Electronic Journal of Differential Equations, Vol. 2004(2004), No. 06, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

TRIPLE POSITIVE SOLUTIONS FOR A CLASS OF TWO-POINT BOUNDARY-VALUE PROBLEMS

ZHANBING BAI, YIFU WANG, & WEIGAO GE

Abstract. We obtain sufficient conditions for the existence of at least three positive solutions for the equation $x''(t) + q(t)f(t, x(t), x'(t)) = 0$ subject to some boundary conditions. This is an application of a new fixed-point theorem introduced by Avery and Peterson [6].

1. Introduction

Recently, the existence and multiplicity of positive solutions for nonlinear ordinary differential equations and difference equations have been studied extensively. To identify a few, we refer the reader to [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. The main tools used in above works are fixed-point theorems. Fixed-point theorems and their applications to nonlinear problems have a long history, some of which is documented in Zeidler's book [14], and the recent book by Agarwal, O'Regan and Wong [1] contains an excellent summary of the current results and applications.

An interest in triple solutions evolved from the Leggett-Williams multiple fixed-point theorem [10]. And lately, two triple fixed-point theorems due to Avery [2] and Avery and Peterson [6] have been applied to obtain triple solutions of certain boundary-value problems for ordinary differential equations as well as for their discrete analogues.

Avery and Peterson [6], generalize the fixed-point theorem of Leggett-Williams by using theory of fixed-point index and Dugundji extension theorem. An application of the theorem be given to prove the existence of three positive solutions to the following second-order discrete boundary-value problem

$$\Delta^2 x(k-1) + f(x(k)) = 0, \quad \text{for all } k \in [a+1, b+1],$$
$$x(a) = x(b+2) = 0,$$

where $f : \mathbb{R} \to \mathbb{R}$ is continuous and nonnegative for $x \geq 0$.

2000 *Mathematics Subject Classification.* 34B15.
Key words and phrases. Triple positive solutions, boundary-value problem, fixed-point theorem.

Submitted November 25, 2003. Published January 2, 2004.
Supported by grants 10371006 from the National Nature Science Foundation of China, and 1999000722 from the Doctoral Program Foundation of Education Ministry of China.

In this paper, we concentrate in getting three positive solutions for the second-order differential equation

$$x''(t) + q(t)f(t, x(t), x'(t)) = 0, \quad 0 < t < 1 \tag{1.1}$$

subject to one of the following two pairs of boundary conditions:

$$x(0) = 0 = x(1), \tag{1.2}$$

$$x(0) = 0 = x'(1). \tag{1.3}$$

We are concerned with positive solutions to the above problem, i.e., $x(t) \geq 0$ on $[0, 1]$. In this article, it is assumed that:

- (C1) $f \in C([0, 1] \times [0, \infty) \times \mathbb{R}, [0, \infty))$;
- (C2) $q(t)$ is nonnegative measurable function defined in $(0, 1)$, and $q(t)$ does not identically vanish on any subinterval of $(0, 1)$. Furthermore, $q(t)$ satisfies $0 < \int_0^1 t(1-t)q(t)dt < \infty$.

Our main results will depend on an application of a fixed-point theorem due to Avery and Peterson which deals with fixed points of a cone-preserving operator defined on an ordered Banach space. The emphasis here is the nonlinear term be involved explicitly with the first-order derivative. To the best of the authors knowledge, there are no results for triple positive solutions by using the Leggett-Williams fixed-point theorem or its generalizations.

2. Background materials and definitions

For the convenience of the reader, we present here the necessary definitions from cone theory in Banach spaces; these definitions can be found in recent literature.

Definition 2.1. Let E be a real Banach space over $\mathbb{R}$. A nonempty convex closed set $P \subset E$ is said to be a cone provided that

- (i) $au \in P$ for all $u \in P$ and all $a \geq 0$ and
- (ii) $u, -u \in P$ implies $u = 0$.

Note that every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if $y - x \in P$.

Definition 2.2. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Definition 2.3. The map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\alpha : P \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$. Similarly, we say the map β is a nonnegative continuous convex functional on a cone P of a real Banach space E provided that $\beta : P \to [0, \infty)$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P. Then for positive real numbers a, b, c, and d, we define the following convex sets:

$$P(\gamma, d) = \{x \in P \mid \gamma(x) < d\},$$
$$P(\gamma, \alpha, b, d) = \{x \in P \mid b \leq \alpha(x), \gamma(x) \leq d\},$$
$$P(\gamma, \theta, \alpha, b, c, d) = \{x \in P \mid b \leq \alpha(x), \theta(x) \leq c, \gamma(x) \leq d\},$$

and a closed set

$$R(\gamma, \psi, a, d) = \{x \in P \mid a \leq \psi(x), \gamma(x) \leq d\}.$$

The following fixed-point theorem due to Avery and Peterson is fundamental in the proofs of our main results.

Theorem 2.4 ([6]). *Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x) \leq \lambda\psi(x)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers M and d,*

$$\alpha(x) \leq \psi(x) \quad \text{and} \quad \|x\| \leq M\gamma(x), \tag{2.1}$$

for all $x \in \overline{P(\gamma, d)}$. Suppose $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is completely continuous and there exist positive numbers a, b, and c with $a < b$ such that

(S1) $\{x \in P(\gamma, \theta, \alpha, b, c, d) \mid \alpha(x) > b\} \neq$ *and* $\alpha(Tx) > b$ *for* $x \in P(\gamma, \theta, \alpha, b, c, d)$;
(S2) $\alpha(Tx) > b$ *for* $x \in P(\gamma, \alpha, b, d)$ *with* $\theta(Tx) > c$;
(S3) $0 \notin R(\gamma, \psi, a, d)$ *and* $\psi(Tx) < a$ *for* $x \in R(\gamma, \psi, a, d)$ *with* $\psi(x) = a$.

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$, such that

$$\gamma(x_i) \leq d \quad \text{for } i = 1, 2, 3;$$
$$b < \alpha(x_1);$$
$$a < \psi(x_2) \quad \text{with } \alpha(x_2) < b;$$
$$\psi(x_3) < a\,.$$

3. Existence of triple positive solutions

In this section, we impose growth conditions on f which allow us to apply Theorem 2.4 to establish the existence of triple positive solutions of Problem (1.1)-(1.2), and (1.1)-(1.3).

We first deal with the boundary-value problem (1.1)-(1.2). Let $X = C^1[0, 1]$ be endowed with the ordering $x \leq y$ if $x(t) \leq y(t)$ for all $t \in [0, 1]$, and the maximum norm,

$$\|x\| = \max\Big\{\max_{0\leq t\leq 1} |x(t)|, \max_{0\leq t\leq 1} |x'(t)|\Big\}.$$

¿From the fact $x''(t) = -f(t, x, x') \leq 0$, we know that x is concave on $[0, 1]$. So, define the cone $P \subset X$ by

$$P = \{x \in X : x(t) \geq 0, x(0) = x(1) = 0, x \text{ is concave on } [0, 1]\} \subset X.$$

Let the nonnegative continuous concave functional α, the nonnegative continuous convex functional θ, γ, and the nonnegative continuous functional ψ be defined on the cone P by

$$\gamma(x) = \max_{0\leq t\leq 1} |x'(t)|, \quad \psi(x) = \theta(x) = \max_{0\leq t\leq 1} |x(t)|, \quad \alpha(x) = \min_{\frac{1}{4}\leq t\leq \frac{3}{4}} |x(t)|.$$

Lemma 3.1. *If $x \in P$, then* $\max_{0\le t\le 1}|x(t)| \le \frac{1}{2}\max_{0\le t\le 1}|x'(t)|$.

Proof. To the contrary, suppose that there exist $t_0 \in (0,1)$ such that $|x(t_0)| > \frac{1}{2}\max_{0\le t\le 1}|x'(t)| =: A$. Then by the mid-value theorem there exist $t_1 \in (0,t_0)$, $t_2 \in (t_0,1)$ such that

$$x'(t_1) = \frac{x(t_0)-x(0)}{t_0} = \frac{x(t_0)}{t_0}, \quad x'(t_2) = \frac{x(1)-x(t_0)}{1-t_0} = \frac{-x(t_0)}{1-t_0}.$$

Thus, $\max_{0\le t\le 1}|x'(t)| \ge \max\{|x'(t_1)|,|x'(t_2)|\} > 2A$, it is a contradiction. The proof is complete. □

By Lemma 3.1 and their definitions, and the concavity of x, the functionals defined above satisfy:

$$\frac{1}{4}\theta(x) \le \alpha(x) \le \theta(x) = \psi(x), \quad \|x\| = \max\{\theta(x),\gamma(x)\} = \gamma(x), \tag{3.1}$$

for all $x \in \overline{P(\gamma,d)} \subset P$. Therefore, Condition (2.1) is satisfied.

Denote by $G(t,s)$ the Green's function for boundary-value problem

$$\begin{gathered} -x''(t) = 0, \quad 0<t<1,\\ x(0)=x(1)=0. \end{gathered}$$

then $G(t,s) \ge 0$ for $0 \le t,s \le 1$ and

$$G(t,s) = \begin{cases} t(1-s) & \text{if } 0 \le t \le s \le 1,\\ s(1-t) & \text{if } 0 \le s \le t \le 1. \end{cases}$$

Let

$$\begin{gathered} \delta = \min\Big\{\int_{1/4}^{3/4} G(1/4,s)q(s)ds, \int_{1/4}^{3/4} G(3/4,s)q(s)ds\Big\},\\ M = \max\Big\{\int_0^1 (1-s)q(s)ds, \int_0^1 sq(s)ds\Big\},\\ N = \max_{0\le t\le 1}\int_0^1 G(t,s)q(s)ds. \end{gathered}$$

To present our main result, we assume there exist constants $0 < a < b \le d/8$ such that

- (A1) $f(t,u,v) \le d/M$, for $(t,u,v) \in [0,1]\times[0,d/]\times[-d,d]$
- (A2) $f(t,u,v) > \frac{b}{\delta}$, for $(t,u,v) \in [1/4,3/4]\times[b,4b]\times[-d,d]$;
- (A3) $f(t,u,v) < \frac{a}{N}$, for $(t,u,v) \in [0,1]\times[0,a]\times[-d,d]$.

Theorem 3.2. *Under assumptions (A1)–(A3), the boundary-value problem* (1.1)-(1.2) *has at least three positive solutions x_1, x_2, and x_3 satisfying*

$$\begin{gathered} \max_{0\le t\le 1}|x_i'(t)| \le d, \quad \textit{for } i=1,2,3;\\ b < \min_{\frac14\le t\le\frac34}|x_1(t)|;\\ a < \max_{0\le t\le 1}|x_2(t)|, \quad \textit{with} \min_{\frac14\le t\le\frac34}|x_2(t)| < b;\\ \max_{0\le t\le 1}|x_3(t)| < a. \end{gathered} \tag{3.2}$$

Proof. Problem (1.1)-(1.2) has a solution $x = x(t)$ if and only if x solves the operator equation

$$x(t) = Tx(t) := \int_0^1 G(t,s)q(s)f(s,x(s),x'(s))ds.$$

It is well know that this operator, $T : P \to P$, is completely continuous. We now show that all the conditions of Theorem 2.4 are satisfied.

If $x \in \overline{P(\gamma,d)}$, then $\gamma(x) = \max_{0\le t\le 1}|x'(t)| \le d$. With Lemma 3.1 and $\max_{0\le t\le 1}|x(t)| \le \frac{d}{2}$, then assumption (A1) implies $f(t,x(t),x'(t)) \le \frac{d}{M}$. On the other hand, for $x \in P$, there is $Tx \in P$, then Tx is concave on $[0,1]$, and $\max_{t\in[0,1]}|(Tx)'(t)| = \max\{|(Tx)'(0)|, |(Tx)'(1)|\}$, so

$$\begin{aligned}
\gamma(Tx) &= \max_{t\in[0,1]}|(Tx)'(t)| \\
&= \max_{t\in[0,1]}\Big| -\int_0^t sq(s)f(s,x(s),x'(s))ds + \int_t^1 (1-s)q(s)f(s,x(s),x'(s))ds\Big| \\
&= \max\Big\{\int_0^1 (1-s)q(s)f(s,x(s),x'(s))ds, \int_0^1 sq(s)f(s,x(s),x'(s))ds\Big\} \\
&\le \frac{d}{M}\cdot\max\Big\{\int_0^1 (1-s)q(s)ds, \int_0^1 sq(s)ds\Big\} \\
&= \frac{d}{M}\cdot M = d.
\end{aligned}$$

Hence, $T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$.

To check condition (S1) of Theorem 2.4, we choose $x(t) = 4b$, $0 \le t \le 1$. It is easy to see that $x(t) = 4b \in P(\gamma,\theta,\alpha,b,4b,d)$ and $\alpha(x) = \alpha(4b) > b$, and so $\{x \in P(\gamma,\theta,\alpha,b,4b,d) \mid \alpha(x) > b\} \ne \emptyset$. Hence, if $x \in P(\gamma,\theta,\alpha,b,4b,d)$, then $b \le x(t) \le 4b, |x'(t)| \le d$ for $1/4 \le t \le 3/4$. From assumption (A2), we have $f(t,x(t),x'(t)) \ge \frac{b}{\delta}$ for $1/4 \le t \le 3/4$, and by the conditions of α and the cone P, we have to distinguish two cases, (i) $\alpha(Tx) = (Tx)(1/4)$ and (ii) $\alpha(Tx) = (Tx)(3/4)$.

In case (i), we have

$$\alpha(Tx) = (Tx)(\frac{1}{4}) = \int_0^1 G(\frac{1}{4},s)q(s)f(s,x(s),x'(s))ds > \frac{b}{\delta}\cdot\int_{1/4}^{3/4} G(\frac{1}{4},s)q(s)ds \ge b\,.$$

In case (ii), we have

$$\alpha(Tx) = (Tx)(\frac{3}{4}) = \int_0^1 G(\frac{3}{4},s)f(s,x(s),x'(s))q(s)ds > \frac{b}{\delta}\cdot\int_{1/4}^{3/4} G(\frac{3}{4},s)q(s)ds \ge b;$$

i.e.,

$$\alpha(Tx) > b, \text{ for all } x \in P(\gamma,\theta,\alpha,b,4b,d).$$

This show that condition (S1) of Theorem 2.4 is satisfied.

Secondly, with (3.1) and $b \le \frac{d}{8}$, we have

$$\alpha(Tx) \ge \frac{1}{4}\theta(Tx) > \frac{4b}{4} = b,$$

for all $x \in P(\gamma,\alpha,b,d)$ with $\theta(Tx) > 4b$. Thus, condition (S2) of Theorem 2.4 is satisfied.

We finally show that (S3) of Theorem 2.4 also holds. Clearly, as $\psi(0) = 0 < a$, there holds that $0 \notin R(\gamma, \psi, a, d)$. Suppose that $x \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$. Then, by the assumption (A3),

$$\begin{aligned}\psi(Tx) &= \max_{0\le t\le 1} |(Tx)(t)| \\ &= \max_{0\le t\le 1} \int_0^1 G(t,s)q(s)f(s,x(s),x'(s))ds \\ &< \frac{a}{N}\cdot \max_{0\le t\le 1}\int_0^1 G(t,s)q(s)ds = a.\end{aligned}$$

So, Condition (S3) of Theorem 2.4 is satisfied. Therefore, an application of Theorem 2.4 imply the boundary-value problem (1.1)-(1.2) has at least three positive solutions x_1, x_2, and x_3 satisfying (3.2). The proof is complete. □

Remark 3.3. *To apply Theorem 2.4, we only need* $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$*, therefore, condition (C1) can be substituted with a weaker condition*

(C1)' $f \in C([0, 1] \times [0, d/2] \times [-d, d], [0,\infty))$

Now we deal with Problem (1.1)-(1.3). The method is just similar to what we have done above. Moreover, the solutions of Problem (1.1)-(1.3) are monotone increasing, which leads to the situation more simple. Define the cone $P_1 \subset X$ by

$$P_1 = \{x \in X \mid x(t) \ge 0, x(0) = x'(1) = 0, x \text{ is concave and increasing on } [0, 1]\}.$$

Let the nonnegative continuous concave functional α_1, the nonnegative continuous convex functional θ_1, γ_1, and the nonnegative continuous functional ψ_1 be defined on the cone P_1 by

$$\gamma_1(x) = \max_{t\in[0,1]} |x'(t)| = x'(0), \quad \psi_1(x) = \theta_1(x) = \max_{t\in[0,1]} |x(t)| = x(1),$$
$$\alpha_1(x) = \min_{t\in[\frac12,1]} |x(t)| = x(\frac{1}{2}), \quad \text{for } x \in P_1 .$$

Lemma 3.4. *If* $x \in P_1$*, then* $x(1) \le x'(0)$.

With Lemma 3.4, their definition, and the concavity of x, the functionals defined above satisfy

$$\frac{1}{2}\theta_1(x) \le \alpha_1(x) \le \theta_1(x) = \psi_1(x), \quad \|x\| = \max\{\theta_1(x), \gamma_1(x)\} \le \gamma_1(x), \tag{3.3}$$

for all $x \in \overline{P_1(\gamma, d)} \subset P_1$.

Denote by $G_1(t, s)$ is Green's function for boundary-value problem

$$-x''(t) = 0, \quad 0 < t < 1,$$
$$x(0) = x'(1) = 0.$$

Then $G_1(t, s) \ge 0$ for $0 \le t, s \le 1$ and

$$G_1(t,s) = \begin{cases} t & \text{if } 0 \le t \le s \le 1, \\ s & \text{if } 0 \le s \le t \le 1. \end{cases}$$

Let

$$\delta_1 = \int_{\frac12}^1 G(1/2, s)q(s)ds = \frac{1}{2}\int_{\frac12}^1 q(s)ds,$$

$$M_1 = \int_0^1 (1-s)q(s)ds,$$
$$N_1 = \int_0^1 sq(s)ds.$$

Suppose there exist constants $0 < a < b \le d/2$ such that

(A4) $f(t,u,v) \le d/M_1$, for $(t,u,v) \in [0,1] \times [0,d] \times [-d,d]$
(A5) $f(t,u,v) > b/\delta_1$, for $(t,u,v) \in [1/2,1] \times [b,2b] \times [-d,d]$
(A6) $f(t,u,v) < \frac{a}{N_1}$, for $(t,u,v) \in [0,1] \times [0,a] \times [-d,d]$.

Theorem 3.5. *Under assumption (A4)–(A6), the boundary-value problem* (1.1)-(1.3) *has at least three positive solutions x_1, x_2, and x_3 satisfying*

$$\max_{0\le t\le 1} |x_i'(t)| \le d, \quad \textit{for } i = 1,2,3;$$
$$b < \min_{\frac{1}{2}\le t\le 1} |x_1(t)|;$$
$$a < \max_{0\le t\le 1} |x_2(t)|, \quad \textit{with } \min_{\frac{1}{2}\le t\le 1} |x_2(t)| < b;$$
$$\max_{0\le t\le 1} |x_3(t)| < a\,.$$

Example. Consider the boundary-value problem

$$\begin{gathered} x''(t) + f(t,x(t),x'(t)) = 0, \quad 0 < t < 1, \\ x(0) = x(1) = 0, \end{gathered} \tag{3.4}$$

where

$$f(t,u,v) = \begin{cases} e^t + \frac{9}{2}u^3 + (\frac{v}{3000})^3 & \text{for } u \le 8, \\ e^t + \frac{9}{2}(9-u)u^3 + (\frac{v}{3000})^3 & \text{for } 8 \le u \le 9, \\ e^t + \frac{9}{2}(u-9)u^3 + (\frac{v}{3000})^3 & \text{for } 9 \le u \le 10, \\ e^t + 4500 + (\frac{v}{3000})^3 & \text{for } u \ge 10. \end{cases}$$

Choose $a = 1, b = 2, d = 3000$, we note $\delta = 1/16, M = 1/2, N = 1/8$. Consequently, $f(t,u,v)$ satisfy

$$f(t,u,v) < \frac{a}{N} = 8, \quad \text{for } 0 \le t \le 1, 0 \le u \le 1, -3000 \le v \le 3000;$$
$$f(t,u,v) > \frac{b}{\delta} = 32, \quad \text{for } 1/4 \le t \le 3/4, 2 \le u \le 8, -3000 \le v \le 3000;$$
$$f(t,u,v) < \frac{d}{M} = 6000, \quad \text{for } 0 \le t \le 1, 0 \le u \le 1500, -3000 \le v \le 3000.$$

Then all assumptions of Theorem 3.2 hold. Thus, with Theorem 3.2, Problem (3.4) has at least three positive solutions x_1, x_2, x_3 such that

$$\max_{0\le t\le 1} |x_i'(t)| \le 3000, \quad \text{for } i = 1,2,3;$$
$$2 < \min_{\frac{1}{4}\le t\le \frac{3}{4}} |x_1(t)|;$$
$$1 < \max_{0\le t\le 1} |x_2(t)|, \quad \text{with } \min_{\frac{1}{4}\le t\le \frac{3}{4}} |x_2(t)| < 2;$$
$$\max_{0\le t\le 1} |x_3(t)| < 1\,.$$

Remark 3.6. The early results, see [1, 2, 3, 5, 6, 10], for example, are not applicable to the above problem. In conclusion, we see that the nonlinear term is involved in first derivative explicitly.

References

[1] R. P. Agarwal, D. O'Regan, and P.J.Y. Wong; *Positive Solutions of Differential, Difference, and Integral Equations*, Kluwer Academic Publishers, Boston, 1999.
[2] R. I. Avery, A generalization of the Leggett-Williams fixed point theorem *Math. Sci. Res. Hot-line* **2** (1998) 9–14.
[3] R. I. Avery and D.R. Anderson; Fixed point theorem of cone expansion and compression of functional type *J. Diff. Equa. Appl.* **8** (2002) 1073–1083.
[4] R. I. Avery and J. Henderson; Three sysmmetric positive solutions for a second-order boundary-value problem *Appl. Math. Lett.* **13** (2000) 1–7.
[5] R. I. Avery and J. Henderson; Two positive fixed points of nonlinear operators on ordered Banach spaces (preprint).
[6] R. I. Avery and A. C. Peterson; Three positive fixed points of nonlinear operators on ordered Banach spaces *Comput. Math. Appl.* **42** (2001) 313–322.
[7] Z. B. Bai and H. Y. Wang; On positive solutions of some nonlinear fourth-order beam equations *J. Math. Anal. Appl.* **270** (2002) 357–368.
[8] D. Guo and V. Lakshmikantham; *Nonlinear Problems in Abstract Cones*, Academic Press, New York, 1988.
[9] J. Henderson and H. Y. Wang; Positive solutions for nonlinear eigenvalue problems *J. Math. Anal. Appl.* **208** (1997) 252–259.
[10] R. W. Leggett and L. R. Williams; Multiple positive fixed points of nonlinear operators on ordered Banach spaces *Indiana Univ. Math. J.* **28** (1979) 673–688.
[11] M. A. Krasnosel'skii; *Positive Solutions of Operator Equations*, Noordhoff, Gronignen, 1964.
[12] W. C. Lian, F. H. Wong, and C. C. Yeh; On the existence of positive solutions of nonlinear second order differential equations *Proc. Amer. Math. Soc.* **124** (1996) 1117–1126.
[13] Z. L. Liu and F. Y. Li; Multiple positive solutions of nonlinear two point boundary-value problem *J. Math. Anal. Appl.* **203** (1996) 610–625.
[14] E. Zeidler, *Nonlinear Analysis and Its Applications I: Fixed-Point Theorems*, Springer-Verlag, New York, (1993).

Zhanbing Bai
Department of Applied Mathematics, Beijing Institute of Technology, Beijing 100081, China, Department of Applied Mathematics, University of Petroleum, Dongying 257061, China
E-mail address: `baizhanbing@263.net`

Yifu Wang
Department of Applied Mathematics, Beijing Institute of Technology, Beijing 100081, China
E-mail address: `yifu-wang@163.com`

Weigao Ge
Department of Applied Mathematics, Beijing Institute of Technology, Beijing 100081, China
E-mail address: `gew@bit.edu.cn`