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MULTIPLE PERIODIC SOLUTIONS OF A DISCRETE TIME PREDATOR-PREY SYSTEMS WITH TYPE IV FUNCTIONAL RESPONSES

ZHIGANG LIU, ANPING CHEN, JINDE CAO, & FENGDE CHEN

Abstract. By using the continuation theorem of Mawhin's coincidence degree theory, some sufficient conditions are obtained ensuring the existence of multiple positive periodic solutions of a discrete time predator-prey systems with type IV functional responses.

1. Introduction

Recently, a Lotka-Volterra model with Holloing Type functional response has been extensively studied by number of papers (see papers [1]-[7], [9]-[12], [15], [18], [21]-[23] and the references cited therein). The model is described by the following system

$$\begin{aligned} x_1'(t) &= x_1(t)\big[b_1(t) - a_1(t)x_1(t) - \frac{c(t)x_2(t)}{m(t)x_2(t)+x_1(t)}\big], \\ x_2'(t) &= x_2(t)\big[-b_2(t) + \frac{a_2(t)x_1(t)}{m(t)x_2(t)+x_1(t)}\big], \end{aligned} \tag{1.1}$$

where $x_1(t)$ and $x_2(t)$ represent the densities of the prey and the predator, respectively, $b_1(t)$, $c(t)$, $b_2(t)$ and $a_2(t)$ are the prey intrinsic growth rate, capture rate, death rate of the predator, and the conversion rate, respectively, $b_1(t)/a_1(t)$ gives the carrying capacity of the prey, and m is the half saturation constant, the functional response $x/(m(t)y+x)$ is ratio-dependent.

When the prey group has defence or toxicity, the functional response in a predator-prey model should be type IV. Kot [19] proposed the following predator-prey model with a type IV functional response

$$\begin{aligned} x_1'(t) &= x_1(t)\big[b_1(t) - a_1(t)x_1(t-\tau_1(t)) - \frac{c(t)x_2(t-\sigma(t))}{\frac{x_1^2(t-\tau_2(t))}{i} + x_1(t-\tau_2(t)) + a}\big], \\ x_2'(t) &= x_2(t)\big[-b_2(t) + \frac{a_2(t)x_1(t-\tau_3(t))}{\frac{x_1^2(t-\tau_4(t))}{i} + x_1(t-\tau_4(t)) + a}\big], \end{aligned} \tag{1.2}$$

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where c, σ, a_i, b_i $(i = 1, 2)$ and τ_j $(j = 1, 2, 3, 4)$ are continuous ω-periodic functions with $c(t) \geq 0$, $\sigma(t) \geq 0$, $a_i(t) \geq 0$ and $\tau_j(t) \geq 0$, $\int_0^\omega c(t)dt > 0$ and $\int_0^\omega b_i(t)dt > 0$, i and a are positive constants.

Recently, many authors studied the existence of positive periodic solutions in population models by using the powerful and effective method of coincidence degree. Chen [8] has established the results of the existence of multiple positive periodic solutions by applying the continuation theorem for system (1.2) in the case $\tau_2(t) = 0$.

When the populations have non-overlapping generations, discrete time model described by difference equations is more appropriate than the continuous one. In [9] and [28], authors studied the periodic solutions of some difference equations by using coincidence degree theory. However, no work has been done for the multiple positive periodic solutions of discrete time predator-prey model with type IV functional responses.

The main purpose of this paper is to propose a discrete analogue of system (1.2) and to obtain sufficient conditions for the existence of its multiple positive periodic solutions by employing coincidence degree theory and some analysis technique. This is the first time that a discrete time predator-prey model with a type IV functional response has been studied by using this way.

The rest of this paper is organized as follows. In Section 2, we propose a discrete predator-prey model with type IV functional responses described by difference equations with the help of differential equations with piecewise constant arguments. In section 3, we shall establish easily verifiable sufficient criteria for the existence of multiple positive periodic solutions of the difference equations derived in Section 2.

2. Discrete analogue of system (1.2)

Let us consider the following equation with piecewise arguments, it is considered as a semi-discretization of (1.2)

$$\begin{aligned}
\frac{1}{x_1(t)}\frac{dx_1(t)}{dt} &= b_1([t]) - a_1([t])x_1([t]-\tau_1([t])) - \frac{c([t])x_2([t]-\sigma([t]))}{\frac{x_1^2([t]-\tau_2([t]))}{i} + x_1([t]-\tau_2([t])) + a}, \\
\frac{1}{x_2(t)}\frac{dx_2(t)}{dt} &= -b_2([t]) + \frac{a_2([t])x_1([t]-\tau_3([t]))}{\frac{x_1^2([t]-\tau_4([t]))}{i} + x_1([t]-\tau_4([t])) + a}, \quad t \neq 0, 1, 2, \cdots,
\end{aligned} \tag{2.1}$$

where $[t]$ denotes the integer part t, $t \in (0, +\infty)$.

By a solution of (2.1), we mean a function $x = (x_1, x_2)^T$, which is defined for $t \in [0, +\infty)$, and possesses the following properties:

(1) x is continuous on $[0, +\infty)$.
(2) The derivative $\frac{dx_1(t)}{dt}$, $\frac{dx_2(t)}{dt}$ exist at each point $t \in [0, +\infty)$ with the possible exception of the points $t \in \{0, 1, 2, \cdots\}$, where left-sided derivatives exist.
(3) The equations in (2.1) are satisfied on each interval $[k, k+1)$ with $k = 0, 1, 2, \cdots$.

For $k \le t < k+1, k = 0, 1, 2, \cdots$, integrating (2.1) from k to t, we obtain,

$$\begin{aligned} x_1(t) &= x_1(k)\exp\Big\{\big[b_1(k) - a_1(k)x_1(k-\tau_1(k)) \\ &\quad - \frac{c(k)x_2(k-\sigma(k))}{\frac{x_1^2(k-\tau_2(k))}{i} + x_1(k-\tau_2(k)) + a}\big](t-k)\Big\}, \\ x_2(t) &= x_2(k)\exp\Big\{\big[-b_2(k) + \frac{a_2(k)x_1(k-\tau_3(k))}{\frac{x_1^2(k-\tau_4(k))}{i} + x_1(k-\tau_4(k)) + a}\big](t-k)\Big\}. \end{aligned} \tag{2.2}$$

Letting $t \to k+1$, we have

$$\begin{aligned} x_1(k+1) &= x_1(k)\exp\Big\{\big[b_1(k) - a_1(k)x_1(k-\tau_1(k)) \\ &\quad - \frac{c(k)x_2(k-\sigma(k))}{\frac{x_1^2(k-\tau_2(k))}{i} + x_1(k-\tau_2(k)) + a}\big]\Big\}, \\ x_2(k+1) &= x_2(k)\exp\Big\{\big[-b_2(k) + \frac{a_2(k)x_1(k-\tau_3(k))}{\frac{x_1^2(k-\tau_4(k))}{i} + x_1(k-\tau_4(k)) + a}\big]\Big\}, \end{aligned} \tag{2.3}$$

for $k = 0, 1, 2, \cdots$. (2.3) is a discrete analogue of system (1.2).

Throughout this paper, we are interested only in solutions $(x_1(k), x_2(k))^T$ of (2.3) with the initial conditions of the form

$$x_i(s) \ge 0, \quad x_i(0) > 0, \quad s = -m, -m+1, \cdots, 0, \quad i = 1, 2, \tag{2.4}$$

where $m = \max_{k\in I_\omega}\{\tau_1(k), \tau_2(k), \tau_3(k), \tau_4(k), \sigma(k)\}$, $\tau_i(k)$ and $\sigma(k)$ are integers. For given initial conditions (2.4), we may prove that (2.3) has a unique solution $(x_1(k), x_2(k))^T$ defined on $\{-m, \cdots, -1, 0, 1, 2, \cdots\}$ and satisfying

$$x_i(k) > 0, \quad i = 1, 2; k = 0, 1, 2, \cdots.$$

3. Existence of multiple positive periodic solutions

In this section, we shall apply the continuation theorem of Mawhin's coincidence degree theory to establish our main results.

Let $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}^2$ denote the sets of all integers, nonnegative integers, real numbers, nonnegative real numbers, and two-dimensional Euclidean vector space, respectively.

Throughout this paper, we will use the following notation:

$$I_\omega = \{0, 1, \cdots, \omega-1\}, \quad \overline{g} = \frac{1}{\omega}\sum_{k=0}^{\omega-1} g(k), \quad \overline{G} = \frac{1}{\omega}\sum_{k=0}^{\omega-1} |g(k)|,$$

where $\{g(k)\}$ is an ω-periodic sequence of real numbers defined for $k \in \mathbb{Z}$.

In system (2.3), we always assume that $b_i : \mathbb{Z} \to \mathbb{R}$ and $c, \sigma, a_i, \tau_j : \mathbb{Z} \to \mathbb{R}^+$ are ω-periodic, i.e.,

$$a_i(k+\omega) = a_i(k), \quad b_i(k+\omega) = b_i(k), \quad c(k+\omega) = c(k),$$
$$\sigma(k+\omega) = \sigma(k), \quad \tau_j(k+\omega) = \tau_j(k),$$

for any $k \in \mathbb{Z}$, $i = 1, 2$; $j = 1, 2, 3, 4$ and $\bar{c} > 0$, $\bar{b}_i > 0$, i and a are positive constants, where ω, a fixed positive integer, denotes the prescribed common period of the parameters in (2.3).

For the reader's convenience, we first summarize a few concepts from the book by Gaines and Mawhin [14].

Let X and Y be normed vector spaces. Let $L : \operatorname{Dom} L \subset X \to Y$ be a linear mapping and $N : X \to Y$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim\ker L = \operatorname{codim}\operatorname{Im} L < \infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Im} P = \ker L$ and $\operatorname{Im} L = \ker Q = \operatorname{Im}(I - Q)$. It follows that $L|\operatorname{Dom} L \cap \ker P : (I - P)X \to \operatorname{Im} L$ is invertible and its inverse is denoted by K_p. If Ω is a bounded open subset of X, the mapping N is called L-compact on $\overline{\Omega}$ if $(QN)(\overline{\Omega})$ is bounded and $K_p(I-Q)N : \overline{\Omega} \to X$ is compact. Because $\operatorname{Im} Q$ is isomorphic to $\ker L$, there exists an isomorphism $J : \operatorname{Im} Q \to \ker L$.

In the proof our existence result, we need the following lemmas.

Lemma 3.1 (Continuation theorem [14]). *Let L be a Fredholm mapping of index zero and N be L-compact on $\overline{\Omega}$. Suppose*

(a) *For each $\lambda \in (0, 1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;*
(b) *$QNx \neq 0$ for each $x \in \partial\Omega \cap \ker L$ and $\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0$.*

Then the operator equation $Lx = Nx$ has at least one solution lying in $\operatorname{Dom} L \cap \overline{\Omega}$.

Lemma 3.2 ([9, Lemma 3.2]). *Let $g : \mathbb{Z} \to \mathbb{R}$ be an ω-periodic, i.e., $g(k+\omega) = g(k)$. Then for any fixed $k_1, k_2 \in I_\omega$, and any $k \in \mathbb{Z}$, one has*

$$g(k) \le g(k_1) + \sum_{s=0}^{\omega-1} |g(s+1) - g(s)|,$$
$$g(k) \ge g(k_2) - \sum_{s=0}^{\omega-1} |g(s+1) - g(s)|.$$

Proof. It is only necessary to prove that the inequalities hold for any $k \in I_\omega$. For the first inequality, it is easy to see the first inequality holds if $k = k_1$. If $k > k_1$, then

$$g(k) - g(k_1) = \sum_{s=k_1}^{k-1} (g(s+1) - g(s)) \le \sum_{s=k_1}^{k-1} |g(s+1) - g(s)| \le \sum_{s=0}^{\omega-1} |g(s+1) - g(s)|,$$

and hence, $g(k) \le g(k_1) + \sum_{s=0}^{\omega-1} |g(s+1) - g(s)|$. If $k < k_1$, then

$$g(k_1) - g(k) = \sum_{s=k}^{k_1-1} (g(s+1) - g(s)) \ge -\sum_{s=k}^{k_1-1} |g(s+1) - g(s)| \ge -\sum_{s=0}^{\omega-1} |g(s+1) - g(s)|,$$

equivalently, $g(k) \le g(k_1) + \sum_{s=0}^{\omega-1} |g(s+1) - g(s)|$. Now we can claim that the first inequality is valid.

The proof of the second inequality is exactly the same as that carried out above and the details are omitted here. The proof is complete. □

Define

$$l_2 = \{y = \{y(k)\} : y(k) \in \mathbb{R}^2, k \in \mathbb{Z}\}.$$

For $a = (a_1, a_2)^T \in \mathbb{R}^2$, define $|a| = \max\{|a_1|, |a_2|\}$. Let $l^\omega \subset l_2$ denote the subspace of all ω-periodic sequences equipped with the usual supremum norm $\|\cdot\|$, i.e., for $y = \{y(k) : k \in \mathbb{Z}\} \in l^\omega$, $\|y\| = \max_{k\in I_\omega} |y(k)|$. It is difficult to show that l^ω is a finite-dimensional Banach space.

Let the linear operator $S : l^\omega \to \mathbb{R}^2$ be defined by

$$S(y) = \frac{1}{\omega}\sum_{k=0}^{\omega-1} y(k), \quad y = \{y(k) : k \in \mathbb{Z}\} \in l^\omega.$$

Then we obtain two subspaces l_0^ω and l_c^ω of l^ω defined by

$$l_0^\omega = \{y = \{y(k)\} \in l^\omega : S(y) = 0\}$$
$$l_c^\omega = \{y = \{y(k)\} \in l^\omega : y(k) \equiv \beta, \text{ for some } \beta \in \mathbb{R}^2 \text{ and for all } k \in \mathbb{Z}\},$$

respectively. Denote by $L : l^\omega \to l^\omega$ the difference operator given by $Ly = \{(Ly)(k)\}$ with

$$(Ly)(k) = y(k+1) - y(k), \quad \text{for } y \in l^\omega \text{ and } k \in \mathbb{Z}.$$

Let a linear operator $K : l^\omega \to l_c^\omega$ be defined by $Ky = \{(Ky)(k)\}$ with

$$(Ky)(k) \equiv S(y), \quad \text{for } y \in l^\omega \text{ and } k \in \mathbb{Z}.$$

Then we have the following lemma. [28].

Lemma 3.3 ([28])**.** (i) *Both l_0^ω and l_c^ω are closed linear subspaces of l^ω and $l^\omega = l_0^\omega \oplus l_c^\omega$, $\dim l_c^\omega = 2$.*
(ii) *L is a bounded linear operator with $\ker L = l_c^\omega$ and $\operatorname{Im} L = l_0^\omega$.*
(iii) *K is a bounded linear operator with $\ker(L+K) = \{0\}$ and $\operatorname{Im}(L+K) = l^\omega$.*

For convenience, we denote $f : y \to \frac{\exp(2y)}{i} + \exp(y) + a$. From now on, we assume that

(H1) $\overline{a}_2 > \overline{b}_2\big(1 + 2\sqrt{\frac{a}{i}}\big)\exp\big[(\overline{B}_1 + \overline{b}_1)\omega\big]$.

For further convenience, we introduce the following six positive numbers:

$$l_\pm = \frac{i\{\overline{a}_2\exp[(\overline{B}_1+\overline{b}_1)\omega] - \overline{b}_2\} \pm \sqrt{i^2\{\overline{a}_2\exp[(\overline{B}_1+\overline{b}_1)\omega] - \overline{b}_2\}^2 - 4ia\overline{b}_2^2}}{2\overline{b}_2},$$

$$\begin{aligned} u_\pm = \Big(&i\{\overline{a}_2 - \overline{b}_2\exp[(\overline{B}_1+\overline{b}_1)\omega]\} \\ &\pm \sqrt{i^2\{\overline{a}_2 - \overline{b}_2\exp[(\overline{B}_1+\overline{b}_1)\omega]\}^2 - 4ia\overline{b}_2^2\exp[2(\overline{B}_1+\overline{b}_1)\omega]}\Big) \\ &\div \big(2\overline{b}_2\exp[(\overline{B}_1+\overline{b}_1)\omega]\big), \end{aligned}$$

$$y_\pm = \frac{i(\overline{a}_2 - \overline{b}_2) \pm \sqrt{i^2(\overline{a}_2-\overline{b}_2)^2 - 4ia\overline{b}_2^2}}{2\overline{b}_2}.$$

It is not difficult to prove that

$$l_- < y_- < u_- < u_+ < y_+ < l_+. \tag{3.1}$$

To state and prove the main result of this paper, we use the hypothesis

(H2) $\overline{a}_1 l_+ \exp[(\overline{B}_1 + \overline{b}_1)\omega] < \overline{b_1}$.

Theorem 3.4. *Under the hypotheses (H1)–(H2), the system (2.3) has at least two ω-periodic positive solutions.*

Proof. we make the change of variables

$$x_i(k) = \exp(y_i(k)), \quad i = 1, 2. \tag{3.2}$$

Then (2.3) is rewritten as

$$\begin{aligned} y_1(k+1)-y_1(k) &= b_1(k)-a_1(k)\exp\{y_1(k-\tau_1(k))\}-\frac{c(k)\exp\{y_2(k-\sigma(k))\}}{f(y_1(k-\tau_2(k)))}, \\ y_2(k+1)-y_2(k) &= -b_2(k)+\frac{a_2(k)\exp\{y_1(k-\tau_3(k))\}}{f(y_1(k-\tau_4(k)))}, \end{aligned} \tag{3.3}$$

If (3.3) has an ω-periodic solution $\{y(k)\}$, then $\{x(k)\}$: $x_i(k)=\exp(y_i(k))$ is a positive ω-periodic solution of (2.3).

Now let we define $X=Y=l^\omega$, $(Ly)(k)=y(k+1)-y(k)$, and

$$\begin{aligned} (Ny)(k) &= \begin{pmatrix} b_1(k)-a_1(k)\exp\{y_1(k-\tau_1(k))\}-\frac{c(k)\exp\{y_2(k-\sigma(k))\}}{f(y_1(k-\tau_2(k)))} \\ -b_2(k)+\frac{a_2(k)\exp\{y_1(k-\tau_3(k))\}}{f(y_1(k-\tau_4(k)))} \end{pmatrix} \\ &:= \begin{pmatrix} \triangle_1(y,k) \\ \triangle_2(y,k) \end{pmatrix}, \end{aligned}$$

for any $y\in X$ and $k\in\mathbb{Z}$. It follows from Lemma 3.3 that L is a bounded linear operator and

$$\ker L = l^\omega_c, \quad \operatorname{Im} L = l^\omega_0, \quad dim\ker L = 2 = \operatorname{codim}\operatorname{Im} L,$$

then it follows that L is a Fredholm mapping of index zero.

Define

$$Py=\frac{1}{\omega}\sum_{s=0}^{\omega-1}y(s), \quad y\in X, \qquad Qz=\frac{1}{\omega}\sum_{s=0}^{\omega-1}z(s), \quad z\in Y.$$

It is not difficult to show that P and Q are two continuous projectors such that

$$\operatorname{Im} P=\ker L \quad\text{and}\quad \operatorname{Im} L=\ker Q=\operatorname{Im}(I-Q).$$

Furthermore, the generalized inverse (of L) K_p: $\operatorname{Im} L\to\ker P\cap\operatorname{Dom} L$ exists and is given by

$$K_p(z)=\sum_{s=0}^{k-1}z(s)-\frac{1}{\omega}\sum_{s=0}^{\omega-1}(\omega-s)z(s).$$

Thus

$$\begin{gathered} QNy=\Big(\frac{1}{\omega}\sum_{k=0}^{\omega-1}\triangle_1(y,k),\frac{1}{\omega}\sum_{k=0}^{\omega-1}\triangle_2(y,k)\Big)^T, \\ K_p(I-Q)Ny=(\Phi_1(y,k),\Phi_2(y,k))^T, \end{gathered}$$

where for $i=1,2$,

$$\Phi_i(y,k)=\sum_{s=0}^{k-1}\triangle_i(y,s)-\frac{1}{\omega}\sum_{s=0}^{\omega-1}(\omega-s)\triangle_i(y,s)-\big(\frac{k}{\omega}-\frac{\omega+1}{2\omega}\big)\sum_{s=0}^{\omega-1}\triangle_i(y,s).$$

Obviously, QN and $K_p(I-Q)N$ are continuous. Since X is a finite-dimensional Banach space, it is not difficult to show that $\overline{K_p(I-Q)N(\overline{\Omega})}$ is compact for any open bounded set $\Omega\subset X$. Moreover, $QN(\overline{\Omega})$ is bounded. Thus, N is L-compact on with any open bounded set $\Omega\subset X$. The isomorphism J of $\operatorname{Im} Q$ onto $\ker L$ can be the identity mapping, since $\operatorname{Im} Q=\ker L$.

From now on, we shall search for at least two appropriate open, bounded subsets Ω_1 and Ω_2 in X. Corresponding to the operator equation $Ly = \lambda Ny, \lambda \in (0,1)$, we have

$$\begin{aligned} y_1(k+1) - y_1(k) &= \lambda\Big[b_1(k) - a_1(k)\exp\{y_1(k-\tau_1(k))\} \\ &\quad - \frac{c(k)\exp\{y_2(k-\sigma(k))\}}{f(y_1(k-\tau_2(k)))}\Big], \\ y_2(k+1) - y_2(k) &= \lambda\Big[-b_2(k) + \frac{a_2(k)\exp\{y_1(k-\tau_3(k))\}}{f(y_1(k-\tau_4(k)))}\Big], \end{aligned} \tag{3.4}$$

Assume that $y = (y_1(k), y_2(k))^T \in X$ is an solution of (3.4) for a certain $\lambda \in (0,1)$. Summing on both sides of (3.4) from 0 to $\omega - 1$ about k, we get

$$\begin{aligned} 0 &= \sum_{k=0}^{\omega-1}(y_1(k+1) - y_1(k)) \\ &= \lambda\sum_{k=0}^{\omega-1}\big[b_1(k) - a_1(k)\exp\{y_1(k-\tau_1(k))\} - \frac{c(k)\exp\{y_2(k-\sigma(k))\}}{f(y_1(k-\tau_2(k)))}\big], \\ 0 &= \sum_{k=0}^{\omega-1}(y_2(k+1) - y_2(k)) = \lambda\sum_{k=0}^{\omega-1}\big[-b_2(k) + \frac{a_2(k)\exp\{y_1(k-\tau_3(k))\}}{f(y_1(k-\tau_4(k)))}\big]; \end{aligned}$$

that is,

$$\begin{aligned} \bar{b}_1\omega &= \sum_{k=0}^{\omega-1}\Big[a_1(k)\exp\{y_1(k-\tau_1(k))\} + \frac{c(k)\exp\{y_2(k-\sigma(k))\}}{f(y_1(k-\tau_2(k)))}\Big], \\ \bar{b}_2\omega &= \sum_{k=0}^{\omega-1}\frac{a_2(k)\exp\{y_1(k-\tau_3(k))\}}{f(y_1(k-\tau_4(k)))}. \end{aligned} \tag{3.5}$$

From the first equation of (3.4), and (3.5), we have

$$\begin{aligned} &\sum_{k=0}^{\omega-1}|y_1(k+1) - y_1(k)| \\ &< \sum_{k=0}^{\omega-1}\big[|b_1(k)| + a_1(k)\exp\{y_1(k-\tau_1(k))\} + \frac{c(k)\exp\{y_2(k-\sigma(k))\}}{f(y_1(k-\tau_2(k)))}\big] \\ &= (\overline{B}_1 + \bar{b}_1)\omega; \end{aligned}$$

that is,

$$\sum_{k=0}^{\omega-1}|y_1(k+1) - y_1(k)| < (\overline{B}_1 + \bar{b}_1)\omega. \tag{3.6}$$

Similarly, it follows from the second equation of (3.4), (3.5) that

$$\sum_{k=0}^{\omega-1}|y_2(k+1) - y_2(k)| < (\overline{B}_2 + \bar{b}_2)\omega. \tag{3.7}$$

Because of $y = \{y(k)\} \in X$, there exist $\xi_i, \eta_i \in I_\omega$ such that

$$y_i(\xi_i) = \min_{k\in I_\omega}\{y_i(k)\}, \quad y_i(\eta_i) = \max_{k\in I_\omega}\{y_i(k)\}, \quad i = 1,2. \tag{3.8}$$

It follows from the second equation of (3.5) and (3.8) that

$$\bar{b}_2\omega \leq \frac{\bar{a}_2\omega\exp\{y_1(\eta_1)\}}{f(y_1(\xi_1))}.$$

So

$$y_1(\eta_1) \geq \ln\big[\frac{\bar{b}_2}{\bar{a}_2}f(y_1(\xi_1))\big]. \tag{3.9}$$

Therefore, Lemma 3.2 and (3.6), (3.9) imply

$$y_1(k) \geq y_1(\eta_1) - \sum_{k=0}^{\omega-1}|y_1(k+1)-y_1(k)| > \ln\big[\frac{\bar{b}_2}{\bar{a}_2}f(y_1(\xi_1))\big] - (\overline{B}_1+\bar{b}_1)\omega. \tag{3.10}$$

In particular, we have $y_1(\xi_1) > \ln\big[\frac{\bar{b}_2}{\bar{a}_2}f(y_1(\xi_1))\big] - (\overline{B}_1+\bar{b}_1)\omega$, or

$$\frac{\bar{b}_2}{i}\exp(2y_1(\xi_1)) - \big[\bar{a}_2\exp(\overline{B}_1+\bar{b}_1)\omega - \bar{b}_2\big]\exp\{y_1(\xi_1)\} + \bar{b}_2 a < 0.$$

Because of (H1), we have

$$\ln l_- < y_1(\xi_1) < \ln l_+. \tag{3.11}$$

Similarly, we have

$$y_1(\eta_1) < \ln u_- \quad \text{or} \quad y_1(\eta_1) > \ln u_+. \tag{3.12}$$

It follows from (3.11), (3.6) and Lemma 3.2 that

$$y_1(k) \leq y_1(\xi_1) + \sum_{s=0}^{\omega-1}|y_1(s+1)-y_1(s)| < \ln l_+ + (\overline{B}_1+\bar{b}_1)\omega := H_{12}. \tag{3.13}$$

On the other hand, it follows from (3.5) and (3.13) that

$$\bar{b}_1\omega \geq \frac{\bar{c}\omega\exp\{y_2(\xi_2)\}}{f(\ln l_+ + (\overline{B}_1+\bar{b}_1)\omega)} \tag{3.14}$$

$$\bar{b}_1\omega \leq \bar{a}_1\omega\exp\big[\ln l_+ + (\overline{B}_1+\bar{b}_1)\omega\big] + \frac{\bar{c}\omega\exp\{y_2(\eta_2)\}}{a}. \tag{3.15}$$

It follows from (3.14) that $y_2(\xi_2) \leq \ln\big\{\frac{\bar{b}_1}{\bar{c}}f(\ln l_+ + (\overline{B}_1+\bar{b}_1)\omega)\big\}$. This, combined with (3.7), gives

$$\begin{aligned} y_2(k) &\leq y_2(\xi_2) + \sum_{s=0}^{\omega-1}|y_2(s+1)-y_2(s)| \\ &< \ln\Big\{\frac{\bar{b}_1}{\bar{c}}f(\ln l_+ + (\overline{B}_1+\bar{b}_1)\omega)\Big\} + (\overline{B}_2+\bar{b}_2)\omega := H_{22}. \end{aligned} \tag{3.16}$$

Moreover, because of (H2), it follows from (3.15) that

$$y_2(\eta_2) \geq \ln\frac{a\{\bar{b}_1 - \bar{a}_1 l_+\exp[(\overline{B}_1+\bar{b}_1)\omega]\}}{\bar{c}}.$$

This, combined with (3.7) again, gives

$$\begin{aligned} y_2(k) &\geq y_2(\eta_2) - \sum_{s=0}^{\omega-1}|y_2(s+1)-y_2(s)| \\ &> \ln\frac{a\{\bar{b}_1 - \bar{a}_1 l_+\exp[(\overline{B}_1+\bar{b}_1)\omega]\}}{\bar{c}} - (\overline{B}_2+\bar{b}_2)\omega := H_{21}. \end{aligned} \tag{3.17}$$

It follows from (3.16) and (3.17) that

$$\max_{k\in I_\omega}|y_2(k)| < \max\{|H_{21}|, |H_{22}|\} := H_2. \tag{3.18}$$

Obviously, $\ln l_\pm$, $\ln u_\pm$, H_{12} and H_2 are independent of λ.

Now, let's consider QNy with $y = (y_1, y_2)^T \in \mathbb{R}^2$. Note that

$$QN(y_1, y_2)^T = \Big(\bar{b}_1 - \bar{a}_1 \exp(y_1) - \frac{\bar{c}\exp(y_2)}{f(y_1)}, -\bar{b}_2 + \frac{\bar{a}_2\exp(y_1)}{f(y_1)}\Big)^T.$$

Because of (H1) and (H2), we can show that $QN(y_1, y_2)^T = 0$ has two distinct solutions $\widetilde{y} = (\ln y_-, \ln\frac{(\bar{b}_1-\bar{a}_1 y_-)f(\ln y_-)}{\bar{c}})$ and $\widehat{y} = (\ln y_+, \ln\frac{(\bar{b}_1-\bar{a}_1 y_+)f(\ln y_+)}{\bar{c}})$. Choose $C > 0$ such that

$$C > \max\Big\{\Big|\ln\frac{(\bar{b}_1-\bar{a}_1 y_-)f(\ln y_-)}{\bar{c}}\Big|, \Big|\ln\frac{(\bar{b}_1-\bar{a}_1 y_+)f(\ln y_+)}{\bar{c}}\Big|\Big\}. \tag{3.19}$$

Let

$$\Omega_1 = \Big\{y = (y_1(k), y_2(k)) \in X : y_1(k) \in (\ln l_-, \ln u_-),\ \max_{k\in I_\omega}|y_2(k)| < H_2 + C\Big\},$$

$$\begin{aligned}\Omega_2 = \Big\{y = (y_1(k), y_2(k)) \in X : &\min_{k\in I_\omega} y_1(k) \in (\ln l_-, \ln l_+),\\ &\max_{k\in I_\omega} y_1(k) \in (\ln u_+, H_{12}),\ \max_{k\in I_\omega}|y_2(k)| < H_2 + C\Big\}.\end{aligned}$$

Then both Ω_1 and Ω_2 are bounded open subsets of X. It follows from (3.1) and (3.19) that $\widetilde{y} \in \Omega_1$ and $\widehat{y} \in \Omega_2$. With the help of (3.1), (3.11)-(3.13) and (3.18)-(3.19), it is easy to that $\Omega_1 \cap \Omega_2 = \phi$ and Ω_i satisfies the requirement (a) Lemma 3.1 for $i = 1, 2$. Moreover, $QNy \neq 0$ for $y \in \partial\Omega \cap \mathbb{R}^2$. A direct computation gives

$$\deg\{JQN, \Omega_i \cap \ker L, 0\} = (-1)^{i+1} \neq 0.$$

Here, J is taken as the identity mapping since $\operatorname{Im} Q = \ker L$. So far we have proved that Ω_i satisfies all the assumptions in Lemma 3.1. Hence, (3.3) has at least two ω-periodic solutions $\{y^*(k)\}$ and $\{y^\dagger(k)\}$ with $y^*(k) \in \operatorname{Dom} L \cap \overline{\Omega}_1$ and $y^\dagger(k) \in \operatorname{Dom} L \cap \overline{\Omega}_2$. Obviously, y^* and $y^\dagger$ are different. Let $x_i^*(k) = \exp(y_i^*(k))$ and $x_i^\dagger(k) = \exp(y_i^\dagger(k)), i = 1, 2$. Then by (3.2), $x^*(k) = (x_1^*(k), x_2^*(k))^T$ and $x^\dagger(k) = (x_1^\dagger(k), x_2^\dagger(k))^T$ are two different positive ω-periodic solutions of (2.3). This completes the proof. □

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Zhigang Liu
Department of Mathematics, Xiangnan University, Chenzhou, Hunan 423000, China
E-mail address: lzglzglzg4359@sina.com

Anping Chen
Department of Mathematics, Xiangnan University, Chenzhou, Hunan 423000, China
E-mail address: anping_chen@hotmail.com

Jinde Cao
Department of Mathematics, Southeast University, Nanjing 210096, China
E-mail address: jdcao@seu.edu.cn

Fengde Chen
Department of Mathematics, Fuzhou University, Fuzhou, Fujian 350002, China
E-mail address: fdchen@fzu.edu.cn