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EXISTENCE OF POSITIVE SOLUTIONS FOR SOME POLYHARMONIC NONLINEAR BOUNDARY-VALUE PROBLEMS

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Abstract. We present existence results for the polyharmonic nonlinear elliptic boundary-value problem

$$(-\Delta)^m u = f(\cdot, u) \quad \text{in } B$$
$$(\frac{\partial}{\partial \nu})^j u = 0 \quad \text{on } \partial B, \quad 0 \le j \le m-1.$$

(in the sense of distributions), where B is the unit ball in $\mathbb{R}^n$ and $n \ge 2$. The nonlinearity $f(x,t)$ satisfies appropriate conditions related to a Kato class of functions $K_{m,n}$. Our approach is based on estimates for the polyharmonic Green function with zero Dirichlet boundary conditions and on the Schauder fixed point theorem.

1. Introduction

Boggio [3] gave an explicit expression for the Green function $G_{m,n}$ of $(-\Delta)^m$ on the unit ball B of $\mathbb{R}^n$ $(n \ge 2)$, with Dirichlet boundary conditions $(\frac{\partial}{\partial \nu})^j u = 0$, $0 \le j \le m-1$. In fact, he proved that for each x, y in B,

$$G_{m,n}(x,y) = k_{m,n}|x-y|^{2m-n} \int_1^{\frac{[x,y]}{|x-y|}} \frac{(v^2-1)^{m-1}}{v^{n-1}}\, dv \tag{1.1}$$

where $\frac{\partial}{\partial \nu}$ is the outward normal derivative, m is a positive integer, $k_{m,n}$ is a positive constant and $[x,y]^2 = |x-y|^2 + (1-|x|^2)(1-|y|^2)$, for x, y in B.

Hence, from its expression, it is clear that $G_{m,n}$ is positive in B^2, which does not hold for the Green function of the biharmonic or m-polyharmonic operator in an arbitrary bounded domain (see for example [7]). Only for the case $m = 1$, we have not this restriction.

Grunau and Sweers [8] derived from Boggio's formula some interesting estimates on the Green function $G_{m,n}$ in B, including a 3G-Theorem, which holds in the case $m = 1$ for the Green function G_Ω of an arbitrary bounded $C^{1,1}$-domain Ω (see [5] and [21]).

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When $m = 1$, the 3G-Theorem has been exploited to introduce the classical Kato class of functions $K_n(\Omega)$, which was used in the study of some nonlinear differential equations (see [15, 20]). Definition and properties of the class $K_n(\Omega)$ can be found in [1, 5].

Recently, Bachar et al [2] improved the inequalities of Grunau and Sweers [8] satisfied by $G_{m,n}$ in B. For instance, they gave a new form of the 3G-Theorem (see inequality (1.2) below and its proof in the Appendix).

Theorem 1.1 (3G-theorem). *There exists $C_{m,n} > 0$ such that for each $x, y, z \in B$, we have*

$$\frac{G_{m,n}(x,z)G_{m,n}(z,y)}{G_{m,n}(x,y)} \leq C_{m,n}\big[\big(\frac{\delta(z)}{\delta(x)}\big)^m G_{m,n}(x,z) + \big(\frac{\delta(z)}{\delta(y)}\big)^m G_{m,n}(y,z)\big], \quad (1.2)$$

where $\delta(x) = 1 - |x|$.

When $m = 1$, this new form of the 3G-Theorem has been proved for the Green function G_Ω in an arbitrary bounded $C^{1,1}$-domain Ω, by Kalton and Verbritsky [11] for $n \geq 3$ and by Selmi [18] for $n = 2$.

In [2], the authors used this 3G-Theorem to define and study a new Kato class of functions on B denoted by $K_{m,n} := K_{m,n}(B)$ (see Definition 1.2 below). In the case $m = 1$, this class was introduced for a bounded $C^{1,1}$-domain Ω in $\mathbb{R}^n$, in [16] for $n \geq 3$ and in [13] and [19] for $n = 2$. Moreover, it has been shown that $K_{1,n}(\Omega)$ contains properly the classical Kato class $K_n(\Omega)$.

Definition 1.2. A Borel measurable function φ defined on B belongs to the class $K_{m,n}$ if φ satisfies the condition

$$\lim_{\alpha \to 0}\Big(\sup_{x \in B}\int_{B \cap B(x,\alpha)}\big(\frac{\delta(y)}{\delta(x)}\big)^m G_{m,n}(x,y)|\varphi(y)|dy\Big) = 0. \quad (1.3)$$

The properties of the class $K_{m,n}$ were used in [2], to study a singular nonlinear differential polyharmonic equation

$$(-\Delta)^m u + \varphi(., u) = 0, \quad \text{in } B\backslash\{0\},$$

with boundary conditions $(\frac{\partial}{\partial \nu})^j u = 0$ on ∂B, $0 \leq j \leq m - 1$. The function φ satisfies $|\varphi(x,t)| \leq tq(x,t)$, where q is a nonnegative Borel measurable function in $B \times (0, \infty)$ which is required to satisfy some other hypotheses related to the class $K_{m,n}$.

The plan for this paper is as follows: In Section 2, we recall some estimates on the Green function $G_{m,n}$ and some properties of functions belonging to the Kato class $K_{m,n}(B)$. In section 3, we study the polyharmonic boundary-value problem

$$\begin{gathered}(-\Delta)^m u = f(\cdot, u) \quad \text{in } B \text{ (in the sense of distributions)}\\ (\frac{\partial}{\partial \nu})^j u = 0 \quad \text{on } \partial B \quad 0 \leq j \leq m - 1.\end{gathered} \quad (1.4)$$

The function f satisfies the following hypotheses:

(H1) The function f is a nonnegative Borel measurable function on $B \times (0, \infty)$, which is continuous and non-increasing with respect to the second variable.
(H2) For each $c > 0$, the function $x \to \frac{f(x, c(\delta(x))^m)}{(\delta(x))^{m-1}}$ is in $K_{m,n}$.
(H3) For each $c > 0$, $f(., c)$ is positive on a set of positive measure.

To study problem (P), we assume $m \geq n \geq 2$. So we show that for $G_{m,n}$ there exists $C > 0$ such that for each $x, y \in B$,

$$\frac{1}{C}(\delta(x))^m G_{m,n}(0,y) \leq G_{m,n}(x,y) \leq C G_{m,n}(0,y),$$

which is a fundamental inequality. Then by similar techniques to those used by Masmoudi and Zribi [17], we prove that (1.4) has a positive continuous solution u satisfying $a(\delta(x))^m \leq u(x) \leq b(\delta(x))^{m-1}$, where a, b are positive constants.

Note that for $m = 1$, using the complete maximum principle argument, which does not hold for $m \geq 2$, Mâagli and Zribi [15] established an existence and an uniqueness result for the problem (1.4) in a bounded $C^{1,1}$ domain Ω of $\mathbb{R}^n$ $(n \geq 3)$, where the function f is required to satisfy the hypotheses (H1), (H3), and

(H0) For each $c > 0$, $f(.,c)$ is in $K_n(\Omega)$.

In section 4, we shall study the following nonlinear polyharmonic problem in B, where $m \geq 1, n \geq 2$,

$$\begin{gathered} (-\Delta)^m u = g(.,u) \quad \text{in } B \text{ (in the sense of distributions)} \\ (\frac{\partial}{\partial \nu})^j u = 0, \quad \text{on} \partial B, \quad 0 \leq j \leq m-1\,. \end{gathered} \tag{1.5}$$

We Assume that g verifies the following hypotheses:

(H4) The function g is nonnegative Borel measurable function on $B \times (0,\infty)$, and is continuous with respect to the second variable.

(H5) There exist $p, q : B \to (0,\infty)$ nontrivial Borel measurable functions and $h, k : (0,\infty) \to [0,\infty)$ nontrivial and nondecreasing Borel measurable functions satisfying

$$p(x)h(t) \leq g(x,t) \leq q(x)k(t),$$

for $(x,t) \in B \times (0,\infty)$, such that

(A1) $p \in L^1_{\rm loc}(B)$.

(A2) The function $\theta(x) := q(x)/(\delta(x))^{m-1}$ is in $K_{m,n}$.

(A3) $\lim_{t\to 0^+} h(t)/t = +\infty$.

(A4) $\lim_{t\to +\infty} k(t)/t = 0$.

Under these hypotheses, we will prove that (1.5) has a positive continuous solution u satisfying $a(\delta(x))^m \leq u(x) \leq b(\delta(x))^{m-1}$, where a, b are positive constants.

This result is a follow up to the one of Dalmasso [6], who studied the problem (1.5) with more restrictive conditions on the function g. Indeed, he assumed that g is nondecreasing with respect to the second variable and satisfies

$$\lim_{t\to 0^+} \min_{x\in \overline{B}} \frac{g(x,t)}{t} = +\infty \quad \text{and} \quad \lim_{t\to +\infty} \max_{x\in \overline{B}} \frac{g(x,t)}{t} = 0.$$

He proved the existence of positive solution and he gave also an uniqueness result for positive radial solution when $g(x,t) = g(|x|,t)$.

On the other hand, we note that when $m = 1$, Brezis and Kamin [4] proved the existence and the uniqueness of a positive solution for the problem

$$\begin{gathered} -\Delta u = \rho(x) u^\alpha \quad \text{in } \mathbb{R}^n, \\ \liminf_{|x|\to\infty} u(x) = 0, \end{gathered}$$

with $0 < \alpha < 1$ and ρ is a nonnegative measurable function satisfying some appropriate conditions.

To simplify our statements, we define the following convenient notations:

- $B = \{x \in \mathbb{R}^n : |x| < 1\}$ with $n \geq 2$.
- $s \wedge t = \min(s,t)$ and $s \vee t = \max(s,t)$, for $s,t \in \mathbb{R}$.
- $C_0(B) = \{w \in C(B) : \lim_{|x|\to 1} w(x) = 0\}$
- For $x, y \in B$, we define: $[x,y]^2 = |x-y|^2 + (1-|x|^2)(1-|y|^2)$, $\delta(x) = 1-|x|$, and $\theta(x,y) = [x,y]^2 - |x-y|^2 = (1-|x|^2)(1-|y|^2)$.
  Note that $[x,y]^2 \geq 1 + |x|^2|y|^2 - 2|x||y| = (1-|x||y|)^2$. So that
$$\delta(x) \vee \delta(y) \leq [x,y]. \tag{1.6}$$
- Let f and g be two positive functions on a set S. We call $f \sim g$, if there is $c > 0$ such that
$$\frac{1}{c} g(x) \leq f(x) \leq c g(x), \quad \text{for all } x \in S.$$
  We call $f \preceq g$, if there is $c > 0$ such that
$$f(x) \leq c g(x), \quad \text{for all } x \in S.$$

The following properties will be used several times.
For $s, t \geq 0$, we have
$$s \wedge t \sim \frac{st}{s+t}, \tag{1.7}$$
$$(s+t)^p \sim s^p + t^p, \quad p \in \mathbb{R}^+. \tag{1.8}$$
Let $\lambda, \mu > 0$ and $0 < \gamma \leq 1$, then we have,
$$1 - t^\lambda \sim 1 - t^\mu, \quad \text{for } t \in [0,1], \tag{1.9}$$
$$\log(1+t) \preceq t^\gamma, \quad \text{for } t \geq 0, \tag{1.10}$$
$$\log(1+\lambda t) \sim \log(1+\mu t), \quad \text{for } t \geq 0, \tag{1.11}$$
$$\log(1+t^\lambda) \sim t^\lambda \log(2+t), \quad \text{for } t \in [0,1]. \tag{1.12}$$
On B^2 (that is $(x,y) \in B^2$), we have
$$\theta(x,y) \sim \delta(x)\delta(y), \tag{1.13}$$
$$[x,y]^2 \sim |x-y|^2 + \delta(x)\delta(y)\,. \tag{1.14}$$

2. Properties of the Green function and Kato class

For this paper to be self contained, we shall recall some results concerning the Green function $G_{m,n}(x,y)$ and the class $K_{m,n}$. The next result is due to Grunau and Sweers in [8].

Proposition 2.1. *On B^2, we have the following statements:*

(1) *For $2m < n$,*
$$G_{m,n}(x,y) \sim |x-y|^{2m-n}\Big(1 \wedge \frac{(\delta(x)\delta(y))^m}{|x-y|^{2m}}\Big).$$
(2) *For $2m = n$,*
$$G_{m,n}(x,y) \sim \log(1 + \frac{(\delta(x)\delta(y))^m}{|x-y|^{2m}}).$$

(3) *For* $2m > n$,

$$G_{m,n}(x,y) \sim (\delta(x)\delta(y))^{m-\frac{n}{2}}\Big(1 \wedge \frac{(\delta(x)\delta(y))^{n/2}}{|x-y|^n}\Big).$$

Corollary 2.2. *On* B^2, *we have*

(1) *If* $2m < n$,

$$G_{m,n}(x,y) \sim \frac{(\delta(x)\delta(y))^m}{|x-y|^{n-2m}\big(|x-y|^2+\delta(x)\delta(y)\big)^m} \sim \frac{(\delta(x)\delta(y))^m}{|x-y|^{n-2m}[x,y]^{2m}}$$

(2) *If* $2m = n$,

$$G_{m,n}(x,y) \sim (1 \wedge \frac{(\delta(x)\delta(y))^m}{|x-y|^{2m}}) \log\big(2+\frac{\delta(x)\delta(y)}{|x-y|^2}\big) \sim \frac{(\delta(x)\delta(y))^m}{[x,y]^{2m}} \log\big(1+\frac{[x,y]^2}{|x-y|^2}\big).$$

(3) *If* $2m > n$,

$$G_{m,n}(x,y) \sim \frac{(\delta(x)\delta(y))^m}{(|x-y|^2+(\delta(x)\delta(y)))^{n/2}} \sim \frac{(\delta(x)\delta(y))^m}{[x,y]^n}.$$

The proof of this corollary follows immediately from Proposition 2.1 and the statements (1.7)–(1.9) and (1.11)–(1.14).

Corollary 2.3. *For each* $x, y \in B$ *such that* $|x-y| \geq r$, *we have*

$$G_{m,n}(x,y) \preceq \frac{(\delta(x)\delta(y))^m}{r^n}. \tag{2.1}$$

Moreover, on B^2 *we have*

$$(\delta(x)\delta(y))^m \preceq G_{m,n}(x,y), \tag{2.2}$$

$$(\delta(x))^m \wedge (\delta(y))^m, \ \textit{if } m \geq n. \tag{2.3}$$

The assertions of this corollary are obviously obtained by using the estimates in Corollary 2.2 and the inequalities (1.6) and $|x-y| \leq [x,y] \preceq 1$.

Now we recall some properties of functions belonging to the class $K_{m,n}$.

Lemma 2.4. *Let* φ *be a function in* $K_{m,n}$. *Then the function* $x \to (\delta(x))^{2m}\varphi(x)$ *is in* $L^1(B)$.

Proof. Let $\varphi \in K_{m,n}$, then by (1.3) there exists $\alpha > 0$ such that for each $x \in B$,

$$\int_{B(x,\alpha)\cap B} (\frac{\delta(y)}{\delta(x)})^m G_{m,n}(x,y)|\varphi(y)|dy \leq 1.$$

Let $x_1, \dots, x_p$ in B such that $B \subset \cup_{1\leq i\leq p} B(x_i,\alpha)$. Then by (2.2), there exists $C > 0$ such that for all $i \in \{1, \dots p\}$ and $y \in B(x_i,\alpha) \cap B$, we have

$$(\delta(y))^{2m} \leq C(\frac{\delta(y)}{\delta(x_i)})^m G_{m,n}(x_i,y).$$

Hence, we have

$$\begin{aligned}\int_B (\delta(y))^{2m}|\varphi(y)|dy &\leq C \sum_{1\leq i\leq p} \int_{B(x_i,\alpha)\cap B} (\frac{\delta(y)}{\delta(x_i)})^m G_{m,n}(x_i,y)|\varphi(y)|dy \\ &\leq Cp < \infty.\end{aligned}$$

This completes the proof. □

Throughout the paper, we will use the notation

$$\|\varphi\|_B := \sup_{x\in B} \int_B \big(\frac{\delta(y)}{\delta(x)}\big)^m G_{m,n}(x,y)|\varphi(y)|dy,$$

for a measurable function φ on B.

Proposition 2.5. *Let φ be a function in $K_{m,n}$, then $\|\varphi\|_B < \infty$.*

Proof. Let $\varphi \in K_{m,n}$ and $\alpha > 0$. Then we have

$$\begin{aligned}\int_B \big(\frac{\delta(y)}{\delta(x)}\big)^m G_{m,n}(x,y)|\varphi(y)|dy &\le \int_{B\cap B(x,\alpha)} \big(\frac{\delta(y)}{\delta(x)}\big)^m G_{m,n}(x,y)|\varphi(y)|dy \\ &\quad + \int_{B\cap B^c(x,\alpha)} \big(\frac{\delta(y)}{\delta(x)}\big)^m G_{m,n}(x,y)|\varphi(y)|dy.\end{aligned}$$

Now, by (2.1), we have

$$\int_{B\cap B^c(x,\alpha)} \big(\frac{\delta(y)}{\delta(x)}\big)^m G_{m,n}(x,y)|\varphi(y)|dy \preceq \frac{1}{\alpha^n}\int_B (\delta(y))^{2m}|\varphi(y)|dy,$$

then the result follows from (1.3) and Lemma 2.4. □

The next result is due to Bachar et al [2]. Since reference [2] is not available, we have chosen to reproduce it here.

Proposition 2.6. *There exists a constant $C > 0$ such that for all $\varphi \in K_{m,n}$ and h a nonnegative harmonic function in B, we have*

$$\int_B G_{m,n}(x,y)(\delta(y))^{m-1}h(y)|\varphi(y)|dy \le C\|\varphi\|_B(\delta(x))^{m-1}h(x), \tag{2.4}$$

for all x in B.

Proof. Let h be a nonnegative harmonic function in B. So by Herglotz representation theorem [10, p, 29], there exists a nonnegative measure μ on ∂B such that

$$h(y) = \int_{\partial B} P(y,\xi)\mu(d\xi),$$

where $P(y,\xi) = \frac{1-|y|^2}{|y-\xi|^n}$, for $y \in B$ and $\xi \in \partial B$. So we need only to verify (2.4) for $h(y) = P(y,\xi)$ uniformly in $\xi \in \partial B$. From expression (1.1) of $G_{m,n}$, it is clear that for each $x, y \in B$, we have

$$G_{m,n}(x,y) \sim \frac{(\theta(x,y))^m}{[x,y]^n}(1 + o(1-|y|^2)).$$

Hence for x, y, z in B,

$$\frac{G_{m,n}(y,z)}{G_{m,n}(x,z)} = \frac{(1-|y|^2)^m[x,z]^n}{(1-|x|^2)^m[y,z]^n}(1 + o(1-|z|^2)),$$

which implies

$$\lim_{z\to\xi}\frac{G_{m,n}(y,z)}{G_{m,n}(x,z)} = \frac{(1-|y|^2)^m}{(1-|x|^2)^m}\frac{|x-\xi|^n}{|y-\xi|^n} \sim \big(\frac{\delta(y)}{\delta(x)}\big)^{m-1}\frac{P(y,\xi)}{P(x,\xi)}. \tag{2.5}$$

Thus by Fatou's lemma and (1.2), we deduce that

$$\begin{aligned}
&\int_B G_{m,n}(x,y)(\frac{\delta(y)}{\delta(x)})^{m-1}\frac{P(y,\xi)}{P(x,\xi)}|\varphi(y)|dy \\
&\preceq \liminf_{z\to\xi}\int_B G_{m,n}(x,y)\frac{G_{m,n}(y,z)}{G_{m,n}(x,z)}|\varphi(y)|dy \\
&\preceq \sup_{x\in B}\int_B(\frac{\delta(y)}{\delta(x)})^m G_{m,n}(x,y)|\varphi(y)|dy = \|\varphi\|_B.
\end{aligned}$$

Which completes the proof. □

For a nonnegative measurable function φ on B and $x\in B$, we define

$$V\varphi(x)=\int_B(\delta(y))^{m-1}G_{m,n}(x,y)\varphi(y)dy.$$

Corollary 2.7. *Let $\varphi\in K_{m,n}$. Then we have*

$$\|V\varphi\|_\infty<\infty. \tag{2.6}$$

Moreover, the function $x\mapsto(\delta(x))^{2m-1}\varphi(x)$ is in $L^1(B)$.

Proof. Put $h\equiv 1$ in (2.4) and using Proposition 2.5, we get (2.6). On the other hand, by (2.2), it follows that

$$\int_B(\delta(y))^{2m-1}|\varphi(y)|dy \preceq \int_B G_{m,n}(0,y)(\delta(y))^{m-1}|\varphi(y)|dy.$$

Hence the result follows from (2.6). □

Example 2.8. If $n\geq 2m$, for $p>\frac{n}{2m}$ we have $L^p(B)\subset K_{m,n}$. Furthermore, if $n<2m$ then for $p>1$ we have

$$\frac{1}{(\delta(.))^{2m-n}}L^p(B)\subset K_{m,n}.$$

Indeed, these inclusions are obtained by using the estimates on Corollary 2.2, (1.6) and the Hölder inequality.

Example 2.9. Let ρ be the function defined in B by $\rho(x)=\frac{1}{\delta(x)^\lambda}$. Then shown in [2], $\rho\in K_{m,n}$ if and only if $\lambda<2m$ and we have the following estimates for $V\rho$ in B,

(1) $\delta(x)^m\preceq V\rho(x)\preceq\delta(x)^{3m-\lambda-1}$, if $2m-1<\lambda<2m$.
(2) $\delta(x)^m\preceq V\rho(x)\preceq\delta(x)^m\log(\frac{2}{\delta(x)})$, if $\lambda=2m-1$.
(3) $V\rho(x)\sim\delta(x)^m$, if $\lambda<2m-1$.

The properties in Propositions 2.11 and 2.12 below are useful for our existence results. However, to establish them we need the next key Lemma.

Lemma 2.10. *Let $x_0\in\overline{B}$, then for each $\varphi\in K_{m,n}$,*

$$\lim_{\alpha\to0}\big(\sup_{x\in B}\int_{B\cap B(x_0,\alpha)}\big(\frac{\delta(y)}{\delta(x)}\big)^m G_{m,n}(x,y)|\varphi(y)|dy\big)=0\,. \tag{2.7}$$

Also for a positive harmonic function h in B, we have

$$\lim_{\alpha\to0}\big(\sup_{x\in B}\int_{B\cap B(x_0,\alpha)}\big(\frac{\delta(y)}{\delta(x)}\big)^{m-1}\frac{h(y)}{h(x)}G_{m,n}(x,y)|\varphi(y)|dy\big)=0. \tag{2.8}$$

Proof. Let $\varepsilon > 0$, then by (1.3), there exists $r > 0$ such that

$$\sup_{z\in B}\int_{B\cap B(z,r)} \big(\frac{\delta(y)}{\delta(z)}\big)^m G_{m,n}(z,y)|\varphi(y)|dy \le \varepsilon$$

Let $x_0 \in \overline{B}$ and $\alpha > 0$. Then by (2.1) we have for each $x \in B$,

$$\begin{aligned}
&\int_{B\cap B(x_0,\alpha)} \big(\frac{\delta(y)}{\delta(x)}\big)^m G_{m,n}(x,y)|\varphi(y)|dy \\
&\le \int_{B\cap B(x,r)} \big(\frac{\delta(y)}{\delta(x)}\big)^m G_{m,n}(x,y)|\varphi(y)|dy \\
&\quad + \int_{B\cap B(x_0,\alpha)\cap B^c(x,r)} \big(\frac{\delta(y)}{\delta(x)}\big)^m G_{m,n}(x,y)|\varphi(y)|dy \\
&\preceq \varepsilon + \int_{B\cap B(x_0,\alpha)} (\delta(y))^{2m}|\varphi(y)|dy.
\end{aligned}$$

Hence, using Lemma 2.4 and letting $\alpha \to 0$, claim (2.7) follows.

Now to prove (2.8), using again Herglotz representation theorem, we need only to verify the assertion for $h(y) = P(y,\xi)$ uniformly in $\xi \in \partial B$, where $P(y,\xi) = \frac{1-|y|^2}{|y-\xi|^n}$, for $y \in B$ and $\xi \in \partial B$.

Let $x \in B$, then by Fatou's Lemma and (2.5), we deduce that

$$\begin{aligned}
&\int_{B\cap B(x_0,\alpha)} \big(\frac{\delta(y)}{\delta(x)}\big)^{m-1} \frac{P(y,\xi)}{P(x,\xi)} G_{m,n}(x,y)|\varphi(y)|dy \\
&\preceq \liminf_{z\to\xi} \int_{B\cap B(x_0,\alpha)} G_{m,n}(x,y)\frac{G_{m,n}(y,z)}{G_{m,n}(x,z)}|\varphi(y)|dy \\
&\preceq \sup_{x\in B}\int_{B\cap B(x_0,\alpha)} \big(\frac{\delta(y)}{\delta(x)}\big)^m G_{m,n}(x,y)|\varphi(y)|dy,
\end{aligned}$$

Then by (2.7), we get (2.8) when $\alpha \to 0$. □

Proposition 2.11. *Let $\varphi \in K_{m,n}$. Then the following function is in $C_0(B)$,*

$$v(x) := \frac{1}{(\delta(x))^{m-1}} V\varphi(x)\,.$$

Proof. Let $x_0 \in B$ and $\alpha > 0$. Let $x, z \in B \cap B(x_0,\alpha)$, then

$$\begin{aligned}
|v(x) - v(z)| &\le \int_B \Big|\frac{G_{m,n}(x,y)}{(\delta(x))^{m-1}} - \frac{G_{m,n}(z,y)}{(\delta(z))^{m-1}}\Big|(\delta(y))^{m-1}|\varphi(y)|dy \\
&\le 2\sup_{\xi\in B}\int_{B\cap B(x_0,2\alpha)} \big(\frac{\delta(y)}{\delta(\xi)}\big)^{m-1} G_{m,n}(\xi,y)|\varphi(y)|dy \\
&\quad + \int_{B\cap B^c(x_0,2\alpha)} \Big|\frac{G_{m,n}(x,y)}{(\delta(x))^{m-1}} - \frac{G_{m,n}(z,y)}{(\delta(z))^{m-1}}\Big|(\delta(y))^{m-1}|\varphi(y)|dy.
\end{aligned}$$

If $|x_0 - y| \ge 2\alpha$ then $|x - y| \ge \alpha$ and $|z - y| \ge \alpha$. Moreover, by (2.1) for all $x \in B \cap B(x_0,\alpha)$ and $y \in \Omega := B \cap B^c(x_0,2\alpha)$, we have

$$\big(\frac{\delta(y)}{\delta(x)}\big)^{m-1} G_{m,n}(x,y) \preceq (\delta(y))^{2m-1}.$$

Since when $y \in \Omega$, the function $x \to \frac{G_{m,n}(x,y)}{(\delta(x))^{m-1}}$ is continuous in $B \cap B(x_0, \alpha)$, then by (2.8), Corollary 2.7 and the dominated convergence theorem, we obtain that

$$\int_B |\frac{G_{m,n}(x,y)}{(\delta(x))^{m-1}} - \frac{G_{m,n}(z,y)}{(\delta(z))^{m-1}}|(\delta(y))^{m-1}|\varphi(y)|dy \to 0$$

as $|x-z| \to 0$. Hence, we deduce that v is continuous in B. Next, we show that $v(x) \to 0$ as $\delta(x) \to 0$. Let $x_0 \in \partial B$, $\alpha > 0$ and $x \in B(x_0, \alpha)$, then

$$\begin{aligned} |v(x)| \le & \int_{B\cap B(x_0,2\alpha)} (\frac{\delta(y)}{\delta(x)})^{m-1} G_{m,n}(x,y)|\varphi(y)|dy \\ & + \int_{B\cap B^c(x_0,2\alpha)} (\frac{\delta(y)}{\delta(x)})^{m-1} G_{m,n}(x,y)|\varphi(y)|dy. \end{aligned}$$

Since $\lim_{\delta(x)\to 0} \frac{G_{m,n}(x,y)}{(\delta(x))^{m-1}} = 0$, as in the above argument, we get $\lim_{x\to x_0} v(x) = 0$. Hence $v \in C_0(B)$. □

For a nonnegative function ρ in $K_{m,n}$, we define

$$M_\rho := \{\varphi \in K_{m,n} : |\varphi| \preceq \rho\}.$$

By similar arguments as in the proof of the above Proposition, we can prove the following statement.

Proposition 2.12. *For any nonnegative function $\rho \in K_{m,n}$, the family of functions $\{V\varphi : \varphi \in M_\rho\}$ is relatively compact in $C_0(B)$.*

3. First existence result

In this section, we consider the case $m \ge n \ge 2$ to study problem (1.4). The main result that we shall prove is the following.

Theorem 3.1. *Assume (H1)–(H3). Then the problem* (1.4) *has a positive continuous solution u. Moreover, there exist two positive constants a and b such that for each $x \in B$,*

$$a(\delta(x))^m \le u(x) \le b(\delta(x))^{m-1}\,.$$

To prove this theorem, we state an existence result for the following boundary-value problem (in the sense of distributions)

$$\begin{gathered} (-\Delta)^m u = f(.,u) \quad \text{in } B \\ u = \lambda \quad \text{on } \partial B, \\ (\frac{\partial}{\partial \nu})^j u = 0, \quad \text{on } \partial B, \quad 1 \le j \le m-1. \end{gathered} \tag{3.1}$$

where $\lambda > 0$. For the next theorem we need the hypothesis

(H2') For each $c > 0$, the function $x \to \frac{f(x,c)}{(\delta(x))^{m-1}}$ is in $K_{m,n}$.

Note that hypothesis (H2) implies (H2').

Proposition 3.2. *Suppose that f satisfies (H1), (H3), and (H2'). Then for each $\lambda > 0$, problem* (3.1) *has a positive solution $u_\lambda \in C(\overline{B})$, such that for each $x \in B$,*

$$u_\lambda(x) = \lambda + \int_B G_{m,n}(x,y) f(y, u_\lambda(y))dy.$$

Proof. Let $\lambda > 0$. Then by (H2'), the function $\rho(y) := \frac{f(y,\lambda)}{(\delta(y))^{m-1}} \in K_{m,n}$ and so by Corollary 2.7, we have $\beta := \lambda + \|V\rho\|_\infty < \infty$. Let Y be the convex set given by

$$Y = \left\{u \in C(\overline{B}) : \lambda \leq u \leq \beta\right\}.$$

We consider the integral operator T on Y, defined by

$$Tu(x) = \lambda + \int_B G_{m,n}(x,y) f(y, u(y)) dy.$$

We shall prove that T has a fixed point in Y. Since for $u \in Y$ and $y \in B$, by (H1) we have

$$\frac{f(y,u(y))}{(\delta(y))^{m-1}} \leq \frac{f(y,\lambda)}{(\delta(y))^{m-1}} = \rho(y),$$

then using (H2'), we deduce that the function $y \to \frac{f(y,u(y))}{(\delta(y))^{m-1}}$ is in M_ρ. So from Proposition 2.12, we deduce that TY is relatively compact in $C(\overline{B})$. In particular, for all $u \in Y$, $Tu \in C(\overline{B})$ and so it is clear that $TY \subset Y$.

Now, we aim to prove the continuity of T in Y. Let $(u_k)_k$ be a sequence in Y which converges uniformly to $u \in Y$. Then since f is continuous with respect to the second variable, we deduce by the dominated convergence theorem that

$$\forall x \in B,\ Tu_k(x) \to Tu(x) \quad \text{as } k \to \infty.$$

As TY is relatively compact in $C(\overline{B})$, then

$$\|Tu_k - Tu\|_\infty \to 0 \quad \text{as } k \to \infty.$$

Thus we have proved that T is a compact mapping from Y to itself. Hence, by Schauder fixed point theorem, there exists a function $u_\lambda \in Y$ such that

$$u_\lambda(x) = \lambda + \int_B G_{m,n}(x,y) f(y, u_\lambda(y)) dy.$$

Finally, we need to verify that u_λ is a solution for problem (3.1). Since by (H1) we have for each $y \in B$, $f(y, u_\lambda(y)) \leq f(y,\lambda) = (\delta(y))^{m-1}\rho(y)$, then we deduce from Corollary 2.7 that the function $y \to f(y, u_\lambda(y))$ is in $L^1_{\mathrm{loc}}(B)$. So it is clear that u_λ satisfies (in the sense of distributions) the elliptic differential equation

$$(-\Delta)^m u_\lambda = f(., u_\lambda) \quad \text{in } B.$$

Furthermore, by (H2'), we have

$$0 \leq \frac{u_\lambda(x) - \lambda}{(\delta(x))^{m-1}} \leq \frac{1}{(\delta(x))^{m-1}} V\rho(x).$$

This implies from Proposition 2.11 that $\lim_{\delta(x)\to 0} \frac{u_\lambda(x)-\lambda}{(\delta(x))^{m-1}} = 0$. Namely, u_λ satisfies the boundary conditions $u_\lambda = \lambda$ and $(\frac{\partial}{\partial\nu})^j u_\lambda = 0$, on ∂B for $1 \leq j \leq m-1$. This ends the proof. □

In the sequel, we consider a sequence $(\lambda_k)_k$ of positive real numbers, decreasing to zero. We denote by u_k the solution of the problem (P_{λ_k}) given by Proposition 3.2 and satisfying for each $x \in B$,

$$u_k(x) = \lambda_k + \int_B G_{m,n}(x,y) f(y, u_k(y)) dy. \tag{3.2}$$

Lemma 3.3. *There exists a positive constant a such that for all $k \in \mathbb{N}$, and $x \in B$, $u_k(x) \geq a(\delta(x))^m$.*

Proof. By (2.2) and (2.3), we remark that on B,

$$G_{m,n}(0,y) \sim (\delta(y))^m.$$

Then by (2.2) and (2.3) again, we deduce that there exists a constant $c > 1$ such that we have for each $x, y \in B$

$$\frac{1}{c}(\delta(x))^m G_{m,n}(0,y) \le G_{m,n}(x,y) \le cG_{m,n}(0,y).$$

This implies by (3.1) that

$$u_k(x) \le c\big(\lambda_k + \int_B G_{m,n}(0,y) f(y,u_k(y))dy\big) = cu_k(0). \tag{3.3}$$

and

$$\begin{aligned} u_k(x) &\ge \frac{1}{c}\big(\delta(x)\big)^m(\lambda_k + \int_B G_{m,n}(0,y) f(y,u_k(y))dy\big) \\ &\ge \frac{1}{c}\big(\delta(x)\big)^m\big(\inf_{k\in\mathbb{N}} u_k(0)\big). \end{aligned}$$

We claim that $a = \frac{1}{c}(\inf_{k\in\mathbb{N}} u_k(0)) > 0$. Assume on the contrary that there exists a subsequence $(u_{k_p}(0))_p$ which converges to zero. In particular, for p large enough, we have $u_{k_p}(0) \le 1$, which implies with (3.3) and (H1) that

$$u_{k_p}(0) = \lambda_{k_p} + \int_B G_{m,n}(0,y) f(y,u_{k_p}(y))dy \ge \lambda_{k_p} + \int_B G_{m,n}(0,y) f(y,c)dy.$$

Thus, by letting p to ∞, we reach a contradiction from hypothesis (H3). This completes the proof. □

Proof of Theorem 3.1. Let a be the constant given in Lemma 3.3, then by hypothesis (H2), we deduce that the function

$$\rho(y) := \frac{f(y, a(\delta(y))^m)}{(\delta(y))^{m-1}} \in K_{m,n}.$$

Since for each $k \in \mathbb{N}$ and $y \in B$, by (H1) we have

$$\frac{f(y,u_k(y))}{(\delta(y))^{m-1}} \le \frac{f(y, a(\delta(y))^m)}{(\delta(y))^{m-1}} = \rho(y)\,.$$

Then the function $y \to \frac{f(y,u_k(y))}{(\delta(y))^{m-1}}$ is in M_ρ. So using Proposition 2.12, we deduce from (3.2) that the family $(u_k)_k$ is relatively compact in $C(\overline{B})$. Then it follows that there exists a subsequence $(u_{k_p})_p$ which converges uniformly to a function $u \in C(\overline{B})$. Moreover, by Lemma 3.3, we have $u(x) \ge a(\delta(x))^m$, for each $x \in B$. Hence, using the continuity of f with respect to the second variable, we apply the dominated convergence theorem in (3.2) to obtain that

$$u(x) = \int_B G_{m,n}(x,y) f(y,u(y))dy.$$

Finally, by Lemma 3.3 and hypothesis (H1), for each $y \in B$, we have

$$f(y,u(y)) \le f(y, a(\delta(y))^m) = (\delta(y))^{m-1}\rho(y).$$

Then we deduce from Corollary 2.7 that the function $y \to f(y,u(y))$ is in $L^1_{\rm loc}(B)$. So u satisfies (in the sense of distributions) the elliptic differential equation

$$(-\Delta)^m u = f(.,u) \quad \text{in } B.$$

Furthermore, we have for $x \in B$,

$$a\delta(x) \le \frac{u(x)}{(\delta(x))^{m-1}} \le \frac{1}{(\delta(x))^{m-1}} V\rho(x),$$

which together with Proposition 2.11 imply that u satisfies the boundary conditions $(\frac{\partial}{\partial\nu})^j u = 0$, on ∂B, for $0 \le j \le m-1$ and that there exists a positive constant b such that

$$a(\delta(x))^m \le u(x) \le b(\delta(x))^{m-1}.$$

This completes the proof. □

Corollary 3.4. *Let $\varphi \in C(\partial B)$ and $\psi \in C^1(\partial B)$ be nonnegative functions on ∂B and f satisfies (H1)–(H3), then the polyharmonic boundary-value problem*

$$(-\Delta)^m u = f(.,u) \quad \text{in } B \text{ (in the sense of distributions)},$$

$$(-\frac{\partial}{\partial\nu})^{m-1} u = \psi, \quad (-\frac{\partial}{\partial\nu})^{m-2} u = \varphi, \quad (\frac{\partial}{\partial\nu})^j u = 0 \quad \text{on } \partial B \quad \text{for } 0 \le j \le m-3, \tag{3.4}$$

has a positive continuous solution u. Moreover there exists a positive constant a such that

$$u(x) \ge a(\delta(x))^m.$$

Proof. Let h be the solution of the Dirichlet problem

$$(-\Delta)^m h = 0 \quad \text{in } B$$

$$(-\frac{\partial}{\partial\nu})^{m-1} h = \psi, \quad (-\frac{\partial}{\partial\nu})^{m-2} h = \varphi, \quad (\frac{\partial}{\partial\nu})^j h = 0, \quad \text{on } \partial B, \text{ for } 0 \le j \le m-3.$$

Then as in [9], for $x \in B$ we have

$$h(x) = \int_{\partial B} K_{m,n}(x,y)\varphi(y)d\omega(y) + \int_{\partial B} L_{m,n}(x,y)\psi(y)d\omega(y),$$

where

$$L_{m,n}(x,y) = \frac{1}{2^m(m-2)!\omega_n} \frac{(1-|x|^2)^m}{|x-y|^{n+2}} [n(1-|x|^2) + (m+2-n)|x-y|^2],$$

$$K_{m,n}(x,y) = \frac{1}{2^{m-1}(m-1)!\omega_n} \frac{(1-|x|^2)^m}{|x-y|^n}$$

for $x, y \in B$, and ω_n denotes the $(n-1)$ dimensional surface area of the unit ball.

For $m \ge n \ge 2$, we have evidently $L_{m,n} > 0$ and so h is nonnegative on B. Using this fact, we can easily see that the function f_0 defined on $B \times (0,\infty)$ by

$$f_0(x,t) = f(x, t + h(x))$$

satisfies (H1)–(H3). Hence by Theorem 3.1, the problem

$$(-\Delta)^m v = f_0(.,v) \quad \text{in } B \text{ (in the sense of distributions)}$$

$$(\frac{\partial}{\partial\nu})^j v = 0, \quad \text{on } \partial B, \quad \text{for } 0 \le j \le m-1\,.$$

has a positive solution $v \in C_0(B)$ satisfying $v(x) \ge a(\delta(x))^m$, where a is a positive constant. Let $u = v + h$. Then u is the desired solution for the problem (3.4). This completes the proof. □

Remark 3.5. Let f satisfy (H1), (H3), and

(H2”) For each $c > 0$, the function $x \to \frac{f(x, c(\delta(x))^m)}{(\delta(x))^{m+n-1}}$ is in $K_{m,n}$.

Then problem (1.4) has a positive solution u satisfying $u(x) \sim (\delta(x))^m$. Indeed, we note that (H2") implies (H2), so by Theorem 3.1, problem (1.4) has a positive solution satisfying that for each $x \in B$

$$u(x) = \int_B G_{m,n}(x,y) f(y,u(y))dy$$

and $u(x) \geq a(\delta(x))^m$. Now, if $m \geq n$, we have by Corollary 2.2 that $G_{m,n}(x,y) \sim \frac{(\delta(x)\delta(y))^m}{[x,y]^n}$, which by (1.6) implies that

$$G_{m,n}(x,y) \preceq (\delta(x))^m (\delta(y))^{m-n}.$$

Hence for each $x \in B$, we have

$$a(\delta(x))^m \leq u(x) \preceq (\delta(x))^m \int_B (\delta(y))^{m-n} f(y, a(\delta(y))^m)dy. \tag{3.5}$$

Since f satisfies (H2"), we deduce by Corollary 2.7, that $u(x) \sim (\delta(x))^m$.

Remark 3.6. Let $\psi(r,.) = \max_{|x|=r} f(x,.)$, for $r \in [0,1]$ and suppose that for all $c > 0$,

$$\int_0^1 r^{n-1}(1-r)^{m-1}\psi(r, c(1-r)^m)dr < \infty. \tag{3.6}$$

Then the solution u of (1.4) satisfies $u(x) \sim (\delta(x))^m$. Indeed, by Theorem 3.1 and (H1), we have

$$a(\delta(x))^m \leq u(x) \leq \int_B G_{m,n}(x,y) f(y, a(\delta(y))^m)dy. \tag{3.7}$$

On the other hand using (1.1), we have

$$G_{m,n}(x,y) \preceq |x-y|^{2m-n}\big(\frac{[x,y]^2}{|x-y|^2} - 1\big)^{m-1} \int_1^{\frac{[x,y]}{|x-y|}} \frac{dv}{v^{n-1}}.$$

Now since $\frac{[x,y]^2}{|x-y|^2} - 1 \sim \frac{\delta(x)\delta(y)}{|x-y|^2}$, we deduce that

$$G_{m,n}(x,y) \preceq (\delta(x)\delta(y))^{m-1} G_{1,n}(x,y).$$

Hence it follows from (3.6) that

$$u(x) \preceq (\delta(x))^{m-1} \int_B (\delta(y))^{m-1} G_{1,n}(x,y)\psi(|y|, a(\delta(y))^m)dy.$$

By similar calculus as in [15, p.538], we have by (3.6) that for $x \in B$,

$$\int_B (\delta(y))^{m-1} G_{1,n}(x,y)\psi(|y|, a(\delta(y))^m)dy \preceq \delta(x).$$

This implies that $u(x) \sim (\delta(x))^m$.

Example 3.7. Let $\alpha > 0$ and $\lambda < m+1$. Let ρ be a nontrivial measurable function in B such that for each $x \in B$

$$0 \leq \rho(x) \leq \frac{1}{(\delta(x))^{\lambda - m\alpha}}.$$

Then the problem

$$\begin{gathered} (-\Delta)^m u = \rho(x) u^{-\alpha} \quad \text{in } B \text{ (in the sense of distributions)} \\ (\frac{\partial}{\partial \nu})^j u = 0 \quad \text{on } \partial B, \quad \text{for } 0 \leq j \leq m-1. \end{gathered}$$

has a positive solution $u \in C_0(B)$ such that for all $x \in B$,

(1) $\delta(x)^m \preceq u(x) \preceq \delta(x)^{2m-\lambda}$, if $m < \lambda < m+1$
(2) $\delta(x)^m \preceq u(x) \preceq \delta(x)^m \log(\frac{2}{\delta(x)})$, if $\lambda = m$
(3) $u(x) \sim \delta(x)^m$, if $\lambda < m$.

4. Second existence result

In this section, we prove the following result about problem (1.5).

Theorem 4.1. *Assume (H4)and (H5). Then problem* (1.5) *has a positive continuous solution u. Moreover there exist positive constants a and b, such that*

$$a(\delta(x))^m \le u(x) \le b(\delta(x))^{m-1}.$$

Proof. By (A2), the function $\theta(x) = q(x)/(\delta(x))^{m-1}$ is in $K_{m,n}$. Then using Proposition 2.11, we have

$$M := \sup_{x \in B}\Big(\frac{1}{(\delta(x))^{m-1}} V\theta(x)\Big) < \infty.$$

By (A4) we have $\lim_{t\to\infty} \frac{k(t)}{t} = 0$, then there exists $b > 0$ such that $Mk(b) \le b$.

On the other hand, by (A1) the function p is a nontrivial nonnegative function in $L^1_{\rm loc}(B)$, then there exists $r \in (0,1)$ such that

$$0 < \int_{B(0,r)} p(y)dy < \infty.$$

Furthermore, from (2.2) there exists $c > 0$ such that for each $x, y \in B$

$$G_{m,n}(x,y) \ge c(\delta(x))^m(\delta(y))^m.$$

Hence, since by (A3) we have $\lim_{t\to 0} \frac{h(t)}{t} = +\infty$, then there exists $a > 0$ such that

$$c(1-r)^m h(a(1-r)^m) \int_{B(0,r)} p(y)dy \ge a.$$

Let Λ be the convex set

$$\Lambda = \{u \in C_0(B) : a(\delta(x))^m \le u(x) \le b(\delta(x))^{m-1}\}$$

and T be the operator defined on Λ by

$$Tu(x) = \int_B G_{m,n}(x,y) g(y,u(y))dy.$$

We shall prove that T has a fixed point. We first note that for $u \in \Lambda$ and $y \in B$, we have by (H5)

$$\frac{g(y,u(y))}{(\delta(y))^{m-1}} \le \frac{q(y)k(u(y))}{(\delta(y))^{m-1}} \le k(b)\frac{q(y)}{(\delta(y))^{m-1}} := k(b)\theta(y).$$

Then we deduce that the function $y \to \frac{g(y,u(y))}{(\delta(y))^{m-1}} \in M_\theta$. Thus by Proposition 2.12, we obtain that the family $T\Lambda$ is relatively compact in $C_0(B)$

We need now to verify that for $u \in \Lambda$, we have

$$a(\delta(x))^m \le Tu(x) \le b(\delta(x))^{m-1}.$$

Let $u \in \Lambda$ and $x \in B$, then by (H5), we have

$$\begin{aligned} Tu(x) &\leq \int_B G_{m,n}(x,y)q(y)k(u(y)) \\ &\leq (\delta(x))^{m-1}\Big[k(b)\int_B (\frac{\delta(y)}{\delta(x)})^{m-1} G_{m,n}(x,y)\theta(y)dy\Big] \\ &\leq Mk(b)(\delta(x))^{m-1} \\ &\leq b(\delta(x))^{m-1}. \end{aligned}$$

On the other hand from (H5) and (2.2), we have

$$\begin{aligned} Tu(x) &\geq c(\delta(x))^m \int_B (\delta(y))^m p(y)h(u(y))dy \\ &\geq (\delta(x))^m\Big[c(1-r)^m h(a(1-r)^m)\int_{B(0,r)} p(y)dy\Big] \\ &\geq a(\delta(x))^m. \end{aligned}$$

Thus we have proved that $T\Lambda \subset \Lambda$.

Now we aim to prove the continuity of T in Λ. We consider a sequence $(u_k)_k$ in Λ which converges uniformly to u in Λ. Then since g is continuous with respect to the second variable, we deduce by the dominated convergence theorem that for all $x \in B$,

$$Tu_k(x) \to Tu(x) \quad \text{as } k \to \infty.$$

Since $T\Lambda$ is relatively compact in $C_0(B)$, we have the uniform convergence. Hence T is a compact mapping from Λ to itself. Then by the Schauder fixed point theorem, we deduce that there exists a function $u \in \Lambda$ such that

$$u(x) = \int_B G_{m,n}(x,y)g(y,u(y))dy.$$

So u satisfies (in the sense of distributions) the elliptic differential equation

$$(-\Delta)^m u = g(.,u) \ in\ B.$$

Moreover, since u satisfies

$$a(\delta(x)) \leq \frac{u(x)}{(\delta(x))^{m-1}} \preceq \frac{1}{(\delta(x))^{m-1}} V\theta(x),$$

we deduce by Proposition 2.11 that $\lim_{\delta(x)\to 0} \frac{u(x)}{(\delta(x))^{m-1}} = 0$ and so u satisfies the boundary conditions $(\frac{\partial}{\partial\nu})^j u = 0$, on ∂B for $0 \leq j \leq m-1$. This completes the proof. ☐

Example 4.2. Let $\lambda < m+1$ and $f : (0,\infty) \to [0,\infty)$ be a nontrivial continuous and nondecreasing function satisfying

$$\lim_{t\to 0} \frac{f(t)}{t} = \infty \quad \text{and} \quad \lim_{t\to\infty} \frac{f(t)}{t} = 0.$$

Then the problem

$$\begin{gathered} (-\Delta)^m u = (\delta(x))^{-\lambda} f(u) \quad \text{in } B \\ (\frac{\partial}{\partial\nu})^j u = 0, \quad \text{on } \partial B \quad \text{for } 0 \leq j \leq m-1, \end{gathered}$$

has a positive solution $u \in C_0(B)$ such that for all $x \in B$,

(1) $(\delta(x))^m \preceq u(x) \preceq (\delta(x))^{2m-\lambda}$, if $m < \lambda < m+1$
(2) $(\delta(x))^m \preceq u(x) \preceq (\delta(x))^m \log(\frac{2}{\delta(x)})$, if $\lambda = m$
(3) $u(x) \sim (\delta(x))^m$, if $\lambda < m$.

5. Appendix

In this section we prove the 3G-theorem. The following Lemma will help us doing so.

Lemma 5.1 ([12, 14])**.** *For $x, y \in B$, we have the following properties:*

(1) *If $\delta(x)\delta(y) \le |x-y|^2$ then $(\delta(x) \vee \delta(y)) \le \frac{(\sqrt{5}+1)}{2}|x-y|$*
(2) *If $|x-y|^2 \le \delta(x)\delta(y)$ then $\frac{(3-\sqrt{5})}{2}\delta(x) \le \delta(y) \le \frac{(3+\sqrt{5})}{2}\delta(x)$*

Proof. 1) We may assume that $(\delta(x) \vee \delta(y)) = \delta(y)$. Then the inequalities $\delta(y) \le \delta(x) + |x-y|$ and $\delta(x)\delta(y) \le |x-y|^2$ imply that

$$(\delta(y))^2 - \delta(y)|x-y| - |x-y|^2 \le 0,$$

i.e.

$$(\delta(y) + \frac{(\sqrt{5}-1)}{2}|x-y|\ )(\delta(y) - \frac{(\sqrt{5}+1)}{2}|x-y|\ ) \le 0.$$

It follows that

$$(\delta(x) \vee \delta(y)) \le \frac{(\sqrt{5}+1)}{2}|x-y|.$$

2) For each $z \in \partial B$, we have $|y-z| \le |x-y| + |x-z|$ and since $|x-y|^2 \le \delta(x)\delta(y)$, we obtain

$$|y-z| \le \sqrt{\delta(x)\delta(y)} + |x-z| \le \sqrt{|x-z||y-z|} + |x-z|,$$

i.e.

$$(\sqrt{|y-z|} + \frac{(\sqrt{5}-1)}{2}\sqrt{|x-z|}\ )(\sqrt{|y-z|} - \frac{(\sqrt{5}+1)}{2}\sqrt{|x-z|}\ ) \le 0.$$

It follows that

$$|y-z| \le \frac{(3+\sqrt{5})}{2}|x-z|.$$

Thus, interchanging the role of x and y, we have

$$(\frac{3-\sqrt{5}}{2})|x-z| \le |y-z| \le (\frac{3+\sqrt{5}}{2})|x-z|.$$

Which implies

$$(\frac{3-\sqrt{5}}{2})\delta(x) \le \delta(y) \le (\frac{3+\sqrt{5}}{2})\delta(x).$$

□

Proof of the 3G-Theorem, [2]. To prove inequality (1.2), we let

$$A(x,y) := \frac{(\delta(x)\delta(y))^m}{G_{m,n}(x,y)}$$

and we claim that A is a quasi-metric, that is for each $x, y, z \in B$,

$$A(x,y) \preceq A(y,z) + A(x,z).$$

To show this claim, we separate the proof into three cases.

Case 1: For $2m < n$, using Proposition 2.1, we have

$$A(x,y) \sim |x-y|^{n-2m}(|x-y|^2 \vee (\delta(x)\delta(y)))^m.$$

We distinguish the following subcases:
• If $\delta(x)\delta(y) \le |x-y|^2$, then we have

$$A(x,y) \sim |x-y|^n \preceq |x-z|^n + |y-z|^n \preceq A(x,z) + A(y,z).$$

• The inequality $|x-y|^2 \le \delta(x)\delta(y)$ implies from Lemma 5.1 that $\delta(x) \sim \delta(y)$. So we deduce that: if $|x-z|^2 \le \delta(x)\delta(z)$ or $|y-z|^2 \le \delta(y)\delta(z)$, then it follows from Lemma 5.1 that $\delta(x) \sim \delta(y) \sim \delta(z)$. Hence,

$$\begin{aligned} A(x,y) &\sim |x-y|^{n-2m}(\delta(x)\delta(y))^m \\ &\preceq (\delta(x)\delta(y))^m(|x-z|^{n-2m} + |y-z|^{n-2m}) \\ &\preceq |x-z|^{n-2m}(\delta(x)\delta(z))^m + |y-z|^{n-2m}(\delta(y)\delta(z))^m \\ &\preceq A(x,z) + A(y,z), \end{aligned}$$

If $|x-z|^2 \ge \delta(x)\delta(z)$ and $|y-z|^2 \ge \delta(y)\delta(z)$. Then using Lemma 5.1, we have

$$(\delta(x) \vee \delta(z)) \preceq |x-z| \quad \text{and} \quad (\delta(y) \vee \delta(z)) \preceq |y-z|.$$

So, we have

$$\begin{aligned} A(x,y) &\sim |x-y|^{n-2m}(\delta(x)\delta(y))^m \\ &\preceq (|x-z|^{n-2m} + |y-z|^{n-2m})(\delta(x)\delta(y))^m \\ &\preceq |x-z|^{n-2m}(\delta(x))^{2m} + |y-z|^{n-2m}(\delta(y))^{2m} \\ &\preceq |x-z|^n + |y-z|^n \\ &\preceq A(x,z) + A(y,z). \end{aligned}$$

Case 2: For $2m = n$, using Proposition 2.1, we have

$$A(x,y) \sim \frac{(\delta(x)\delta(y))^m}{\log\big(1 + \frac{(\delta(x)\delta(y))^m}{|x-y|^{2m}}\big)}. \tag{5.1}$$

Since for each $t \ge 0$, $\frac{t}{1+t} \preceq \log(1+t) \preceq t$, we deduce that

$$|x-y|^{2m} \preceq A(x,y) \preceq |x-y|^{2m} + (\delta(x)\delta(y))^m. \tag{5.2}$$

So we distinguish the following subcases:
• If $\delta(x)\delta(y) \le |x-y|^2$, then by (1.8), we have

$$A(x,y) \preceq |x-y|^{2m} \preceq |x-z|^{2m} + |y-z|^{2m} \preceq A(x,z) + A(y,z).$$

• If $|x-y|^2 \le \delta(x)\delta(y)$, it follows from Lemma 5.1 that $\delta(x) \sim \delta(y)$.
If $|x-z|^2 \le \delta(x)\delta(z)$ or $|y-z|^2 \le \delta(y)\delta(z)$, so from Lemma 5.1, we deduce that $\delta(x) \sim \delta(y) \sim \delta(z)$. Since

$$|x-y|^{2m} \preceq |x-z|^{2m} + |y-z|^{2m} \preceq (|x-z|^{2m} \vee |y-z|^{2m}),$$

we obtain that

$$\Big(\log\big(1 + \frac{(\delta(x)\delta(z))^m}{|x-z|^{2m}}\big) \wedge \log\big(1 + \frac{(\delta(y)\delta(z))^m}{|y-z|^{2m}}\big)\Big) \preceq \log\big(1 + \frac{(\delta(x)\delta(y))^m}{|x-y|^{2m}}\big),$$

which together with (1.7) imply $A(x,y) \preceq A(y,z) + A(x,z)$.
If $|x-z|^2 \geq \delta(x)\delta(z)$ and $|y-z|^2 \geq \delta(y)\delta(z)$, then by Lemma 5.1, it follows that

$$(\delta(x) \vee \delta(z)) \preceq |x-z| \quad \text{and} \quad (\delta(y) \vee \delta(z)) \preceq |y-z|.$$

Hence, by (5.2) we have

$$\begin{aligned} A(x,y) &\preceq (\delta(x)\delta(y))^m \\ &\preceq (\delta(x))^{2m} + (\delta(y))^{2m} \\ &\preceq |x-z|^{2m} + |y-z|^{2m} \\ &\preceq A(x,z) + A(y,z). \end{aligned}$$

Case 3: For $2m > n$, from Proposition 2.1, we have

$$A(x,y) \sim (|x-y|^2 \vee (\delta(x)\delta(y)))^{1/2}.$$

Then the result holds by similar arguments as in case 1. The proof is complete. $\square$

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