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BLOW UP OF SOLUTIONS TO SEMILINEAR WAVE EQUATIONS

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Abstract. This work shows the absence of global solutions to the equation

$$u_{tt} = \Delta u + p^{-k}|u|^m,$$

in the Minkowski space $\mathbb{M}_0 = \mathbb{R} \times \mathbb{R}^N$, where $m > 1$, $(N-1)m < N+1$, and p is a conformal factor approaching 0 at infinity. Using a modification of the method of conformal compactification, we prove that any solution develops a singularity at a finite time.

1. Introduction

This note presents nonexistence results of the problem

$$u_{tt} = \Delta u + p^{-k}|u|^m, \tag{1.1}$$

posed in the Minkowski space $\mathbb{M}_0 = \mathbb{R} \times \mathbb{R}^N$, $N \geq 1$, with the initial condition

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^N. \tag{1.2}$$

Here p is a conformal factor approaching 0 at infinity, the parameter $m > 1$ satisfies $(N-1)m < N+1$. The constant $k = sm - (N+3)/2$, where $s = (N-1)/2$. The initial data u_0, u_1 belong to $X := \{f : f \in C_0^\infty(\mathbb{R}^N); 0 \not\equiv f \geq 0\}$. Note that the factor p^{-k} approaches 0 as $|x|$ tends to infinity for $(N-1)m < N+1$.

This work is motivated by a recent paper by Belchev, Kepka and Zhou [3] in which Problem (1.1),(1.2) with $1 < m < 1 + (2/N)$ is considered. The authors proved the following theorem using a modification of the technique of conformal compactification due to Penrose [6] and developed by Christodolou [4] and Baez *et al.* [5].

Theorem 1.1. *Let* $1 < m < 1 + (2/N)$ *and* u *be a solution to* (1.1),(1.2) *with* $u_0, u_1 \in X$. *Then* u *blows up in finite time.*

Attention will be given to show that (1.1),(1.2) does not possess global solutions for $m > 1$ and $(N-1)m < N+1$, complementing in this way the results in [3]. Theorem 1.1 is also announced in [1] and the proof is similar to the one given in [3]. Our main result is the following:

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Theorem 1.2. *Let $m > 1, (N-1)m < N+1$ and u be a solution to* (1.1),(1.2) *with $u_0, u_1 \in X$. Then u blows up in finite time.*

The proof of this theorem is given in Section 2 which contains also a result of the nonexistence of global solutions in the case $u_1 \leq 0$.

2. Proof of the main result

Notation and preliminary results. To clarify the proof, we consider as in [3] the conformal map c from the Minskowski space $\mathbb{M}_0$ to the Einstein universe $\mathbb{E} := \mathbb{R} \times S^N$. Here S^N is the unit sphere in $\mathbb{R}^{N+1}$ and

$$c(t,x) := c(t,x_1,x_2,\dots,x_N) = (T,Y_1,Y_2,\dots,Y_{N+1}),$$

where

$$\begin{gathered}
\sin T = pt,\ \cos T = p\big(1 - \frac{t^2-x^2}{4}\big), \quad T \in (-\pi,\pi),\\
Y_j = px_j,\ j = 1,\dots,N, \quad Y_{N+1} = p\big(1 + \frac{t^2-x^2}{4}\big),\\
p = \Big(t^2 + \big(1 - \frac{t^2-x^2}{4}\big)^2\Big)^{-1/2}.
\end{gathered}$$

The space $\mathbb{M}_0$ is equipped with the Minkowski metric:

$$g = dt^2 - dx^2,$$

and the space $\mathbb{E}$ with the metric

$$\tilde{g} = dT^2 - dS^2,$$

where dS^2 is the canonical metric on S^N. Therefore, c is a conformal map between the Lorentz manifolds $(\mathbb{M}_0, g)$ and $(\mathbb{E}, \tilde{g})$, with the conformal factor p; that is, $c^\star \tilde{g} = p^2 g$.

Next, we consider as in [3], the function v defined in $\mathbb{E}$ by

$$u = R^{-2/(m-1)} p^s v, \quad R > 0,\ s = \frac{N-1}{2},$$

where u is a solution to (1.1), (1.2). Then v satisfies

$$\begin{gathered}
(\mathcal{L}_c + s^2)v = |v|^m, \quad \text{on } \mathbb{E},\\
v(0,.) = R^{2/(m-1)} p_0^{-s} u_0 \circ c^{-1},\\
v_T(0,.) = R^{(m+1)/(m-1)} p_0^{-(s+1)} u_1 \circ c^{-1},
\end{gathered} \tag{2.1}$$

where $p_0 = \cos^2 \frac{\rho}{2}$, $\rho \in [0,\pi)$ is the distance on S^N from the north pole $T = Y_j = 0,\ j = 1,\dots,N,\quad Y_{N+1} = 1$ and $\mathcal{L}_c$ denotes the d'Alembertien in $\mathbb{E}$ relative to the metric $\tilde{g}$. Then the function $H(T) = \int_{S^N} v(T,.)dS$ satisfies (see [3])

$$H'' \geq (C_0|H|^{m-1} - s^2)|H|, \tag{2.2}$$

for some positive constant C_0 independent of the parameter R. At the origin we have

$$\begin{aligned}
H(0) &= R^{2/(m-1)-N} \int_{\mathbb{R}^N} \big(1 + \frac{r^2}{4R^2}\big)^{-(N+1)/2} u_0 dx,\\
&\geq R^{2/(m-1)-N} \int_{\mathbb{R}^N} \big(1 + \frac{r^2}{4}\big)^{-(N+1)/2} u_0 dx,
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned} H'(0) &= R^{(m+1)/(m-1)-N}\int_{\mathbb{R}^N}\big(1+\frac{r^2}{4R^2}\big)^{-(N-1)/2}u_1dx,\\ &\geq R^{(m+1)/(m-1)-N}\int_{\mathbb{R}^N}\big(1+\frac{r^2}{4}\big)^{-(N-1)/2}u_1dx, \quad r=|x|,\, R\geq 1. \end{aligned} \tag{2.4}$$

Proposition 2.1. *Let H be a solution to* (2.2) *where $H'(0)\geq 0$ and $H(0)>(\frac{s^2}{C_0})^{1/(m-1)}$. Then H cannot be a global solution.*

Proof. By contradiction and assume that H is global. By (2.2) we have $H''(0)>0$. It follows that $H'>0$ and then $H>(\frac{s^2}{C_0})^{1/(m-1)}$ on $(0,\varepsilon),\varepsilon$ small. Arguing in the same way, we deduce that $H'>0$ and $H>(\frac{s^2}{C_0})^{1/(m-1)}$ on $(\varepsilon,\varepsilon+\varepsilon^\star)$. This shows, in particular that

$$H'(T)>0,\quad H(T)>\big(\frac{s^2}{C_0}\big)^{1/(m-1)}\quad\text{and}\quad H''(T)>0,$$

for all $T>0$. Next we claim that $H(T)$ goes to infinity with T. First note that $H(T)$ has a limit as T tends to infinity. Assume that this limit is finite. Since H'' is positive, $H'(T)$ goes to 0 as T tends to infinity. Integrating inequality (2.2) over $(0,T)$ and passing to the limit yield

$$-H'(0)\geq\int_0^\infty(C_0H^{m-1}-s^2)HdT.$$

The left side of the last inequality is non-positive while the right hand side is positive. This is impossible. Now using (2.2) and the fact that $H(\infty)=\infty$,

$$H''\geq C_1H^m,\quad \forall\, T>T_0,$$

holds for some T_0 large and for some positive constant C_1. Therefore, H develops a singularity since $m>1$. □

Remark 2.2. Note that, as inequality (2.2) is autonomous, if there exists T_0 such that $H(T_0)>(\frac{s^2}{C_0})^{1/(m-1)}$ and $H'(T_0)\geq 0$ the conclusion of the preceding proposition remains valid.

Remark 2.3. The condition $H(0)>(\frac{s^2}{C_0})^{1/(m-1)}$ can be replaced by $H(0)\geq(\frac{s^2}{C_0})^{1/(m-1)}$ if $H'(0)>0$.

Remark 2.4. In the case $1<m<1+\frac{2}{N}$ we have

$$lim_{R\to\infty}R^{2/(m-1)-N}\int_{\mathbb{R}^N}\big(1+\frac{r^2}{4}\big)^{-(N+1)/2}u_0\,dx=\infty.$$

Hence we can choose $R>R_0$ such that $H(0)>(\frac{s^2}{C_0})^{1/(m-1)}$; therefore using Proposition 2.1 we deduce Theorem 1.1 for $1<m<1+\frac{2}{N}$.

Proof of Theorem 1.2. Let u be a local solution to (1.1), (1.2) where $(N-1)m<N+1$, $m>1$. Using the fact that

$$\lim_{R\to\infty}R^{(m+1)/(m-1)-N}\int_{\mathbb{R}^N}\big(1+\frac{r^2}{4}\big)^{-(N-1)/2}u_1dx=\infty, \tag{2.5}$$

we deduce from (2.4), that $H'(0) > Q$, for $R > R_0$ large, where

$$Q^2 := \frac{m-1}{m+1} C_0^{-2/(m-1)} s^{2(m+1)/(m-1)}. \tag{2.6}$$

Hence Theorem 1.2 is a direct consequence of the following result which is valid for any $m > 1$. $\square$

Proposition 2.5. *Let $m > 1$ and H be a solution to* (2.2) *where $H(0) \geq 0$ and $H'(0) > Q$. Then there exists $T_1 > 0$ such that $H(T_1) \geq (\frac{s^2}{C_0})^{1/(m-1)}$, $H'(T_1) > 0$ and hence H is not a global solution.*

Proof. Let H be a solution to (2.2) such that $H(0) \geq 0$ and $H'(0) > Q$. Let us suppose that $H(0) < (\frac{s^2}{C_0})^{1/(m-1)}$, otherwise the proof follows from Proposition 2.1. Therefore, there exists $T_0 \leq \infty$ such that $0 < H(T) < (\frac{s^2}{C_0})^{1/(m-1)}$ and $H'(T) > 0$ for all T in $(0, T_0)$. Assume first that T_0 is finite and $H'(T_0) = 0$. Since the function

$$F(T) = \frac{1}{2}(H'(T))^2 - \frac{C_0}{m+1} H^{m+1}(T) + \frac{s^2}{2} H^2(T)$$

is strictly increasing on $(0, T_0)$, thanks to (2.2), we get $F(T) \leq F(T_0) \leq \frac{1}{2}Q^2$, for all $0 \leq T < T_0$, in particular $F(0) \leq \frac{1}{2}Q^2$ which yields to $H'(0) \leq Q$. A contradiction.

Next we suppose that $T_0 = \infty$. Since H is monotone and bounded, there exists $0 < L \leq (\frac{s^2}{C_0})^{1/(m-1)}$ such that $\lim_{T\to\infty} H(T) = L$ and then there exists T_n converging to infinity with n such that $H'(T_n) \to 0$, as $n \to \infty$. Using again the function F we deduce that $F(0) \leq \lim_{n\to\infty} F(T_n)$. Hence $H'(0) \leq Q$, a contradiction. Then there exists $T_1 > 0$ such that $H(T_1) > (\frac{s^2}{C_0})^{1/(m-1)}$, $H'(T_1) > 0$ and hence H is not global thanks to Proposition 2.1 and Remark 2.2. $\square$

Corollary 2.6. *Let $m > 1$ and let u_0, u_1 be in X such that, for some positive R, one of the following two conditions is satisfied*

(1) $R^{2/(m-1)-N} \int_{\mathbb{R}^N} \big(1 + \frac{r^2}{4R^2}\big)^{-(N+1)/2} u_0 dx > \big(\frac{s^2}{C_0}\big)^{1/(m-1)}$,

(2) $R^{(m+1)/(m-1)-N} \int_{\mathbb{R}^N} \big(1 + \frac{r^2}{4R^2}\big)^{-(N-1)/2} u_1 dx > Q$.

Then Problem (1.1),(1.2) *has no global solution.*

Case $u_1 \leq 0$. In what follows we shall see that solutions to (1.1) may blow up in the case where $u_1 \in C_0^\infty(\mathbb{R}^N)$ is non-positive.

Theorem 2.7. *Let $m > 1$ and $u_0, -u_1$ in X be such that*

$$(H'(0))^2 - \frac{2C_0}{m+1} H^{m+1}(0) + s^2 H^2(0) \leq Q, \quad H(0) > \big(\frac{s^2}{C_0}\big)^{1/(m-1)}, \tag{2.7}$$

where Q is given by (2.6),

$$H(0) = R^{\frac{m+1}{m-1}} \int_{\mathbb{R}^N} \big(R^2 + \frac{r^2}{4}\big)^{-\frac{N+1}{2}} u_0 dx$$

and

$$H'(0) = R^{2/(m-1)} \int_{\mathbb{R}^N} \big(R^2 + \frac{r^2}{4}\big)^{-\frac{N-1}{2}} u_1 dx,$$

for some fixed $R > 0$. Then Problem (1.1),(1.2) *has no global solution.*

Proof. Assume that u_0 and u_1 satisfy (2.7) and are such that (1.1) has a global solution. Using Proposition 2.1 we easily deduce that the function H is strictly decreasing and $H > (\frac{s^2}{C_0})^{1/(m-1)}$ on $(0, T_0)$, for some $0 < T_0 \le \infty$. Now, a simple analysis shows that $H(T_0) = (\frac{s^2}{C_0})^{1/(m-1)}$. Next, since $H' < 0$ the function

$$F(T) = \frac{1}{2}(H'(T))^2 - \frac{C_0}{m+1}H^{m+1}(T) + \frac{s^2}{2}H^2(T)$$

is decreasing on $(0, T_0)$, thanks to (2.2). Therefore $F(0) > F(T_0) \ge \frac{1}{2}Q$, which contradicts (2.7). $\square$

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