{N}$, then the signed spectral multiplicity of $H(\mu\vec{a})$ crossing E as μ increases from 0 to λ is given by

$$\mathcal{M}(\lambda; \mathcal{B}, E) = \sum_{j=1}^{\ell(\lambda)} \sum_{s=1}^{r_j} \sigma_{js} m_{js}, \tag{2.7}$$

where the sign-factors $\sigma_{js} \in \{-1, 0, 1\}$ are defined as follows: if $k_{js} \in \mathbb{N}$ is the order of the first non-zero derivative of ϕ_{js} at μ_j (i.e., $\phi_{js}^{(k_{js})}(\mu_j) \neq 0$ while $\phi_{js}^{(m)}(\mu_j) = 0$, for $0 < m < k_{js}$), we let $\sigma_{js} := 0$ if k_{js} is even, and $\sigma_{js} = \operatorname{sgn} \phi_{js}^{(k_{js})}(\mu_j)$, for k_{js} odd. In other words, strict crossings in upward direction give a positive contribution to $\mathcal{M}(\lambda; \mathcal{B}, E)$, strict downward crossings are counted negatively, while direction changes are not counted at all.

It is easy to see that $\mathcal{M}(\lambda; \mathcal{B}, E)$ is monotonically decreasing in $E \in (a, b)$. More precisely, for λ fixed, $\mathcal{M}(\lambda; \mathcal{B}, .)$ is constant if E varies between two consecutive eigenvalues of $H(\lambda\vec{a})$ while $\mathcal{M}(\lambda; \mathcal{B}, E)$ decreases by $\dim\ker(H(\lambda\vec{a}) - \eta)$ if E crosses an eigenvalue $\eta \in (a, b)$ of $H(\lambda\vec{a})$ in upward direction.

Approximating Operators on B_n. We finally describe the modifications that we need to make in the approach developed in [7, 10, 1, 11] for Schrödinger operators $H_\lambda = -\Delta + V - \lambda W$, $\lambda \in \mathbb{R}$. As above, let $(a,b) \cap \sigma(H) = \emptyset$ and $E \in (a,b)$. The basic idea translates to the magnetic case as follows: in order to find solutions of the equation

$$H(\lambda\vec{a})u = Eu \tag{2.8}$$

on $\mathbb{R}^2$, we look for solutions of suitably defined problems on B_n, and let n tend to ∞. The main technical difficulty comes from the "surface states" that may appear inside the gap of H upon the introduction of a Dirichlet boundary condition along ∂B_n. In the non-magnetic Schrödinger case, one can use a (λ-independent) projection plus cut-offs to eliminate these surface states. Since we wish to employ gauge transformations in the regions

$$G_n = B_n \cap G = B_n \setminus \bar{\Omega}, \tag{2.9}$$

we have to make sure that the modifications near ∂B_n will cooperate with such transformations. We first define

$$H_n(\lambda\vec{a}) = (-\mathrm{i}\nabla - \lambda\vec{a})^2 + V(x), \tag{2.10}$$

acting in $L_2(B_n)$, with a Dirichlet boundary condition along ∂B_n; if $\vec{b}$ is some bounded vector potential of class C^1, $H_n(\vec{b})$ will be defined accordingly. We then choose a subinterval $[a', b'] \subset (a,b)$ such that $E \in (a', b')$, and a continuous function $\mathcal{P} : \mathbb{R} \to \mathbb{R}$ satisfying

$$0 \le \mathcal{P} \le 1, \quad \operatorname{supp} \mathcal{P} \subset (a,b), \quad \mathcal{P}(x) = 1 \quad \text{for } a' \le x \le b'. \tag{2.11}$$

One should think of $\mathcal{P}$ as a smoothed characteristic function so that $\mathcal{P}(H_n(\lambda\vec{a}))$ is close to a spectral projection; note that $\mathcal{P}(H_n(\lambda\vec{a}))$ depends continuously on λ in operator norm. We distinguish between two classes of eigenfunctions of $H_n(\lambda\vec{a})$ that contribute to $\mathcal{P}(H_n(\lambda\vec{a}))$:

(1) there may be eigenfunctions produced by the Dirichlet boundary condition on ∂B_n (and the interaction with $\lambda\vec{a}$); we expect these eigenfunctions to be concentrated near ∂B_n.

(2) there may be eigenfunctions produced by the magnetic field; we expect these eigenfunctions to be concentrated close to the set Ω.

It will turn out that this classification is exhaustive, for n large.

To conclude our construction, we define a family of cut-off functions as follows: fix $\varphi \in C_c^\infty(\mathbb{R}^2)$ with the properties

$$0 \le \varphi \le 1, \quad \varphi(x) = \begin{cases} 1, & \text{for } |x| \le 1/3 \\ 0, & \text{for } |x| \ge 2/3 \end{cases} \tag{2.12}$$

and let

$$\varphi_k(x) = \varphi(x/k), \quad \psi_k = 1 - \varphi_k, \quad k \in \mathbb{N}. \tag{2.13}$$

The functions φ_k satisfy the familiar estimates

$$\|\nabla\varphi_k\|_\infty \le c/k, \quad \|\Delta\varphi_k\|_\infty \le c/k^2, \tag{2.14}$$

for a constant c. Then $\psi_n \mathcal{P}(H_n(\lambda\vec{a}))\psi_n$ singles out that part of the range of $\mathcal{P}(H_n(\lambda\vec{a}))$ which is supported close to the boundary ∂B_n. We choose a constant $\gamma > b - a$ and define

$$\tilde{H}_n(\lambda\vec{a}) = H_n(\lambda\vec{a}) + \gamma\psi_n \mathcal{P}(H_n(\lambda\vec{a}))\psi_n. \tag{2.15}$$

The family $(\tilde{H}_n(\lambda\vec{a}))^{-1}$ is norm-continuous in λ.

More generally, if $\vec{b}$ is a bounded vector potential, we define

$$\tilde{H}_n(\vec{b}) = H_n(\vec{b}) + \gamma\psi_n \mathcal{P}(H_n(\vec{b}))\psi_n. \tag{2.16}$$

In the sequel, we will tacitly assume that n is so large that ψ_n vanishes in a neighborhood of $\overline{\Omega}$.

As a (partial) motivation for the above definition of $\tilde{H}_n(\lambda\vec{a})$, we point out that $\tilde{H}_n(0)$ has no spectrum in $[a'', b''] \subset (a', b')$, for n sufficiently large (cf. [10, 1, 11]), i.e., there exists $n' \in \mathbb{N}$ such that

$$\sigma(\tilde{H}_n(0)) \cap [a'', b''] = \emptyset, \quad n \geq n'. \tag{2.17}$$

We certainly cannot expect the same to be true for $\tilde{H}_n(\lambda\vec{a})$. However, it can be shown that $\tilde{H}_n(\lambda\psi_k\vec{a})$ has no spectrum in $[a'', b'']$, for k sufficiently large and all $n \geq n_k$:

Proposition 2.2. *Suppose that (a, b) is a spectral gap of H, let $[a', b'] \subset (a, b)$ and $[a'', b''] \subset (a', b')$, and let $\vec{b}$ be a continuous vector potential tending to 0 at infinity. For $\mu \geq 0$, define $\tilde{H}_n(\mu\psi_k\vec{b})$ as in (2.16), with $\mathcal{P}$ as in (2.11). Then for $M > 0$ fixed, there exists $k_0 \in \mathbb{N}$ such that*

$$\sigma(\tilde{H}_n(\mu\psi_k\vec{b})) \cap [a'', b''] = \emptyset, \quad k \geq k_0, \quad n \geq n_k, \quad 0 \leq \mu \leq M, \tag{2.18}$$

for some $n_k > k$.

A proof of this result is given in Section 6.

In counting the eigenvalues of $\tilde{H}_n(\lambda\vec{a})$, we will need two related operators which appear in the process of taking the limit $\lambda \to \infty$ in $\tilde{H}_n(\lambda\vec{a})$.

Here we first introduce the Dirichlet Laplacian $-\Delta_{G_n}$ on $G_n = B_n \setminus \bar{\Omega}$, and the operator

$$H_{G_n}(\lambda\vec{a}) = (-\mathrm{i}\nabla - \lambda\vec{a}(x))^2 + V(x), \tag{2.19}$$

acting in $L_2(G_n)$, with Dirichlet boundary conditions (i.e., with $C_c^\infty(G_n)$ as a form core). We have two choices to modify $H_{G_n}(\lambda\vec{a})$ near the boundary ∂B_n: following the construction of $\tilde{H}_n(\lambda\vec{a})$, we first define

$$\tilde{H}_{G_n}(\lambda\vec{a}) = H_{G_n}(\lambda\vec{a}) + \gamma\psi_n \mathcal{P}(H_{G_n}(\lambda\vec{a}))\psi_n. \tag{2.20}$$

Alternatively, instead of $\mathcal{P}(H_{G_n}(\lambda\vec{a}))$ we may use $\mathcal{P}(H_n(\lambda\vec{a}))$ to produce

$$\hat{H}_{G_n}(\lambda\vec{a}) = H_{G_n}(\lambda\vec{a}) + \gamma\psi_n \mathcal{P}(H_n(\lambda\vec{a}))\psi_n; \tag{2.21}$$

we will need to use both operators since taking $\lambda \to \infty$ in $\tilde{H}_n(\lambda\vec{a})$ leads us to $\hat{H}_{G_n}(\lambda\vec{a})$, while the natural gauge transformation operates on $\tilde{H}_{G_n}(\lambda\vec{a})$. Fortunately, the norm difference between the two correction terms tends to zero in operator norm, as $n \to \infty$; cf. Proposition 8.1.

3. Schrödinger Operators with High Barriers

In this section, we recall some basic facts about the eigenvalues and eigenvalue counting functions associated with the family of Schrödinger operators

$$H + \mu\chi_U, \quad \mu \geq 0, \tag{3.1}$$

acting in $L_2(\mathbb{R}^2)$, where $U \subset \mathbb{R}^2$ is open and bounded. Let $M = \mathbb{R}^2 \setminus U$ and $H_M = -\Delta_M + V$; for simplicity of notation, we again assume that $V \geq 1$. It is well-known that $H+\mu\chi_U$ converges to H_M in norm resolvent sense; this follows from the monotonicity of the associated quadratic forms and the fact that the difference of resolvents $H^{-1} - H_M^{-1}$ is compact (cf., e.g., [14]). Compactness also implies that $\sigma_{\rm ess}(H_M) = \sigma_{\rm ess}(H)$ so that H_M has only discrete spectrum in $\mathbb{R} \setminus \sigma_{\rm ess}(H)$.

Let $(a,b) \cap \sigma(H) = \emptyset$ and let $E \in (a,b)$. The eigenvalues of $H + \mu\chi_U$ in (a,b) are analytic, monotonically increasing functions of the coupling μ. Any compact subset of $\mathbb{R} \times (a,b)$ is intersected by at most a finite number of these analytic functions. For $E \in (a,b)$ and $\lambda > 0$, we let

$$\mathcal{N}(\lambda; U, E) = \sum_{0<\mu<\lambda} \dim\ker(H + \mu\chi_U - E) \tag{3.2}$$

denote the total spectral multiplicity crossing E at couplings $\mu \in (0,\lambda)$. Obviously, $\mathcal{N}(\lambda; U, E)$ is monotonically increasing w.r.t. λ and monotonically decreasing w.r.t. E. Monotonicity with respect to U is more subtle:

Lemma 3.1. *Let $U \subset U'$ be open and bounded subsets of $\mathbb{R}^2$ and let $E \in \mathbb{R} \setminus \sigma(H)$. Then*

$$\mathcal{N}(\lambda; U, E) \le \mathcal{N}(\lambda; U', E), \quad \lambda \ge 0.$$

Proof. By the Birman-Schwinger principle, $\mathcal{N}(\lambda; U, E)$ is equal to the number of eigenvalues less than $-1/\lambda$ of the (compact and symmetric) *Birman-Schwinger operator*

$$\mathcal{K}_U = \chi_U (H - E)^{-1} \chi_U. \tag{3.3}$$

Now suppose $\mathcal{L}$ is a linear subspace of $L_2(U)$ such that $\langle \mathcal{K}_U u, u\rangle < (-1/\lambda)\|u\|^2$, for all $0 \ne u \in \mathcal{L}$. Then $U' \supset U$ implies that we also have $\langle \mathcal{K}_{U'} u, u\rangle < (-1/\lambda)\|u\|^2$, for all $0 \ne u \in \mathcal{L}$. Therefore, the min-max principle of Weyl ([28, Thm. XIII-2]) yields $\dim_{-1/\lambda}(\mathcal{K}_{U'}) \ge \dim_{-1/\lambda}(\mathcal{K}_U)$, and the result follows. □

The eigenvalues of $H + \mu\chi_U$ either cross the gap or they are asymptotic to some eigenvalue of H_M in the gap, as $\mu \to \infty$; in fact, for each $\eta \in \sigma_{\rm disc}(H_M)$ there exists an eigenvalue branch of $H + \mu\chi_U$ that is asymptotic to η, as $\mu \to \infty$. Finally, we let

$$\mathcal{N}(U, E) = \limsup_{\lambda\to\infty} \mathcal{N}(\lambda; U, E) \tag{3.4}$$

denote the total number of eigenvalues (counting multiplicities) of $H + \mu\chi_U$ that cross E as μ ranges from 0 to ∞. $\mathcal{N}(U, E)$ is finite for bounded U and there exists $\lambda_0 \ge 0$ such that $\mathcal{N}(U, E) = \mathcal{N}(\lambda; U, E)$, for all $\lambda \ge \lambda_0$. Upper and lower bounds for $\mathcal{N}(U, E)$ are discussed below.

Central to our approach are approximations on large disks B_n and we need to study the convergence of such approximations to the corresponding problem on $\mathbb{R}^2$. For the case of potential barriers, we consider $\tilde{H}_n = \tilde{H}_n(0)$, as defined in Section 2, and we let

$$\mathcal{N}_n(U, E) = \sum_{0<\mu<\infty} \dim\ker(\tilde{H}_n(0) + \mu\chi_U - E) \tag{3.5}$$

denote the number of eigenvalues (counting multiplicities) of $\tilde{H}_n(0) + \mu\chi_U$ that cross E at positive couplings. It follows by regular perturbation theory and monotonicity that

$$\mathcal{N}_n(U, E) = \dim_E(\tilde{H}_n) - \dim_E(\hat{H}_{G_n}), \tag{3.6}$$

with $\hat{H}_{G_n} = \hat{H}_{G_n}(0)$ as in (2.21). For later use we note that, for $E \notin \sigma(H_M)$ and n large, we have $\dim_E(\hat{H}_{G_n}) = \dim_E(\tilde{H}_{G_n})$, as $\|\psi_n(\mathcal{P}(H_n) - \mathcal{P}(H_{G_n}))\psi_n\| \to 0$, $n \to \infty$, by Proposition 8.1. Therefore,

$$\mathcal{N}_n(U,E) = \dim_E(\tilde{H}_n) - \dim_E(\tilde{H}_{G_n}), \tag{3.7}$$

for n large, provided $E \notin \sigma(H_M)$.

The main result of this section reads as follows.

Proposition 3.2. *Let $U \subset \mathbb{R}^2$ be open and bounded, let $M = \mathbb{R}^2 \setminus U$, and suppose that $E \in \mathbb{R}$, $E \notin \sigma(H) \cup \sigma(H_M)$. We then have*

$$\mathcal{N}_n(U,E) \to \mathcal{N}(U,E), \quad n \to \infty. \tag{3.8}$$

Proof. (1) We denote the associated Birman-Schwinger operators as

$$\mathcal{K} = \chi_U(H-E)^{-1}\chi_U, \quad \mathcal{K}_n = \chi_U(\tilde{H}_n - E)^{-1}\chi_U; \tag{3.9}$$

then [11, Corollary 2.2] implies

$$\|\mathcal{K}_n - \mathcal{K}\| \to 0, \quad n \to \infty, \tag{3.10}$$

where K_n has been extended by zero outside of B_n.

(2) As $\mathcal{N}(U,E)$ is finite, there exists $\eta_0 > 0$ such that $\mathcal{K}$ has no spectrum in the interval $(-\eta_0, 0)$. There also exist $\lambda_0 > 0$ and $\delta_0 > 0$ such that $H + \mu\chi_U$ has no eigenvalues in the interval $(E-\delta_0, E+\delta_0)$, for $\mu \geq \lambda_0$.

(3) We now claim that there exist $\eta_1 > 0$ and $n_1 \in \mathbb{N}$ such that the Birman-Schwinger operators $\mathcal{K}_n$ have no spectrum in the interval $(-\eta_1, 0)$, for $n \geq n_1$.

Assuming the claim not to be true, it follows that there exist sequences $(n_j) \subset \mathbb{N}$, $(\lambda_j) \subset (0,\infty)$ such that $E \in \sigma(\tilde{H}_{n_j} + \lambda_j\chi_U)$ and $n_j \to \infty$, $\lambda_j \to \infty$. Let $u_j \in D(\tilde{H}_{n_j})$ satisfy $\|u_j\| = 1$ and $(\tilde{H}_{n_j} + \lambda_j\chi_U)u_j = Eu_j$. We now consider $v_j = \phi_{n_j/3}u_j$ and $w_j = u_j - v_j = \psi_{n_j/3}u_j$, where ϕ_k, ψ_k are as in (2.12), (2.13). By the familiar calculations we find (assuming that n_j is so large that $\chi_U\psi_{n_j/3} = 0$, without restriction)

$$(\tilde{H}_{n_j} - E)(w_j) = (\tilde{H}_{n_j} + \lambda_j\chi_U - E)(w_j) = -2\nabla\psi_{n_j/3}\cdot\nabla u_j - (\Delta\psi_{n_j/3})u_j. \tag{3.11}$$

By construction of $\tilde{H}_n$, there exists a constant $\beta > 0$ such that $\|(\tilde{H}_n - E)w\| \geq \beta\|w\|$, for all $w \in D(\tilde{H}_n)$, and so we can use (2.14) to find a constant C such that

$$\beta\|w_j\| \leq Cn_j^{-1}\left(\|\nabla u_j\| + \|u_j\|\right). \tag{3.12}$$

It is easy to see that there is a constant $C' > 0$ such that $\|\nabla u_j\| \leq C'$, since $\lambda_j\chi_U \geq 0$. As a consequence,

$$\|w_j\| \to 0, \quad \|v_j\| \to 1, \quad j \to \infty. \tag{3.13}$$

By a similar calculation we also find that

$$(\tilde{H}_{n_j} + \lambda_j\chi_U - E)v_j \to 0, \quad j \to \infty. \tag{3.14}$$

Obviously, $v_j \in D(H)$ and $(H + \lambda_j\chi_U - E)v_j = (\tilde{H}_{n_j} + \lambda_j\chi_U - E)v_j \to 0$. Since $\|v_j\| \to 1$, it follows that there exists a sequence (ϵ_j) of positive numbers, $\epsilon_j \to 0$, such that $H + \lambda_j\chi_U$ has spectrum in the interval $(E - \epsilon_j, E + \epsilon_j)$, in contradiction to what was obtained in (1).

(4) Let $\eta = \min\{\eta_0, \eta_1\}$. Then (2) and (3) imply that the Birman-Schwinger operators $\mathcal{K}$ and $\mathcal{K}_n$, $n \geq n_1$, have no eigenvalues in the interval $(-\eta, 0)$, and the desired result follows from (3.10). □

Upper and lower estimates for $\mathcal{N}(U,E)$ are discussed, e.g., in [11, 12]. The simplest result reads as follows:

Proposition 3.3. *(i) There exist constants C_1, C_2 such that*

$$\mathcal{N}(U,E) \le C_1R^2 + C_2, \quad R > 0, \tag{3.15}$$

for all open and bounded sets $U \subset \mathbb{R}^2$ satisfying $U \subset B_R$.
(ii) Suppose H satisfies Assumption 1.6. Then there exist constants $c_1 > 0$, $c_2 \ge 0$ such that

$$\mathcal{N}(U,E) \ge c_1R^2 - c_2, \quad R > 0, \tag{3.16}$$

for all open and bounded $U \subset \mathbb{R}^2$ that satisfy $U \supset B_R$.

Proof. By the monotonicity property of Lemma 3.1, we only need estimates for $U = B_R$. For the statement (i), we can directly refer to [12, Theorem 3.6], since $\dim_E(H_R) \le \dim_E(-\Delta_{B_R}) \le cR^2$. Similarly, Assumption 1.6 and [12, Theorem 4.4] yield (ii). □

Remark 3.4. It is not easy to give sharp upper or lower bounds for $\mathcal{N}(U,E)$ since $\mathcal{N}(U,E)$ is more or less unrelated to the volume of U (cf. [11, 12]). On the one hand, the counting function will not be affected in the limit $\lambda \to \infty$ if U has many tiny holes ("Swiss cheese"). Hence there may be a large number of eigenvalues crossing E although U has small volume. Conversely, if U consists of small and well separated pieces, the volume of U may be large while $\mathcal{N}(U,E) = 0$.

4. Eigenvalue Counting on Large Disks

In this section, we create the link that connects the magnetic problem with the high barrier case: we will show that, for large couplings λ that satisfy $\lambda\Phi \in 2\pi\mathbb{Z}$, the signed spectral flow across E on B_n is the same for magnetic perturbations as for potential barriers. We will use the function

$$\mathcal{M}_n(\lambda;\mathcal{B},E) = \dim_E(\tilde{H}_n(0)) - \dim_E(\tilde{H}_n(\lambda\vec{a})), \quad \lambda > 0, \tag{4.1}$$

which corresponds to the loss of spectral multiplicity below E for the operators $\tilde{H}_n(\mu\vec{a})$ as the coupling μ increases from 0 to λ.

Proposition 4.1. *Let $E \in \mathbb{R}$, $E \notin \sigma(H) \cup \sigma(H_M)$ and let $\eta > 0$ be so small that $[E-\eta, E+\eta]$ does not intersect $\sigma(H) \cup \sigma(H_M)$. Then there exists $\Lambda \ge 0$ such that for all $\lambda \ge \Lambda$ that satisfy $\lambda\Phi \in 2\pi\mathbb{Z}$ there exists n_λ such that*

$$\mathcal{N}_n(\Omega, E+\eta) \le \mathcal{M}_n(\lambda;\mathcal{B},E) \le \mathcal{N}_n(\Omega, E-\eta), \quad n \ge n_\lambda. \tag{4.2}$$

The proof, given at the end of this section, relies on the following two basic facts:
(1) By the results of [14, 18], the resolvent of $\tilde{H}_{G_n}(\lambda\vec{a})$ yields a good approximation of the resolvent of $\tilde{H}_n(\lambda\vec{a})$, for λ large. (Here the resolvent of $\tilde{H}_{G_n}(\lambda\vec{a})$ acts on $L_2(G_n)$, with $G_n = B_n \setminus \bar{\Omega}$, and we take the direct sum with the zero operator on $L_2(\bar{\Omega})$ without making this explicit in the notation.)
(2) Second, for $\lambda\Phi \in 2\pi\mathbb{Z}$ we may use a gauge transformation to eliminate the vector potential in the region G_n.

Our first lemma is an adaptation of results of [14, 18] on the emergence of Dirichlet boundary conditions at the boundary of a strong magnetic field. There are two additional difficulties: we need estimates that are uniform in n, for n large, and we have to take care of the non-local terms $\psi_n\mathcal{P}(H_n(\lambda\vec{a}))\psi_n$.

Lemma 4.2. *Suppose $\vec{a}$ is a bounded vector potential of class C^1 such that $\mathcal{B} = \operatorname{curl}\vec{a}$ has compact support and Assumption 1.2 is satisfied.*

For any $\epsilon > 0$ there exists $\Lambda_\epsilon \geq 0$ such that

$$\|\tilde{H}_n(\lambda\vec{a})^{-1} - \tilde{H}_{G_n}(\lambda\vec{a})^{-1}\| < \epsilon, \quad \lambda \geq \Lambda_\epsilon, \tag{4.3}$$

for all sufficiently large n (i.e., $n \geq N_{\epsilon,\lambda}$).

Proof. By Proposition 7.2, for $\epsilon > 0$ given there exists Λ_ϵ such that

$$\|H_n(\lambda\vec{a})^{-1} - H_{G_n}(\lambda\vec{a})^{-1}\| < \epsilon, \tag{4.4}$$

for all $\lambda \geq \Lambda_\epsilon$ and $n \geq N_{\epsilon,\lambda}$. We now add the term $\gamma\psi_n\mathcal{P}(H_n(\lambda\vec{a}))\psi_n$ to both operators in (4.4) to produce $\tilde{H}_n(\lambda\vec{a})$ and $\hat{H}_{G_n}(\lambda\vec{a})$, as defined in (2.15), (2.21). It follows from (4.4) and Lemma 4.3, below, that

$$\|\tilde{H}_n(\lambda\vec{a})^{-1} - \hat{H}_{G_n}(\lambda\vec{a})^{-1}\| < 4\epsilon, \quad \lambda \geq \Lambda_\epsilon, \quad n \geq N_{\epsilon,\lambda}. \tag{4.5}$$

By (2.15), (2.21), we have

$$\tilde{H}_{G_n}(\lambda\vec{a}) - \hat{H}_{G_n}(\lambda\vec{a}) = \gamma\psi_n\left(\mathcal{P}(H_{G_n}(\lambda\vec{a})) - \mathcal{P}(H_n(\lambda\vec{a}))\right)\psi_n, \tag{4.6}$$

and Proposition 8.1 concludes the proof. □

We have singled out the following lemma from the proof of Lemma 4.2.

Lemma 4.3. *Let A, B denote self-adjoint operators that satisfy $A \geq c$ and $B \geq c$, for some $c > 0$, $D(A) = D(B)$, and $\|A^{-1} - B^{-1}\| \leq \epsilon$, for some $\epsilon \in (0,1)$. Let C denote a bounded, symmetric operator. If $\|C\| \leq c/2$, then*

$$\|(A+C)^{-1} - (B+C)^{-1}\| \leq 4\epsilon. \tag{4.7}$$

Proof. First note that $A + C \geq c/2$ implies that $A + C$ is invertible with $\|(A + C)^{-1}\| \leq 2/c$. By the second resolvent equation, we have

$$(A+C)^{-1} - (B+C)^{-1} = -(A+C)^{-1}AA^{-1}(A-B)B^{-1}B(B+C)^{-1}, \tag{4.8}$$

and the desired result is now immediate from $\|A(A+C)^{-1}\| \leq 2$, $\|B(B+C)^{-1}\| \leq 2$, and $B^{-1}(A-B)A^{-1} = B^{-1} - A^{-1}$. □

We next employ a gauge transformation on $G_n = B_n \setminus \bar{\Omega}$ to eliminate the vector potential $\lambda\vec{a}$, for $\lambda\Phi \in 2\pi\mathbb{Z}$.

Lemma 4.4. *If $\lambda \in \mathbb{R}$ satisfies $\lambda\Phi \in 2\pi\mathbb{Z}$, then the operators $\tilde{H}_{G_n}(\lambda\vec{a})$ and $\tilde{H}_{G_n}(0)$ are unitarily equivalent.*

Proof. Fixing a base point $x_0 \in G_n$, we define a (multi-valued) function F by a line integral,

$$F(x) = \int_{\gamma_x} \vec{a}(y) \cdot \mathrm{d}y, \tag{4.9}$$

where γ_x is a smooth path connecting x_0 and x within G_n. The values of F at $x \in G_n$ differ by integer multiples of the flux Φ. Each branch of F is (locally) C^2 and $\nabla F = \vec{a}$. Thus, for $\lambda\Phi \in 2\pi\mathbb{Z}$, the function $u_\lambda = \mathrm{e}^{\mathrm{i}\lambda F}$ is in $C^2(G_n)$ and has modulus 1. Hence multiplication by u_λ defines a unitary operator U_λ on $L_2(G_n)$.

To check that U_λ establishes a unitary equivalence between $\tilde{H}_{G_n}(\lambda\vec{a})$ and $\tilde{H}_{G_n}(0)$, we first note that $C_c^1(G_n)$, the space of C^1-functions of compact support in G_n, is a core for the quadratic forms of both operators and that $C_c^1(G_n)$ is invariant under U_λ. Furthermore, it is easy to see that for all f, $g \in C_c^1(G_n)$,

$$\langle u_\lambda^{-1}(-\mathrm{i}\nabla - \lambda\vec{a})u_\lambda f, u_\lambda^{-1}(-\mathrm{i}\nabla - \lambda\vec{a})u_\lambda g\rangle = \langle -\mathrm{i}\nabla f, -\mathrm{i}\nabla g\rangle. \tag{4.10}$$

From this it easily follows that $U_\lambda^{-1} H_{G_n}(\lambda\vec{a}) U_\lambda = H_{G_n}(0)$. Therefore, by the Spectral Calculus,

$$U_\lambda^{-1} \mathcal{P}(H_{G_n}(\lambda\vec{a})) U_\lambda = \mathcal{P}(H_{G_n}(0)). \tag{4.11}$$

Finally, $[U_\lambda, \psi_n] = 0$ since U_λ is a multiplication operator, and we are done. □

Proof of Proposition 4.1. It follows from Lemma 4.2 that there exists $\Lambda \geq 0$ such that

$$\dim_E(\tilde{H}_n(\lambda\vec{a})) \leq \dim_{E+\eta}(\tilde{H}_{G_n}(\lambda\vec{a})), \quad \lambda \geq \Lambda, \tag{4.12}$$

for n large, $n \geq n_\lambda$, say. By Lemma 4.4, $\tilde{H}_{G_n}(\lambda\vec{a})$ is unitarily equivalent to $\tilde{H}_{G_n} = \tilde{H}_{G_n}(0)$, provided $\lambda\Phi \in 2\pi\mathbb{Z}$, and we conclude that for $\lambda \geq \Lambda$, $\lambda\Phi \in 2\pi\mathbb{Z}$, and $n \geq n_\lambda$ we have

$$\begin{aligned} \mathcal{M}_n(\lambda; \mathcal{B}, E) &= \dim_E(\tilde{H}_n) - \dim_E(\tilde{H}_n(\lambda\vec{a})) \\ &\geq \dim_E(\tilde{H}_n) - \dim_{E+\eta}(\tilde{H}_{G_n}) \\ &= \dim_{E+\eta}(\tilde{H}_n) - \dim_{E+\eta}(\tilde{H}_{G_n}), \end{aligned} \tag{4.13}$$

as $\dim_t(\tilde{H}_n)$ is constant for $a < t < b$. Recalling (3.7), this proves the first inequality.

The proof of the second inequality in Proposition 4.1 is analogous and omitted. □

Combining the convergence results of Section 3 with Proposition 4.1 we obtain the following preliminary result.

Proposition 4.5. *Let $E \in \mathbb{R}$, $E \notin \sigma(H) \cup \sigma(H_M)$. Then there exists $\Lambda_0 \in \mathbb{N}$ such that for $\lambda \geq \Lambda_0$ satisfying $\lambda\Phi \in 2\pi\mathbb{Z}$, there exists n_λ such that*

$$\mathcal{M}_n(\lambda; \mathcal{B}, E) = \mathcal{N}(\Omega, E), \quad n \geq n_\lambda. \tag{4.14}$$

Proof. Let $\eta > 0$ be such that $[E-\eta, E+\eta]$ doesn't intersect the spectrum of either H or H_M; in particular, $\mathcal{N}(\Omega, E\pm\eta) = \mathcal{N}(\Omega, E)$. By Proposition 3.2, $\mathcal{N}_n(\Omega, E\pm\eta) \to \mathcal{N}(\Omega, E \pm \eta) = \mathcal{N}(\Omega, E)$. Now Proposition 4.1 implies that $\mathcal{M}_n(\lambda; \mathcal{B}, E) = \mathcal{N}(\Omega, E)$, for $\lambda \geq \Lambda$, $\lambda\Phi \in 2\pi\mathbb{Z}$, and for all $n \geq n_\lambda$, and we are done. □

5. The Convergence Step

In Section 4, we have seen that $\lim_{n\to\infty} \mathcal{M}_n(\lambda; \mathcal{B}, E)$ exists and is equal to $\mathcal{N}(\Omega, E)$, provided $\lambda \geq \Lambda$ and $\lambda\Phi \in 2\pi\mathbb{Z}$. Nowe, we show that $\lim_{n\to\infty} \mathcal{M}_n(\lambda; \mathcal{B}, E)$ is equal to $\mathcal{M}(\lambda; \mathcal{B}, E)$ for most $\lambda \geq 0$. The main result of this section reads as follows.

Proposition 5.1. *Let $\mathcal{B}$ be a continuous function of compact support, and let $\vec{a} \in C^1(\mathbb{R}^2; \mathbb{R}^2)$ be such that $\operatorname{curl} \vec{a} = \mathcal{B}$, with $\vec{a}$ tending to zero at infinity and $\operatorname{div} \vec{a}(x) = 0$ outside some ball B_R. Let $E \in \mathbb{R} \setminus \sigma(H)$ and define $\mathcal{M}_n(\lambda; \mathcal{B}, E)$ and $\mathcal{M}(\lambda; \mathcal{B}, E)$ as before. Let $\lambda \geq 0$ be such that $E \notin \sigma(H(\lambda\vec{a}))$. Then*

$$\mathcal{M}_n(\lambda; \mathcal{B}, E) \to \mathcal{M}(\lambda; \mathcal{B}, E), \quad n \to \infty. \tag{5.1}$$

Proof. (1) Fix $a < a' < b' < b$ such that $(a, b) \cap \sigma(H) = \emptyset$ and $a' < E < b'$. Also fix $\lambda > 0$ such that $E \notin \sigma(H(\lambda\vec{a}))$. By analyticity and compactness, the spectrum of $H(\mu\vec{a})$ inside the compact set $K_\lambda = [0, \lambda + 1] \times [a', b'] \subset [0, \infty) \times (a, b)$ is the union of the graphs $\Gamma(\phi_j)$ of a finite number of analytic functions ϕ_j, defined on suitable intervals of the real line, with values in (a, b). Regular perturbation theory yields a constant C such that $|\phi_j'| \le C$, for all j.

By compactness and analyticity again, the number of points in K_λ where the graphs of two or more of the functions ϕ_j intersect is finite. Similarly, the number of points in K_λ where any of the functions ϕ_j has derivative zero, is finite too.

Therefore, we can find levels $E_+ > E$ and $E_- < E$ with the following properties:

- $E_\pm \in (a', b')$ and $\mathcal{M}(\lambda; \mathcal{B}, E_\pm) = \mathcal{M}(\lambda; \mathcal{B}, E)$;
- $E_\pm \notin \sigma(H(\lambda\vec{a}))$;
- For all j we have $\phi_j'(t) \neq 0$ whenever $t \in (0, \lambda+1]$ is such that $\phi_j(t) = E_\pm$;
- For any $t \in (0, \lambda + 1]$, there is at most one j such that $\phi_j(t) = E_+$ or $\phi_j(t) = E_-$.

(2) We now consider E_+. A simple compactness argument and the properties collected in (1) imply that there exists a finite number of open rectangles

$$\mathcal{R}_k := (t_k - \tau_k, t_k + \tau_k) \times (E_+ - \eta_k, E_+ + \eta_k), \quad k = 1, \dots, k_\lambda, \tag{5.2}$$

with $\tau_k, \eta_k > 0$, such that $0 \le t_k < t_{k+1}$, and
(a) the union of the intervals $(t_k - \tau_k, t_k + \tau_k)$, $k = 1, \dots, k_\lambda$, covers $[0, \lambda]$;
(b) for each k, there is *at most one* $j = j_k$ such that $\Gamma(\phi_{j_k}) \cap \mathcal{R}_k \neq \emptyset$.
(c) suppose there exists $j = j_k$ such that $\Gamma(\phi_{j_k})$ intersects $\mathcal{R}_k$; then we may assume that $\phi_{j_k}(t_k) = E_+$, i.e., ϕ_{j_k} crosses the level E_+ at the center of $\mathcal{R}_k$. In addition, we may assume that $\phi_{j_k}'(t) \neq 0$, for $t_k - \tau_k < t < t_k + \tau_k$, and that $|\phi_{j_k}(t_k \pm \tau_k) - E_+| < \eta_k/2$.
(d) By decreasing the τ_k's, if necessary, we may assume without restriction that $t_{k-1} < t_k - \tau_k < t_k + \tau_k < t_{k+1}$.
(3) Suppose now that $\mathcal{R}_k \cap \Gamma(\phi_j) = \emptyset$, for all j. Then it is a direct consequence of Lemma 5.2, below, that $(E_+ - \eta_k/2, E_+ + \eta_k/2)$ doesn't contain any eigenvalues of $\tilde{H}_n(\mu\vec{a})$, for $|\mu - t_k| < \tau_k$ and for $n \ge n_k$. Therefore, the counting functions $\mathcal{M}(.; \mathcal{B}, E)$ and $\mathcal{M}_n(.; \mathcal{B}, E)$ are constant in the interval $(t_k - \tau_k, t_k + \tau_k)$, for n large.
(4) We next consider the case where for some $j = j_k$ we have $\Gamma(\phi_{j_k}) \cap \mathcal{R}_k \neq \emptyset$. Here we define the band of width $\delta < \eta_k$ around the graph of ϕ_{j_k},

$$\Gamma_\delta(\phi_{j_k}) = \{(t, e) : |t - t_k| < \tau_k, |e - \phi_{j_k}(t)| < \delta\}. \tag{5.3}$$

Lemma 5.2 implies that there exists a sequence of positive numbers $\delta_n \downarrow 0$ such that $\{(t, e); |t - t_k| < \tau_k, e \in \sigma(\tilde{H}_n(t\vec{a})) \cap (E_+ - 3\eta_k/4, E_+ + 3\eta_k/4)\}$ is contained in $\Gamma_{\delta_n}(\phi_{j_k}) \cap \mathcal{R}_k$, for $n \ge n_k$. In view of (5.7), below, we also assume that $\delta_n \ge 1/\log n$, without restriction.

According to the properties gathered in part (c) of (2), ϕ_{j_k}' is either strictly positive or strictly negative in $(t_k - \tau_k, t_k + \tau_k)$. We now consider the case where $\phi_{j_k}'(t_k) > 0$, so that for n large

$$\begin{aligned} m_k &:= \mathcal{M}(t_k + \tau_k; \mathcal{B}, E) - \mathcal{M}(t_k - \tau_k; \mathcal{B}, E) \\ &= \dim_{(E_+ - \delta_n, E_+ + \delta_n)}(H(t_k\vec{a})). \end{aligned} \tag{5.4}$$

We also let $m_k^{(n)}$ denote the loss or gain in spectral multiplicity below E_+ for $\tilde{H}_n(t\vec{a})$, as t ranges between $t_k - \tau_k$ and $t_k + \tau_k$, i.e.,

$$\begin{aligned} m_k^{(n)} &= \mathcal{M}_n(t_k + \tau_k; \mathcal{B}, E) - \mathcal{M}_n(t_k - \tau_k; \mathcal{B}, E) \\ &= \dim_{E_+}(\tilde{H}_n((t_k - \tau_k)\vec{a})) - \dim_{E_+}(\tilde{H}_n((t_k + \tau_k)\vec{a})). \end{aligned} \tag{5.5}$$

Here we conclude from the above that

$$m_k^{(n)} = \dim_{(E_+-\delta_n, E_++\delta_n)}(\tilde{H}_n(t_k\vec{a})), \tag{5.6}$$

for n large. In fact, the number of eigenvalues of $\tilde{H}_n(t\vec{a})$ inside the intervals $(\phi_{j_k}(t)-\delta_n, \phi_{j_k}(t)+\delta_n)$ is constant for $t_k-\tau_k < t < t_k+\tau_k$, while at the same time $\phi_{j_k}(t_k-\tau_k)+\delta_n < E_+$ and $\phi_{j_k}(t_k+\tau_k)-\delta_n > E_+$, for n large. Hence all eigenvalue branches of $\tilde{H}_n(t\vec{a})$ that intersect $\tilde{\mathcal{R}}_k = (t_k-\tau_k, t_k+\tau_k) \times (E_+-3\eta_k/4, E_++3\eta_k/4)$ enter below the level E_+ (at $t = t_k-\tau_k$) and leave $\tilde{\mathcal{R}}_k$ above the level E_+ (at $t = t_k+\tau_k$). Now Lemma 5.3 and Equations (5.4), (5.6) imply that $\limsup_{n\to\infty} m_k^{(n)} \le m_k$.

In the other direction, we apply simple cut-offs to a basis of $\ker(H(t_k\vec{a}) - E_+)$ to show that, for n large,

$$\dim_{(E_+-\delta_n, E_++\delta_n)}(\tilde{H}_n(t_k\vec{a})) \ge m_k, \tag{5.7}$$

and so $m_k^{(n)} \ge m_k$, for n large. In conclusion, we have shown that $m_k^{(n)} = m_k$, for n large.

(5) From the above we conclude that $\mathcal{M}(\lambda;\mathcal{B},E_+) = \mathcal{M}_n(\lambda;\mathcal{B},E_+)$, for n large.

By monotonicity, we have $\mathcal{M}_n(\lambda;\mathcal{B},E_+) \le \mathcal{M}_n(\lambda;\mathcal{B},E)$, while $E_\pm$ have been chosen so that $\mathcal{M}(\lambda;\mathcal{B},E_\pm) = \mathcal{M}(\lambda;\mathcal{B},E)$. As a consequence, $\liminf \mathcal{M}_n(\lambda;\mathcal{B},E) \ge \liminf \mathcal{M}_n(\lambda;\mathcal{B},E_+) = \lim \mathcal{M}_n(\lambda;\mathcal{B},E_+) = \mathcal{M}(\lambda;\mathcal{B},E_+) = \mathcal{M}(\lambda;\mathcal{B},E)$.

(6) Similarly, using E_- in place of E_+, we obtain $\limsup \mathcal{M}_n(\lambda;\mathcal{B},E) \le \mathcal{M}(\lambda;\mathcal{B},E)$, and we are done. □

In the proof of Proposition 5.1, we have been using two lemmas where the finer parts of our construction come into play. The first lemma does not take care of multiplicities.

Lemma 5.2. *Under the assumptions of Proposition 5.1, let $(a,b)\cap\sigma(H) = \emptyset$, let $\lambda > 0$ and suppose we are given sequences $(\mu_n) \subset (0,\lambda]$ and $(E_n) \subset (a,b)$ such that E_n is an eigenvalue of $\tilde{H}_n(\mu_n\vec{a})$, for all $n \in \mathbb{N}$.*

If $\mu_n \to \mu_0 \in [0,\lambda]$ and $E_n \to E_0 \in (a,b)$, as $n\to\infty$, then E_0 is an eigenvalue of $H(\mu_0\,\vec{a})$.

Proof. Let $[a',b'] \subset (a,b)$ and $[a'',b''] \subset (a',b')$ such that $E_0 \in (a'',b'')$. By assumption, there exist $u_n \in D(\tilde{H}_n) = D(H_n)$, $\|u_n\| = 1$, such that

$$\tilde{H}_n(\mu_n\vec{a})u_n = E_nu_n, \quad n \in \mathbb{N}. \tag{5.8}$$

With ϕ_k and ψ_k as in (2.12), (2.13), we let

$$v_n = \phi_{n/3}u_n, \quad w_n = \psi_{n/3}u_n = 1 - v_n. \tag{5.9}$$

Clearly, $v_n \in D(H)$. Below, we will prove that

$$\|v_n\| \to 1, \quad (H(\mu_0\vec{a}) - E_0)v_n \to 0, \quad n\to\infty. \tag{5.10}$$

It is then immediate from (5.10) that E_0 belongs to the spectrum of $H(\mu_0\vec{a})$.

For the proof of (5.10) we first observe that $[\psi_n\mathcal{P}(H_n(\mu_n\vec{a}))\psi_n, \psi_{n/3}] = 0$ because $(1-\psi_{n/3})\psi_n = 0$. We next compute

$$\begin{aligned}(\tilde{H}_n(\mu_n\vec{a}) - E_n)(\psi_{n/3}u_n) &= [\tilde{H}_n(\mu_n\vec{a}), \psi_{n/3}]u_n \\ &= [H_n(\mu_n\vec{a}), \psi_{n/3}]u_n \\ &= -2\mathrm{i}\nabla\psi_{n/3}\cdot(-\mathrm{i}\nabla - \mu_n\vec{a})u_n - \Delta\psi_{n/3}u_n.\end{aligned} \tag{5.11}$$

Since $\|(-\mathrm{i}\nabla - \mu_n\vec{a})u_n\|^2$ is clearly bounded while the $\psi_{n/3}$ satisfy the estimates given in (2.14), we see that

$$(\tilde{H}_n(\mu_n\vec{a}) - E_n)(\psi_{n/3}u_n) \to 0, \quad n \to \infty. \tag{5.12}$$

In (5.12), we are now going to replace $\vec{a}$ with $\psi_k\vec{a}$, for suitable k. This is the most subtle point of our construction and it is here that the specific definition of the operators $\tilde{H}_n(\vec{b})$ bears fruit.

First, Proposition 2.2 yields a $k \in \mathbb{N}$ such that the interval $[a'', b'']$ is free of spectrum of $\tilde{H}_n(\mu\psi_k\vec{a})$, for $n \geq n_k$, and for all $0 \leq \mu \leq \lambda$.

Second, for $n \geq 9k$, we have $\psi_k\psi_{n/3} = \psi_{n/3}$ so that $H_n(\mu_n\vec{a})(\psi_{n/3}u_n) = H_n(\mu_n\psi_k\vec{a})(\psi_{n/3}u_n)$.

Third, Proposition 8.1 implies that

$$\|\psi_n\left[\mathcal{P}(H_n(\mu_n\vec{a})) - \mathcal{P}(H_n(\mu_n\psi_k\vec{a}))\right]\psi_n\| \to 0, \quad n \to \infty, \tag{5.13}$$

and we conclude from (5.12), (5.13) that

$$(\tilde{H}_n(\mu_n\psi_k\vec{a}) - E_n)(w_n) \to 0, \quad n \to \infty. \tag{5.14}$$

Let $\eta > 0$ denote the smaller of the numbers $b'' - E_0$ and $E_0 - a''$. By the Spectral Theorem, we have

$$\eta\|v\| \leq \|(\tilde{H}_n(\mu\psi_k\vec{a}) - E_0)v\|, \quad v \in D(H_n), \quad n \geq n_k, \tag{5.15}$$

and it now follows from (5.14) that $w_n \to 0$, proving the first part of (5.10).

As for the second part of (5.10), we first compute

$$\begin{aligned}(H(\mu_n\vec{a}) - E_n)((1 - \psi_{n/3})u_n) &= (\tilde{H}_n(\mu_n\vec{a}) - E_n)((1 - \psi_{n/3})u_n) \\ &= -[\tilde{H}_n(\mu_n\vec{a}), \psi_{n/3}]u_n.\end{aligned} \tag{5.16}$$

It was shown above that $[\tilde{H}_n(\mu_n\vec{a}), \psi_{n/3}]u_n \to 0$, as $n \to \infty$, and it follows that $(H(\mu_n\vec{a}) - E_n)v_n \to 0$, as $n \to \infty$. We finally observe that

$$\begin{aligned}\|\left(H(\mu_n\vec{a}) - H(\mu_0\vec{a})\right)v_n\| \leq& 2|\mu_n - \mu_0|^2\||\vec{a}|^2\, v_n\| \\ &+ |\mu_n - \mu_0|\,\left(2\|\vec{a}\cdot\nabla v_n\| + \|(\operatorname{div}\vec{a})v_n\|\right).\end{aligned} \tag{5.17}$$

Since $\vec{a}$ and $\operatorname{div}\vec{a}$ are bounded, the RHS of (5.17) tends to zero, as $n \to \infty$, and it follows from (5.16) that $(H(\mu_0\vec{a}) - E_0)v_n \to 0$, as claimed. $\square$

The next lemma is a variant of Lemma 5.2, where we keep track of multiplicities. The proof is similar to the proof of Lemma 5.2 and it is omitted.

Lemma 5.3. *Let $\mu_0 > 0$, $E_0 \in (a, b)$ and assume that*

$$\dim_{(E_0-\delta_n, E_0+\delta_n)}(\tilde{H}_n(\mu_0\vec{a})) = m, \quad n \geq n_0, \tag{5.18}$$

for some $m, n_0 \in \mathbb{N}$ and for a sequence of positive numbers δ_n such that $\delta_n \to 0$, as $n \to \infty$. Then

$$\dim\ker(H(\mu_0\vec{a}) - E_0) \geq m. \tag{5.19}$$

Before proceeding to the proof of our main result, Theorem 1.3, we need yet another lemma.

Lemma 5.4. *Let $E \notin \sigma(H) \cup \sigma(H_M)$. Then there exists $\Lambda' \geq 0$ such that $E \notin \sigma(H(\lambda\vec{a}))$, for $\lambda \geq \Lambda'$ satisfying $\lambda\Phi \in 2\pi\mathbb{Z}$.*

Proof. Let $\gamma > 0$ denote the distance from E to $\sigma(H_M)$. For λ large, $\lambda\Phi \in 2\pi\mathbb{Z}$, the spectrum of $H(\lambda\vec{a})$ inside the gap (a, b) is contained in a $\gamma/2$-neighborhood of $\sigma(H_M)$: in fact, Proposition 7.1 implies that

$$\|(H(\lambda\vec{a}))^{-1} - (H_M(\lambda\vec{a}))^{-1}\| \to 0, \quad \lambda \to \infty, \tag{5.20}$$

while, by a simple variant of Lemma 4.4, $H_M(\lambda_k\vec{a})$ is unitarily equivalent to H_M, provided $\lambda\Phi \in 2\pi\mathbb{Z}$, and so $\sigma(H_M(\lambda\vec{a})) = \sigma(H_M)$. This implies the desired result. □

We are now ready for the proof of our main results as stated in Section 1.

Proof of Theorem 1.3. By Proposition 4.5 we have for all $\lambda \geq \Lambda$ that satisfy $\lambda\Phi \in 2\pi\mathbb{Z}$

$$\mathcal{M}_n(\lambda; \mathcal{B}, E) = \mathcal{N}(\Omega, E), \quad n \geq n_\lambda. \tag{5.21}$$

By Lemma 5.4 we see that $E \notin \sigma(H(\lambda\vec{a}))$, for λ large, $\lambda\Phi \in 2\pi\mathbb{Z}$, and we may apply Proposition 5.1 to the effect that $\mathcal{M}_n(\lambda; \mathcal{B}, E) \to \mathcal{M}(\lambda; \mathcal{B}, E)$, as $n \to \infty$. □

Corollaries 1.5 and 1.7 follow directly from Theorem 1.3 and Proposition 3.3.

Remark 5.5. In this remark, we give some indications on the proof of (1.7). Let again $\Phi > 0$ and $E \notin \sigma(H) \cup \sigma(H_M)$. Also recall that $\mathcal{M}(\lambda_k; \mathcal{B}, E) = \mathcal{N}(\Omega, E)$, according to Theorem 1.3, for $\lambda_k := 2k\pi/\Phi$ large enough.

Writing $d := \operatorname{dist}(E, \sigma(H) \cup \sigma(H_M))$, we let $E' = E + d/3$ and $E'' = E + 2d/3$. For any $\epsilon > 0$, Lemma 4.2 yields an estimate

$$\|\tilde{H}_n(\lambda\vec{a})^{-1} - \tilde{H}_{G_n}(\lambda\vec{a})^{-1}\| < \epsilon, \tag{5.22}$$

for $\lambda \geq \Lambda_\epsilon$, $n \geq N(\epsilon, \lambda)$, and it follows that, for $\lambda \geq \Lambda'$ and $n \geq n'(\lambda)$,

$$\dim_E(\tilde{H}_n(\lambda\vec{a})) \leq \dim_{E'}(\tilde{H}_{G_n}(\lambda\vec{a})) = \dim_{E'}(\tilde{H}_{G_n}((\lambda - \lambda_k)\vec{a})), \tag{5.23}$$

for $k \in \mathbb{N}$, by a simple variant of Lemma 4.4. For any $\lambda \geq 0$ we can pick k such that $|\lambda - \lambda_k| \leq \pi/\Phi =: \mu_0$.

The electric potential $q(x) := |\vec{a}(x)|^2$ is continuous and satisfies $0 \leq q(x) \leq C(1 + |x|)^{-2}$, by Lemma 2.1. By (6.1), we have

$$\tilde{H}_{G_n}(\kappa\vec{a}) \geq (1 - \epsilon)\tilde{H}_{G_n}(0) - (1 + 1/(2\epsilon))\mu_0^2\, q, \quad 0 \leq \kappa \leq \mu_0, \tag{5.24}$$

in the sense of quadratic forms. We now fix some $0 < \epsilon_0 \leq d$ with the property that $\dim_E(\tilde{H}_{G_n}(0)) = \dim_{E''}(\tilde{H}_{G_n}(0)) = \dim_{E'}\Big((1 - \epsilon_0)\tilde{H}_{G_n}(0)\Big)$, for n large; such an ϵ_0 exists since the spectrum of $\tilde{H}_{G_n}$ approaches the spectrum of H_M, as $n \to \infty$. Now (5.24) implies (writing $\eta_0 := (1 + 1/(2\epsilon_0))\mu_0^2$)

$$\begin{aligned} \dim_{E'}(\tilde{H}_{G_n}(\kappa\vec{a})) &\leq \dim_{E'}\Big((1 - \epsilon_0)\tilde{H}_{G_n}(0) - \eta_0 q\Big) \\ &\leq \dim_{E'}((1 - \epsilon_0)\tilde{H}_{G_n}(0)) + C(\eta_0, q) \\ &= \dim_E(\tilde{H}_{G_n}(0)) + C(\eta_0, q), \end{aligned} \tag{5.25}$$

with $C(\eta_0, q)$ denoting the number of eigenvalues of the family $(1 - \epsilon_0)\tilde{H}_{G_n}(0) - \xi q$ that cross E' at couplings $\xi \in (0, \eta_0)$. There are several possibilities to obtain bounds on $C(\eta_0, q)$ by using the Birman-Schwinger principle, e.g. We conclude that

$$\begin{aligned} \mathcal{M}_n(\lambda; \mathcal{B}, E) &= \dim_E(\tilde{H}_n(0)) - \dim_E(\tilde{H}_n(\lambda\vec{a})) \\ &\geq \dim_E(\tilde{H}_n(0)) - \dim_E(\tilde{H}_{G_n}(0)) - C(\eta_0, q) \\ &= \mathcal{N}_n(\Omega, E) - C(\eta_0, q), \end{aligned} \tag{5.26}$$

for λ large, where we have also used (3.7). The upper estimate is obtained in the same way. Now $n \to \infty$ and Propositions 5.1 and 3.2 yield (1.7).

We conclude this section with a comment on the case where the flux Φ is zero.

Remark 5.6. Our main result (Theorem 1.3) includes the case $\Phi = 0$. If the flux Φ is zero, substantial simplifications of the proof are possible under additional smootheness assumptions on ∂G that allow to pass to an equivalent vector potential $\vec{b}$ such that $\operatorname{curl} \vec{b} = \mathcal{B}$ and $\vec{b}(x) = 0$ for a.e. $x \in M$. A suitable gauge transformation can be constructed by starting from a line integral of $\vec{a}$ in the open set G, as in the proof of Lemma 4.4, and then extending first to the closed set $\overline{G}$ and then to $\mathbb{R}^2$; both extension steps are non-trivial and require additional (but mild) assumptions ([32, 14]).

Assuming for the moment that a vector potential $\vec{b}$ with the above properties exists, we may work with the λ-independent "correction term" $\psi_n \mathcal{P}(H_n(0))\psi_n$ throughout. From [14] and a simple variant of Proposition 7.2, we then infer that

$$\|\hat{H}_n(\lambda\vec{b})^{-1} - \hat{H}_{G_n}^{-1}(0)\| < \epsilon, \quad \lambda \geq \Lambda_\epsilon, \quad n \geq n_{\epsilon,\lambda}, \tag{5.27}$$

and (3.6) establishes the desired link with the case of a high barrier on Ω.

6. Appendix A: Proof of Proposition 2.2

We let $\mathbf{h}$ and $\mathbf{h}_{\vec{b}}$ denote the quadratic forms of the operators $H = -\Delta + V$ and $H(\vec{b}) = (-\mathrm{i}\nabla - \vec{b})^2 + V$, respectively, where $\vec{b} = (b_1(x), b_2(x))$ denotes a bounded vector potential. Recall that we always assume $V \geq 1$. Introducing the electric potential $q = q_{\vec{b}}(x) := b_1^2(x) + b_2^2(x)$, it is easy to see that

$$|\mathbf{h}[u] - \mathbf{h}_{\vec{b}}[u]| \leq \epsilon \mathbf{h}[u] + (1 + 1/(2\epsilon)) \langle qu, u\rangle, \quad u \in \mathcal{H}^1, \quad \epsilon > 0. \tag{6.1}$$

In fact, (6.1) follows from the elementary estimate

$$\big|\langle \vec{b}u, \nabla u\rangle\big| \leq \sum_{j=1}^{2} \int |b_j u\, \partial_j u|\, \mathrm{d}x \leq \sum_{j=1}^{2} \big(\epsilon \|\partial_j u\|^2 + \frac{1}{4\epsilon}\langle b_j u, b_j u\rangle\big), \quad \epsilon > 0. \tag{6.2}$$

The following notation will be useful: Given a spectral gap (a, b) of H, we choose a sequence of (non-empty) subintervals (a_j, b_j) satisfying $[a_j, b_j] \subset (a_{j-1}, b_{j-1})$, with $a_0 = a$ and $b_0 = b$, $j \in \mathbb{N}$. (6.1) and [20, Thm. VI-3.9] yield the following lemma.

Lemma 6.1. *Suppose $(a, b) \cap \sigma(H) = \emptyset$, and let (a_1, b_1) as above. Then there exists a constant $\eta > 0$ such that $\sigma(H(\vec{b})) \cap [a_1, b_1] = \emptyset$, for all $\vec{b}$ satisfying $\|\vec{b}\|_\infty \leq \eta$.*

Proof of Proposition 2.2. By the construction of $\tilde{H}_n(0)$ we have (cf. [7, 1, 10, 11])

$$\sigma(\tilde{H}_n(0)) \cap [a_1, b_1] = \emptyset, \quad n \geq n_1. \tag{6.3}$$

We will pass from $\tilde{H}_n(0)$ to $\tilde{H}_n(\psi_k\vec{b})$ by using as an intermediate the operators

$$K_{k,n} = (-\mathrm{i}\nabla - \psi_k\vec{b})^2 + V(x) + \gamma\psi_n\mathcal{P}(H_n(0))\psi_n, \quad k, n \in \mathbb{N}. \tag{6.4}$$

As $\vec{b}$ tends to zero at infinity, we find as before that the magnetic terms obtained from $\psi_k\vec{b}$ satisfy the following estimate: for $\epsilon > 0$, there exists k_ϵ such that

$$|\langle\psi_k\vec{b}u,\nabla u\rangle| + \|\psi_k\vec{b}u\|^2 \le \epsilon\langle H_n(0)u,u\rangle \le \epsilon\langle\tilde{H}_n(0)u,u\rangle, \quad u\in\mathcal{H}_0^1(B_n), \tag{6.5}$$

for $k \ge k_\epsilon$ and for all $n \in \mathbb{N}$; here terms like $\langle H_n(0)u,u\rangle$ should be read in the sense of quadratic forms. As before, (6.5) and [20, Thm. VI-3.9] imply that the operator $K_{k,n}$ has no spectrum in the interval $[a_2, b_2]$, for $k \ge k_1$, and all $n \ge n_1(k)$.

We next provide an estimate for the difference of $\tilde{H}_n(\psi_k\vec{b})$ and $K_{k,n}$ which we denote as

$$D_{k,n} = \tilde{H}_n(\psi_k\vec{b}) - K_{k,n} = \gamma\psi_n\Big(\mathcal{P}(H_n(\psi_k\vec{b})) - \mathcal{P}(H_n(0))\Big)\psi_n. \tag{6.6}$$

As $\|\psi_k\vec{b}\|_\infty \to 0$, as $k\to\infty$, it follows again from [20, Thm. VI-3.9] that, for any $\epsilon > 0$ there exists k'_ϵ such that for all n

$$\|H_n(\psi_k\vec{b})^{-1} - H_n(0)^{-1}\| < \epsilon, \quad k \ge k'_\epsilon. \tag{6.7}$$

As a consequence, we find that, for $\epsilon > 0$ given, there exists $k''_\epsilon \in \mathbb{N}$ such that

$$\|\mathcal{P}(H_n(\psi_k\vec{b})) - \mathcal{P}(H_n(0))\| < \epsilon, \quad k \ge k''_\epsilon, \tag{6.8}$$

and for all n. Going from (6.7) to (6.8) is not entirely trivial. Proceeding as in the proof of [27, Thm. VIII.18] we can infer from (6.7) that the resolvents of $A = H_n(\psi_k\vec{b})^{-1}$ and $B = H_n(0)^{-1}$ satisfy a uniform estimate

$$\|(A-\zeta)^{-1} - (B-\zeta)^{-1}\| < \epsilon, \quad \zeta\in\mathbb{C}, \quad |\zeta| = 2, \tag{6.9}$$

for k large. We may then express any continuous function of A or B as a contour integral of the resolvents along $|\zeta| = 2$.

We finally conclude that $\tilde{H}_n(\psi_k\vec{b}) = K_{k,n} + D_{k,n}$ has no spectrum inside the interval $[a_3, b_3]$, for k sufficiently large and $n \ge n_1$. □

7. Appendix B: Strong magnetic fields and Dirichlet boundaries

Our starting point in this appendix is a result of [14, 18] on the emergence of a Dirichlet boundary condition along the boundary of the support of a strong magnetic field. We let $H_M(\lambda\vec{a})$ denote the operator $(-\mathrm{i}\nabla - \lambda\vec{a})^2 + V$, acting in $L_2(M)$ with Dirichlet boundary condition. Here V is a bounded, measurable potential; for simplicity, we again assume $V \ge 1$. We use the notation of Section 2 for G, G_n, H_M, H_{G_n} etc. The following proposition is derived in [18] in the case $V = 0$ only; it easy to see that one might include a potential V with positive part locally integrable and negative part in the Kato class.

Proposition 7.1 ([18])**.** *Suppose $\vec{a}$ is a (bounded) vector potential of class C^1 such that $\mathcal{B} = \operatorname{curl}\vec{a}$ has compact support and let $M = \{x : \mathcal{B}(x) = 0\}$. We then have*

$$\|(H(\lambda\vec{a}))^{-1} - (H_M(\lambda\vec{a}))^{-1}\| \to 0, \quad \lambda\to\infty. \tag{7.1}$$

In view of Lemma 4.2, we need an analogous result for operators acting on B_n instead of $\mathbb{R}^2$, with estimates that are uniform in n large, for λ fixed (and sufficiently large). The precise result is as follows:

Proposition 7.2. *Let $\vec{a}$ and $\mathcal{B}$ as above and let $\epsilon > 0$. Then there exists Λ_ϵ such that for all $\lambda \ge \Lambda_\epsilon$ there exists $n_{\epsilon,\lambda} \in \mathbb{N}$ such that*

$$\|H_n(\lambda\vec{a})^{-1} - H_{G_n}(\lambda\vec{a})^{-1}\| < \epsilon, \quad n \ge n_{\epsilon,\lambda}. \tag{7.2}$$

For the proof, we will show that the difference of the terms on the left-hand side of (7.1) and (7.2) tends to zero, as $n \to \infty$. Here we first deal with the case $\vec{a} = 0$ and $V = 1$ and then use the Feynman-Kac-Itô formula to reduce the magnetic case to the situation of Lemma 7.3.

Lemma 7.3. *Let $-\Delta_G$, $-\Delta_{G_n}$ etc. as in Section 2. Then, as $n \to \infty$,*

$$\| \big((-\Delta+1)^{-1} - (-\Delta_G+1)^{-1}\big) - \big((-\Delta_{B_n}+1)^{-1} - (-\Delta_{G_n}+1)^{-1}\big) \| \to 0. \tag{7.3}$$

Proof. We multiply with $1 = \phi_n + \psi_n$, with ϕ_n as in (2.12), (2.13), and rearrange the 8 terms in form of the following 4 differences,

$$\phi_n(-\Delta+1)^{-1} - \phi_n(-\Delta_{B_n}+1)^{-1}, \quad \phi_n(-\Delta_G+1)^{-1} - \phi_n(-\Delta_{G_n}+1)^{-1},$$

$$\psi_n(-\Delta+1)^{-1} - \psi_n(-\Delta_G+1)^{-1}, \quad \psi_n(-\Delta_{B_n}+1)^{-1} - \psi_n(-\Delta_{G_n}+1)^{-1};$$

again, resolvents are extended by zero whenever necessary. As an example, we treat the third of the four terms; the argument in the other cases is similar and omitted. Let

$$v = (-\Delta+1)^{-1} f - (-\Delta_G+1)^{-1} f, \quad f \in L_2, \tag{7.4}$$

and find by the usual calculation $(-\Delta+1)\psi_n v = [-\Delta, \psi_n]v$, whence

$$\psi_n v = (-\Delta+1)^{-1}[-\Delta, \psi_n]v. \tag{7.5}$$

It is easy to see that $\|(-\Delta+1)^{-1}[-\Delta, \psi_n]\| \le c/n$, and the result follows. □

With some more effort one could obtain an exponentially small bound in Lemma 7.3. We are now ready for the proof of Proposition 7.2.

Proof of Proposition 7.2. The proof relies on arguments used in the proof of [14, Lemma 1.3]. In the following, it will be enough to consider $0 \le f \in C_c^\infty(\mathbb{R}^d)$. We start from the Feynman-Kac-Itô-representation for the difference of the semi-groups associated with $H(\lambda\vec{a})$ and $H_M(\lambda\vec{a})$,

$$\begin{aligned} D_\lambda f(x;t) :&= \big(\mathrm{e}^{-tH(\lambda\vec{a})} f - \mathrm{e}^{-tH_M(\lambda\vec{a})} f\big)(x) \\ &= \mathbb{E}_x \Big\{ \mathrm{e}^{\mathrm{i}\Phi(\lambda\vec{a},\omega)} \mathrm{e}^{-\int_0^t V(\omega(s))ds} \chi_W(\omega) f(\omega(t)) \Big\}, \quad t > 0, \end{aligned} \tag{7.6}$$

where $W = \{\omega\,;\,\omega(0) = x,\, \exists s \in [0,t) : \omega(s) \in \Omega\}$ denotes the set of those Brownian paths ω (starting at x at time $t = 0$) that enter Ω before time t; $\Phi(\lambda\vec{a}, \omega)$ is a real phase, given by (15.2) in [31].

We may write down a completely analogous formula for the difference of the semi-groups associated with the operators $H_n(\lambda\vec{a})$ and $H_{G_n}(\lambda\vec{a})$; instead of the set W we will now have $W_n = \{\omega\,;\,\omega(0) = x,\, \forall s \in [0,t) : \omega(s) \in B_n,\ \exists s \in [0,t) : \omega(s) \in \Omega\}$, which consists of those paths ω starting at x that do not leave B_n and that enter Ω before time t. The corresponding difference of semi-groups will be denoted as $D_{\lambda;n}$.

For the difference $D_\lambda - D_{\lambda;n}$, the paths ω that remain in the play are given by the set

$$W \setminus W_n = \{\omega\,;\,\exists s_1, s_2 \in [0,t) : \omega(s_1) \in \Omega,\, \omega(s_2) \notin B_n\}. \tag{7.7}$$

Taking the Laplace-transform, we find for a.e. x

$$\begin{aligned} &|H(\lambda\vec{a})^{-1}f(x) - H_M(\lambda\vec{a})^{-1}f(x) - \big(H_n(\lambda\vec{a})^{-1}f(x) - H_{G_n}(\lambda\vec{a})^{-1}f(x)\big)\,| \\ &\le \int_0^\infty \Big|\mathbb{E}_x\{\mathrm{e}^{\mathrm{i}\Phi(\lambda\vec{a},\omega)-\int_0^t V(\omega(s))ds}\chi_{W\setminus W_n}(\omega)f(\omega(t))\}\Big|\,dt \\ &\le \int_0^\infty \mathbb{E}_x\left\{\mathrm{e}^{-t}\chi_{W\setminus W_n}(\omega)f(\omega(t))\right\}\mathrm{d}t, \end{aligned} \tag{7.8}$$

as $|\mathrm{e}^{\mathrm{i}\Phi(\vec{a},\omega)}| = 1$ and $\mathrm{e}^{-\int_0^t V(\omega(s))ds} \le \mathrm{e}^{-t}$ (recall that $V \ge 1$). By the Feynman-Kac formula, we can rewrite the last line of (7.8):

$$\begin{aligned} &\int_0^\infty \mathbb{E}_x\left\{\mathrm{e}^{-t}\chi_{W\setminus W_n}(\omega)f(\omega(t))\right\}\mathrm{d}t \\ &= (-\Delta+1)^{-1}f(x) - (-\Delta_M+1)^{-1}f(x) \\ &\quad - \left((-\Delta_n+1)^{-1}f - (-\Delta_{G_n}+1)^{-1}f\right)(x). \end{aligned} \tag{7.9}$$

The result now follows from Lemma 7.3 and Proposition 7.1. □

8. Appendix C

In this appendix we show that the difference of the operators $\tilde{H}_{G_n}(\lambda\vec{a})$ and $\hat{H}_{G_n}(\lambda\vec{a})$ is small for n large. We prove here a result that is slightly more general than what we need in the preceding sections: we compare two magnetic Schrödinger operators on open sets G_n, $G_n' \subset B_n$ such that $G_n, G_n' \supset B_n \setminus B_R$, for some $0 < R < n$. These operators have bounded magnetic and electric potentials $\vec{a}$, V and $\vec{a}'$, V', respectively, which are defined on all of $\mathbb{R}^d$ and which satisfy $\vec{a}(x) = \vec{a}'(x)$, $V(x) = V'(x)$, for $x \in B_R^C$. We write $h_n = (-\mathrm{i}\nabla - \vec{a})^2 + V$, acting in $L_2(G_n)$, with Dirichlet boundary conditions (cf. the definition of $H_{G_n}(\lambda\vec{a})$ in (2.19)); h_n' is defined accordingly. Note that we need not assume that $\vec{a}$, $\vec{a}'$ tend to zero at infinity nor that the associated magnetic fields have support inside B_R. Furthermore, the result of Proposition 8.1 does not depend on the presence of a spectral gap of $-\Delta + V$. For simplicity of notation, we again assume $V, V' \ge 1$. We then have:

Proposition 8.1. *Let h_n, h_n' be as above and let $p \in C_0([1,\infty))$, the space of continuous functions on $[0,\infty)$ tending to 0 at infinity. Then*

$$\|\psi_n\,(p(h_n) - p(h_n'))\,\psi_n\| \to 0, \quad n \to \infty. \tag{8.1}$$

Furthermore, for any $\kappa > 0$ fixed, (8.1) holds uniformly for $\mu\vec{a}$, $\mu\vec{a}'$ in place of $\vec{a}$, $\vec{a}'$, provided $0 \le \mu \le \kappa$.

As in Section 7, we first use the Feynman-Kac-Itô formula to eliminate the magnetic vector potentials. In the following lemma, we write $-\Delta_n = -\Delta_{B_n}$ and $-\Delta_{R,n} = -\Delta_{B_n\setminus B_R}$.

Lemma 8.2. *With the notation and assumptions of Proposition 8.1, we have for $t > 0$*

$$\left|\mathrm{e}^{-th_n}f(x) - \mathrm{e}^{-th_n'}f(x)\right| \le 2\big(\mathrm{e}^{-t(-\Delta_n+1)}|f|(x) - \mathrm{e}^{-t(-\Delta_{R,n}+1)}|f|(x)\big), \tag{8.2}$$

for all $f \in C_c^\infty$ and a.e. $x \in \mathbb{R}^d$.

Proof. It is enough to consider $f \geq 0$. Writing down the Feynman-Kac-Itô formula for $\mathrm{e}^{-th_n} f(x)$ and for $\mathrm{e}^{-th'_n} f(x)$ (cf. (15.2) in [31], e.g.) and taking the difference, we see that all paths drop out from the integration that do not enter the set B_R, for $s \in [0,t)$. We write for fixed $x \in B_n$

$$W = \{\omega; \omega(0) = x, \exists s \in [0,t) : \omega(s) \in B_R, \ \forall \tau \in [0,t) : \omega(\tau) \in B_n\} \tag{8.3}$$

and obtain for $f \geq 0$

$$\begin{aligned} &\left|\mathrm{e}^{-th_n} f(x) - \mathrm{e}^{-th'_n} f(x)\right| \\ &\leq \mathbb{E}_x \Big\{ \big(\left|\mathrm{e}^{\mathrm{i}\Phi(\vec{a},\omega) - \int_0^t V(\omega(s))ds}\right| + \left|\mathrm{e}^{\mathrm{i}\Phi(\vec{a}',\omega) - \int_0^t V'(\omega(s))ds}\right|\big) \chi_W(\omega) f(\omega(t)) \Big\}. \end{aligned} \tag{8.4}$$

As in the proof of Proposition 7.2, we find

$$\begin{aligned} \left|\mathrm{e}^{-th_n} f(x) - \mathrm{e}^{-th'_n} f(x)\right| &\leq 2\mathbb{E}_x \left\{ \mathrm{e}^{-t} \chi_W(\omega) f(\omega(t)) \right\} \\ &= 2\big(\mathrm{e}^{-t(-\Delta_n + 1)} - \mathrm{e}^{-t(-\Delta_{R,n}+1)}\big) f(x), \end{aligned} \tag{8.5}$$

by the Feynman-Kac-formula for $\mathrm{e}^{-t(-\Delta_n+1)}$ and $\mathrm{e}^{-t(-\Delta_{R,n}+1)}$. □

As before, we employ the Laplace-transform to pass from the semi-group estimates of Lemma 8.2 to resolvent estimates:

Lemma 8.3. *Under the above assumptions, we have for $k \in \mathbb{N}$*

$$\|\psi_n \left(h_n^{-k} - (h'_n)^{-k}\right) \psi_n\| \leq 2 \|\psi_n \left((-\Delta_n + 1)^{-k} - (-\Delta_{R,n} + 1)^{-k}\right) \psi_n\|. \tag{8.6}$$

Lemma 8.4. *With the above notation and assumptions, there exist constants $\alpha > 0$, $c \geq 0$ such that*

$$\|\psi_n \left((-\Delta_n + 1)^{-k} - (-\Delta_{R,n} + 1)^{-k}\right) \psi_n\| \leq c\, \mathrm{e}^{-\alpha(n-R)}. \tag{8.7}$$

Proof. We first consider $k = 1$. For $u \in C_c^\infty(B_n)$, write

$$v = \left((-\Delta_n + 1)^{-1} - (-\Delta_{R,n} + 1)^{-1}\right) \psi_n u. \tag{8.8}$$

Fix $k_0 \in \mathbb{N}$ such that $\phi_{k_0} = 1$ on B_R and consider n large enough to have $\psi_n \psi_{k_0} = \psi_n$. We then have $\psi_{k_0} v \in D(-\Delta_n)$ and $(-\Delta_n + 1)(\psi_{k_0} v) = -2\nabla\psi_{k_0} \cdot \nabla v - \Delta\psi_{k_0} v$, or

$$\psi_n v = \psi_n \psi_{k_0} v = \psi_n (-\Delta_n + 1)^{-1} \left(-2\nabla\psi_{k_0} \cdot \nabla v - \Delta\psi_{k_0} v\right). \tag{8.9}$$

By construction, the set $\{x; \nabla\psi_{k_0} \neq 0\}$ has distance $c(n - R)$ from the support of ψ_n. By the maximum principle, the Green's function of $-\Delta_n + 1$ is pointwise bounded by the Green's function of $-\Delta + 1$, which decays exponentially off the diagonal. This implies the desired estimate. The proof for $k > 1$ is similar and omitted. □

Proof of Proposition 8.1. If p is a polynomial in $1/x$, then (8.1) holds by Lemmas 8.3, 8.4. The general result follows via the Stone-Weierstraß-theorem. □

In the following proposition we combine an estimate from Section 6 with Proposition 8.1.

Proposition 8.5. *Under the assumptions of Proposition 2.2 (in particular, $\vec{b} \to 0$ at ∞), we have for $\mathcal{P}$ as in (2.11) and $H_n(\vec{b})$ as in (2.10),*

$$\|\psi_n \big(\mathcal{P}(H_n(0)) - \mathcal{P}(H_n(\vec{b}))\big) \psi_n\| \to 0, \quad n \to \infty. \tag{8.10}$$

Since $\vec{b}$ decays at infinity, it is natural to expect that, for n large, there shouldn't be much difference between the various correction terms.

Proof. Given $\epsilon > 0$, it follows from (6.7) that we can find $k \in \mathbb{N}$ and $n_0 = n_0(\epsilon, k)$ such that

$$\|\mathcal{P}(H_n(\psi_k \vec{b})) - \mathcal{P}(H_n(0))\| < \epsilon, \quad n \geq n_0. \tag{8.11}$$

On the other hand, Proposition 8.1 yields that

$$\|\psi_n\big(\mathcal{P}(H_n(\vec{b})) - \mathcal{P}(H_n(\psi_k \vec{b}))\big)\psi_n\| \to 0, \quad n \to \infty, \tag{8.12}$$

and the result follows. □

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