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STRONGLY INDEFINITE FUNCTIONALS WITH PERTURBED SYMMETRIES AND MULTIPLE SOLUTIONS OF NONSYMMETRIC ELLIPTIC SYSTEMS

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Abstract. We prove a critical-point result which provides conditions for the existence of infinitely many critical points of a strongly indefinite functional with perturbed symmetries. Then we apply this result to obtain infinitely many solutions of non-symmetric super-quadratic noncooperative elliptic systems, allowing some supercritical growth.

1. Introduction

Consider the noncooperative elliptic system

$$\begin{gathered} -\Delta u=|u|^{p-2}u+f_u(x,u,v) \quad \text{in } \Omega \\ \Delta v=|v|^{q-2}v+f_v(x,u,v) \quad \text{in } \Omega \\ u=0, \quad v=0 \quad \text{on } \partial\Omega\,, \end{gathered} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded smooth domain, $N\geq 3$, $p\in(2,2^*)$, $q\in[2,\infty)$, and $f\in C^1(\overline{\Omega}\times\mathbb{R}^2,\mathbb{R})$ is a lower order term, which is not necessarily symmetric in (u,v). As usual, $2^*:=\frac{2N}{N-2}$ denotes the critical Sobolev exponent.

In the previous decades there has been a great amount of activity in the study of elliptic systems. Elliptic systems leading to strongly indefinite functionals have been studied, for example, in [4, 5, 6, 11, 12, 15, 16, 17, 18]. However, only subcritical systems have been considered in these papers, and the multiplicity results therein require some symmetry assumption on f. Recently De Figueiredo and Ding [14] considered the case $q\geq 2^*$. Under appropriate growth conditions, they established the existence of infinitely many solutions of ($\wp$) for functions f which are even in (u,v). Here we will show that one can do without the symmetry assumption. Namely, we prove the following.

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Theorem 1.1. *If $p \in (2, \frac{2N-2}{N-2})$, $q \in [p, \infty)$, and $f \in C^1(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$ satisfies*

$$|f_u(x,u,v)| \le c(|u|^{\gamma p-1} + |v|^{\sigma-1} + 1)$$
$$|f_v(x,u,v)| \le c(|u|^{\gamma p-1} + |v|^{\gamma q-1} + 1)$$

for all $(x,u,v) \in \Omega \times \mathbb{R}^2$, and some $c > 0$, $0 \le \sigma - 1 \le \frac{q}{p}(\gamma p - 1)$, and $\frac{1}{p} \le \gamma < \min\{(\frac{1}{p} - \frac{1}{2^*})N, \frac{q}{q-1}(\frac{2^*-1}{2^*})\}$, then (1.1) has infinitely many solutions.

This result is a special case of a stronger result (Theorem 3.1) which is obtained as an application of an abstract critical point theorem for strongly indefinite functionals with perturbed symmetries which we state and prove in section 2.

Variational methods for establishing existence of infinitely many solutions of an elliptic equation with perturbed symmetries were first introduced by Bahri and Berestycki [1], Struwe [26] and Rabinowitz [22] in the early eighties, further developed by Bahri and Lions [3] and Tanaka [27] and, more recently, by Bolle, Ghoussoub and Tehrani [7, 8], among others.

On the other hand, various methods for dealing with symmetric strongly indefinite functionals are now well known. The first one is due to Rabinowitz [21] who reduced the indefinite problem to a finite-dimensional one. Another useful approach is due to Benci and Rabinowitz [6] who showed that the original methods of critical point theory still work if one restricts the class of deformations appropriately. A different approach, based on a Galerkin type approximation, was given by Bartsch and Clapp in [4].

The abstract result we present here is also based on a Galerkin type approximation which reduces the study of strongly indefinite functionals with perturbed symmetries to a semidefinite situation, thus allowing the use of Morse theory methods as in [3] and [27]. However, unlike the symmetric case or the semidefinite case, the strongly indefinite perturbed case requires fine knowledge on the topology of the sublevel sets of the approximations of the associated symmetric functional (see Remark *(e)* at the end of section 2). The key step in the proof of Theorem 1.1 consists in a careful study of such sublevel sets. Our description will yield, in addition, estimates for the energy of the solutions of the system $(\wp)$, similar to those given by Bahri and Lions [3] in the symmetric single equation case, and recently extended by Castro and Clapp [9] to the perturbed single equation case (see Theorem 3.1).

This paper is organized as follows. In section 2 we state and prove an abstract critical point result for strongly indefinite functionals with perturbed symmetries, and in section 3 we apply this result to prove Theorem 1.1.

2. Strongly indefinite functionals with perturbed symmetries

Let X be a Banach space with a direct sum decomposition $X = X^+ \oplus X^* \oplus X^-$. According to this decomposition, a point in X will be denoted $u = (u^+, u^*, u^-)$. Let

$$X_1^+ \subset X_2^+ \subset \cdots \subset X^+, \quad X_1^* \subset X_2^* \subset \cdots \subset X^*, \quad X_1^- \subset X_2^- \subset \cdots \subset X^-$$

be sequences of finite dimensional linear subspaces of X^+, X^* and X^- such that $\dim X_k^+ = k$. For $k, n \ge 1$ we write

$$X^n := X^+ \oplus X_n^* \oplus X_n^- \quad \text{and} \quad X_k^n := X_k^+ \oplus X_n^* \oplus X_n^-.$$

Let $\iota : X \to X$ be the involution

$$\iota(u^+, u^*, u^-) = (-u^+, u^*, -u^-).$$

Then $X^* = \{u \in X : \iota u = u\}$ is the fixed point set of ι. We say that a subspace V of X is ι-invariant if $\iota u \in V$ for every $u \in V$, and we say that a map $\sigma : V \to W$ between two ι-invariant subspaces is ι-equivariant if $\sigma(\iota u) = \iota(\sigma(u))$ for every $u \in V$.

Let $\Phi : X \times [0,1] \to \mathbb{R}$ be a C^1-functional, and let $\Phi^n : X^n \times [0,1] \to \mathbb{R}$ be its restriction to $X^n \times [0,1]$. We think of Φ and Φ^n as being paths of functionals

$$\Phi_t : X \to \mathbb{R}, \quad \Phi_t(u) = \Phi(u,t), \quad 0 \le t \le 1,$$
$$\Phi_t^n : X^n \to \mathbb{R}, \quad \Phi_t^n(u) = \Phi^n(u,t), \quad 0 \le t \le 1,\ n \ge 1,$$

and write

$$\Phi_t'(u) := \frac{\partial}{\partial u}\Phi(u,t), \quad (\Phi_t^n)'(u) := \frac{\partial}{\partial u}\Phi^n(u,t).$$

for their derivatives with respect to u. We assume that Φ satisfies the following assumptions.

- (H1) Every sequence (u_k, t_k) in $X \times [0,1]$ with $u_k \in X^{n_k}$, $n_k \to \infty$, $t_k \to t$, $\Phi_{t_k}(u_k) \to c$, $\|(\Phi_{t_k}^{n_k})'(u_k)\| \to 0$, has a subsequence converging in X to a critical point of Φ_t.
- (H2) For every $n \in \mathbb{N}$ large enough and $b \in \mathbb{R}$ there is a constant $C = C(n,b)$ such that
$$\Big|\frac{\partial}{\partial t}\Phi(u,t)\Big| \le C(\|(\Phi_t^n)'(u)\| + 1)(\|u\| + 1) \quad \text{if } u \in X^n,\ |(\Phi_t^n)(u)| \le b.$$
- (H3) There exist two continuous functions $\theta_1, \theta_2 : [0,1] \times \mathbb{R} \to \mathbb{R}$, $\theta_1 \le \theta_2$, which are Lipschitz continuous in the second variable and such that
$$\theta_1(t, \Phi_t(u)) \le \frac{\partial}{\partial t}\Phi(u,t) \le \theta_2(t, \Phi_t(u)) \quad \text{if } \Phi_t'(u) = 0.$$
- (H4) For every finite dimensional subspace W of X and $a \in \mathbb{R}$ there exists an $R > 0$ such that $\Phi_t(w) \le a$ for every $t \in [0,1]$, $w \in W$ with $\|w\| \ge R$.
- (H5) $\Phi_0(\iota u) = \Phi_0(u)$ for every $u \in X$, and there exists an $M \ge 0$ such that $\Phi_t(u^*) \le M$ for every $t \in [0,1]$, $u^* \in X^*$.
- (H6) $\sup\{\Phi_0(u) : u \in X_k^+ \oplus X^* \oplus X^-\} =: M_k < \infty$ for every $k \ge 1$.
- (H7) For each $k \ge 1$ there exist $n_k \ge 1$ and a nondecreasing function $\ell_k : \mathbb{R} \to \mathbb{R}$ with the following property: Given $n \ge n_k$, an ι-equivariant map $\sigma \in C^0(X_k^n, X^n)$ and an $R > 0$ such that $\sigma(u) = u$ if $\|u\| > R$, there exist an ι-equivariant map $\widetilde{\sigma} \in C^0(X_{k+1}^n, X^n)$ and an $\widetilde{R} > R$ such that $\widetilde{\sigma}(u) = \sigma(u)$ if $u \in X_k^n$, $\widetilde{\sigma}(u) = u$ if $\|u\| > \widetilde{R}$, and
$$\sup \Phi_0(\widetilde{\sigma}(X_{k+1}^n)) \le \ell_k(\sup \Phi_0(\sigma(X_k^n))).$$

We shall prove the following statement.

Theorem 2.1. *Assume that Φ satisfies (H1)-(H7). Then there exists a sequence (c_k) of real numbers such that, if the sequence*

$$\Big(\frac{c_{k+1} - c_k}{\max_{0\le t\le 1} |\theta_1(t, c_{k+1})| + \max_{0\le t\le 1} |\theta_2(t, c_k)| + 1}\Big) \tag{2.1}$$

is unbounded, then Φ_1 has an unbounded sequence of critical values.

The numbers c_k are defined as follows (see (2.2) below): For each $n \in \mathbb{N}$, let

$$c_k^n := \inf_{\sigma \in \Gamma_k^n} \sup_{u \in X_k^n} \Phi_0(\sigma(u))$$

where Γ_k^n is the set of maps $\sigma \in C^0(X_k^n, X^n)$ with the following three properties:

(i) σ is ι-equivariant, that is, $\sigma(\iota(u)) = \iota(\sigma(u))$ for each $u \in X_k^n$,

(ii) There exists $R > 0$ such that $\sigma(u) = u$ if $\|u\| > R$.

(iii) $\sigma(u) = u$ for each $u \in X_n^*$.

These values have the following linking property for Φ_0.

Lemma 2.2. *Let $e \in X_{k+1}^+ \setminus X_k^+$ and let*

$$\vartheta : \{v + se \in X_{k+1}^n : v \in X_k^n,\ s \in [0, \infty)\} \to X^n$$

be a continuous map with the following properties:

(i) *$\vartheta \mid X_k^n$ is ι-equivariant.*

(ii) *There exists $R > 0$ such that $\vartheta(u) = u$ if $\|u\| > R$.*

(iii) *$\vartheta(v) = v$ for all $v \in X_n^*$.*

Then there exists $(v_0, s_0) \in X_k^n \times [0, \infty)$ such that

$$\Phi_0(\vartheta(v_0 + s_0 e)) \geq c_{k+1}^n.$$

Proof. We extend ϑ to a map $\widetilde{\vartheta} : X_{k+1}^n \to X^n$ as follows:

$$\widetilde{\vartheta}(v + se) := \iota\vartheta(\iota v - se) \quad \text{if } (v, s) \in X_k^n \times (-\infty, 0].$$

Since $\dim X_{k+1}^n = \dim X_k^n + 1$ and since $\vartheta \mid X_k^n$ is ι-equivariant, $\widetilde{\vartheta}$ is well defined. By definition, $\widetilde{\vartheta} \in \Gamma_{k+1}^n$. Hence there exists $u_0 \in X_{k+1}^n$ with $\Phi_0(\widetilde{\vartheta}(u_0)) \geq c_{k+1}^n$. But $\Phi_0 \circ \iota = \Phi_0$ by assumption (H5). Therefore ιu_0 also satisfies $\Phi_0(\widetilde{\vartheta}(\iota u_0)) \geq c_{k+1}^n$. So, without loss of generality, $u_0 = v_0 + s_0 e$ with $(v_0, s_0) \in X_k^n \times [0, \infty)$. □

Since the inclusion $X_k^n \hookrightarrow X^n$ belongs to Γ_k^n, assumption (H6) guarantees that

$$c_k^n \leq \sup_{u \in X_k^n} \Phi_0(u) \leq M_k < \infty \quad \text{for all } k, n \geq 1.$$

The numbers c_k in Theorem 2.1 are defined as follows:

$$c_k := \limsup_{n \to \infty} c_k^n. \tag{2.2}$$

Note that $c_k \leq c_{k+1}$ for all $k \geq 1$.

We shall assume without loss of generality that $c_k^n \to c_k$ as $n \to \infty$. We also assume from now on that (H1)-(H7) hold and that the sequence (2.1) is unbounded. We wish to show that the linking property of Φ_0 is preserved by the flow of the path of functionals Φ_t in a suitable sense. We start with the following easy observation.

Lemma 2.3. *For every $b, \delta > 0$ there exist $n_0 \in \mathbb{N}$ and $\rho > 0$ such that*

$$\theta_1(t, \Phi_t(u)) - \delta < \frac{\partial}{\partial t}\Phi(u, t) < \theta_2(t, \Phi_t(u)) + \delta$$

for every $(u, t) \in X^n \times [0, 1]$, with $n \geq n_0$, $|\Phi_t(u)| < b$ and $\|(\Phi_t^n)'(u)\| < \rho$.

The statement of the above lemma is an immediate consequence of (H1) and (H3).

Fix $\delta > 0$. For θ_1 and θ_2 as in (H3) we consider the flows $\zeta_1, \zeta_2 : [0,1] \times \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{gathered}\zeta_1(0,s) = s \\ \frac{\partial}{\partial t}\zeta_1(t,s) = \theta_1(t,\zeta_1(t,s)) - \delta\end{gathered}$$

and

$$\begin{gathered}\zeta_2(0,s) = s \\ \frac{\partial}{\partial t}\zeta_2(t,s) = \theta_2(t,\zeta_2(t,s)) + \delta\,.\end{gathered}$$

Following Bolle [7] we prove the following deformation lemma for the path of functionals Φ_t.

Lemma 2.4. *For every pair of real numbers $d_1 \le d_2$ there exist $n_0 \in \mathbb{N}$ and, for each $n \ge n_0$ and each $\nu = 1, 2$, there exists a homotopy $\eta_\nu^n : X^n \times [0,1] \to X^n$ with the following properties:*

(i) *$\eta_\nu^n(u,0) = u$ for all $u \in X^n$.*
(ii) *$\eta_\nu^n(u,t) = u$ if either $\Phi_t(u) \le \min_{0\le t\le 1} \zeta_\nu(t,d_1) - 1$ or $\Phi_t(u) \ge \max_{0\le t\le 1} \zeta_\nu(t,d_2) + 1$.*
(iii) *$\eta_\nu^n(\cdot,t) : X^n \to X^n$ is a homeomorphism for every $t \in [0,1]$.*
(iv) *If $c \in [d_1,d_2]$ and $\Phi_0(u) \ge c$, then $\Phi_t(\eta_1^n(u,t)) \ge \zeta_1(t,c)$ for all $t \in [0,1]$.*
(v) *If $c \in [d_1,d_2]$ and $\Phi_0(u) \le c$, then $\Phi_t(\eta_2^n(u,t)) \le \zeta_2(t,c)$ for all $t \in [0,1]$.*

Proof. We extend Bolle's argument [7] to Banach spaces and C^1-functionals as follows. Let $M^n = \{(u,t) \in X^n \times [0,1] : (\Phi_t^n)'(u) \ne 0\}$ and let $W^n : M^n \to X^n$ be a pseudogradient vector field for the map $(u,t) \mapsto (\Phi_t^n)'(u)$, that is, W^n is locally Lipschitz continuous and satisfies

$$\|W^n(u,t)\| \le 2\|(\Phi_t^n)'(u)\| \quad \text{and} \quad \langle (\Phi_t^n)'(u), W^n(u,t)\rangle \ge \|(\Phi_t^n)'(u)\|^2 \tag{2.3}$$

for every $(u,t) \in M^n$ [28, Lemma 2.2]. Set $\alpha_\nu = \min\{\zeta_\nu(t,d_1) : 0 \le t \le 1\}$ and $\beta_\nu = \max\{\zeta_\nu(t,d_2) : 0 \le t \le 1\}$. For $b = \max\{|\alpha_\nu|, |\beta_\nu| : \nu = 1,2\}$ and δ as above we choose $n_0 \in \mathbb{N}$ and $\rho > 0$ as in Lemma 2.3. Let $\lambda_\nu, \mu \in C^\infty(\mathbb{R},[0,1])$ be such that $\lambda_\nu \equiv 0$ on $(-\infty, \alpha_\nu - \frac12] \cup [\beta_\nu + \frac12, \infty)$ and $\lambda_\nu \equiv 1$ on $[\alpha_\nu, \beta_\nu]$, and $\mu \equiv 0$ on $[-\frac{\rho}{2}, \frac{\rho}{2}]$ and $\mu \equiv 1$ on $(-\infty,-\rho] \cup [\rho,\infty)$.

Fix $n \ge n_0$ and consider the vector fields $V_\nu^n : X^n \times [0,1] \to X^n$ given by

$$\begin{aligned}
&V_1^n(u,t)\\
&= 4\Big((\frac{\partial}{\partial t}\Phi)^-(u,t) + 1 + \theta_1^+(t,\zeta_1(t,c))\Big)\lambda_1(\Phi_t(u))\mu(\|W^n(u,t)\|)\frac{W^n(u,t)}{\|W^n(u,t)\|^2},\\
&V_2^n(u,t)\\
&= -4\Big((\frac{\partial}{\partial t}\Phi)^+(u,t) + 1 + \theta_2^-(t,\zeta_2(t,c))\Big)\lambda_2(\Phi_t(u))\mu(\|W^n(u,t)\|)\frac{W^n(u,t)}{\|W^n(u,t)\|^2},
\end{aligned}$$

where $h^\pm := \max\{\pm h, 0\} \ge 0$. Note that $V_\nu^n(u,t) = 0$ if $\Phi_t(u) \notin [\alpha_\nu - \frac12, \beta_\nu + \frac12]$ or $\|W^n(u,t)\| \le \frac{\rho}{2}$. On the other hand, if $\Phi_t(u) \in [\alpha_\nu - \frac12, \beta_\nu + \frac12]$ and $\|W^n(u,t)\| \ge \frac{\rho}{2}$

then, conditions (H2) and (2.3) imply that

$$\begin{aligned}\|V_\nu^n(u,t)\| &\le \frac{4(|(\frac{\partial}{\partial t}\Phi)(u,t)|+1+|\theta_\nu(t,\zeta_\nu(t,c))|)}{\|W^n(u,t)\|}\\ &\le \frac{\widetilde{C}(\|(\Phi_t^n)'(u)\|+1)(\|u\|+1)}{\|W^n(u,t)\|}\\ &\le \widehat{C}(\|u\|+1)\end{aligned}$$

for some positive constants $\widetilde{C}$ and $\widehat{C}$. This, and the fact that V_ν^n is locally Lipschitz continuous, imply the existence of a global flow $\eta_\nu^n : X^n\times[0,1]\to X^n$ for V_ν^n given by

$$\begin{gathered}\eta_\nu^n(u,0)=u\\ \frac{\partial}{\partial t}\eta_\nu^n(u,t)=V_\nu^n(\eta_\nu^n(u,t),t)\end{gathered}$$

Properties (i)-(iii) are immediate. We prove (iv): Let $u\in X^n$ satisfy $\Phi_0(u)\ge c$ for some $c\in[d_1,d_2]$. Set $f(t):=\Phi_t(\eta_1^n(u,t))$. Since $f(0)=\Phi_0(u)\ge c=\zeta_1(0,c)$ it suffices to show that

$$f(t)=\zeta_1(t,c)\Longrightarrow f'(t)>\frac{\partial}{\partial t}\zeta_1(t,c)=\theta_1(t,\zeta_1(t,c))-\delta. \tag{2.4}$$

So let us assume $f(t)=\zeta_1(t,c)$. Then $\lambda_1(f(t))=1$. Hence, setting

$$\varphi(t):=(\frac{\partial}{\partial t}\Phi)^-(\eta_1^n(u,t),t)+1+\theta_1^+(t,\zeta_1(t,c)),$$

we obtain

$$\begin{aligned}f'(t)&=\langle(\Phi_t^n)'(\eta_1^n(u,t)),V_1^n(\eta_1^n(u,t),t)\rangle+\frac{\partial}{\partial t}\Phi(\eta_1^n(u,t),t)\\ &=4\varphi(t)\mu(\|W^n(\eta_1^n(u,t),t)\|)\frac{\langle(\Phi_t^n)'(\eta_1^n(u,t)),W^n(\eta_1^n(u,t),t)\rangle}{\|W^n(\eta_1^n(u,t),t)\|^2}\\ &\quad+\frac{\partial}{\partial t}\Phi(\eta_1^n(u,t),t)\\ &\ge\varphi(t)\mu(\|W^n(\eta_1^n(u,t),t)\|)+\frac{\partial}{\partial t}\Phi(\eta_1^n(u,t),t)\end{aligned}$$

If $\|W^n(\eta_1^n(u,t),t)\|<\rho$ then $\|(\Phi_t^n)'(\eta_1^n(u,t))\|<\rho$ and, by Lemma 2.3,

$$f'(t)\ge\frac{\partial}{\partial t}\Phi(\eta_1^n(u,t),t)>\theta_1(t,\zeta_1(t,c))-\delta.$$

If $\|W^n(\eta_1^n(u,t),t)\|\ge\rho$ then $\mu(\|W^n(\eta_1^n(u,t),t)\|)=1$, hence,

$$f'(t)\ge\varphi(t)+\frac{\partial}{\partial t}\Phi(\eta_1^n(u,t),t)\ge\theta_1(t,\zeta_1(t,c))>\theta_1(t,\zeta_1(t,c))-\delta.$$

This proves (2.4). Therefore, η_1^n satisfies (iv). Similarly, η_2^n satisfies (v). □

Given a subset A of X^n we define

$$\begin{aligned}\Gamma_k^n(A):=\big\{\tau\in C^0(X^n,X^n):\tau(u)=u \text{ if either } u\in A,\\ \text{or } u\in X_{k+1}^n \text{ and } \|u\| \text{ is large}\big\}.\end{aligned}$$

Let $M\ge 0$ be as in (H5), and let $n_k\ge 1$ and $\ell_k:\mathbb{R}\to\mathbb{R}$ be as in (H7). We now prove a linking property for Φ_1.

Lemma 2.5. *For every $k \in \mathbb{N}$ such that*
$$M + 1 \leq \zeta_2(t, c_k) < \zeta_1(t, c_{k+1}) \quad \text{for all } t \in [0,1],$$
there exist $\varepsilon_k > 0$, $m_k \geq n_k$ and, for each $n \geq m_k$, two subsets $A_k^n \subset B_k^n$ of X^n with the following properties:

(a) $\sup \Phi_1(A_k^n) \leq \zeta_2(1, c_k + \varepsilon_k) < \zeta_1(1, c_{k+1} - \varepsilon_k)$.
(b) $\sup \Phi_1(B_k^n) \leq \zeta_2(1, \ell_k(c_k + \varepsilon_k))$.
(c) $\inf_{\tau \in \Gamma_k^n(A_k^n)} \sup \Phi_1(\tau(B_k^n)) \geq \zeta_1(1, c_{k+1} - \varepsilon_k)$.

Proof. Fix $0 < \varepsilon_k < 1$ such that
$$\zeta_2(t, c_k + \varepsilon_k) < \zeta_1(t, c_{k+1} - \varepsilon_k) \quad \text{for all } t \in [0,1]. \tag{2.5}$$
Let $m_k \geq n_k$ be such that, for each $n \geq m_k$,
$$c_k^n < c_k + \varepsilon_k \quad \text{and} \quad c_{k+1}^n > c_{k+1} - \varepsilon_k,$$
and there exist homotopies $\eta_\nu^n : X^n \times [0,1] \to X^n$ which satisfy (i)-(v) of Lemma 2.4 for $d_1 = d_2 = c_{k+1} - \varepsilon_k$ if $\nu = 1$, and for $d_1 = c_k + \varepsilon_k$, $d_2 = \ell_k(c_k + \varepsilon_k)$ if $\nu = 2$. In particular,
$$\Phi_t(\eta_1^n(u,t)) \geq \zeta_1(t, c_{k+1} - \varepsilon_k) \quad \text{if } \Phi_0(u) \geq c_{k+1} - \varepsilon_k \tag{2.6}$$
$$\Phi_t(\eta_2^n(u,t)) \leq \zeta_2(t, c) \quad \text{if } \Phi_0(u) \leq c \quad \text{and} \quad c_k + \varepsilon_k \leq c \leq \ell_k(c_k + \varepsilon_k) \tag{2.7}$$
for all $t \in [0,1]$. Fix $n \geq m_k$ and choose $\sigma \in \Gamma_k^n$ such that
$$\sup \Phi_0(\sigma(X_k^n)) \leq c_k + \varepsilon_k.$$
By assumption (H7), σ has an extension $\widetilde{\sigma} \in \Gamma_{k+1}^n$ such that
$$\sup \Phi_0(\widetilde{\sigma}(X_{k+1}^n)) \leq \ell_k(c_k + \varepsilon_k).$$
Inequalities (2.5), (2.6) and (2.7) imply that
$$(\eta_2^n)_t(\sigma(X_k^n)) \cap (\eta_1^n)_t(\Phi_0^n)^{>c_{k+1}-\varepsilon_k} = \emptyset \quad \text{for all } t \in [0,1], \tag{2.8}$$
where $(\Phi_0^n)^{>c_{k+1}-\varepsilon_k} := \{u \in X^n : \Phi_0(u) > c_{k+1} - \varepsilon_k\}$. Choose $e \in X_{k+1}^+ \setminus X_k^+$ and define
$$A_k^n := \{(\eta_2^n)_1(\sigma(u)) : u \in X_k^n\}$$
$$B_k^n := \{(\eta_2^n)_1(\widetilde{\sigma}(u + te)) : u \in X_k^n,\ t \geq 0\}.$$
It follows from (2.7) that
$$\sup \Phi_1(A_k^n) \leq \zeta_2(1, c_k + \varepsilon_k) \quad \text{and} \quad \sup \Phi_1(B_k^n) \leq \zeta_2(1, \ell_k(c_k + \varepsilon_k)).$$
Thus (a) and (b) hold. Let us prove (c). For every $\tau \in \Gamma_k^n(A_k^n)$, the function $\vartheta : \{u + te : u \in X_k^n,\ t \in [0,\infty)\} \to X^n$, defined by
$$\vartheta(u + te) = \begin{cases} (\eta_1^n)_{2t}^{-1} \circ (\eta_2^n)_{2t} \circ \sigma(u) & \text{if } 0 \leq t \leq 1/2 \\ (\eta_1^n)_1^{-1} \circ \tau \circ (\eta_2^n)_1 \circ \widetilde{\sigma}(u + (2t-1)e) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$
satisfies the hypotheses of Lemma 2.2. Hence there exists $(u_0, t_0) \in X_k^n \times [0, \infty)$ with
$$\Phi_0(\vartheta(u_0 + t_0 e)) \geq c_{k+1}^n > c_{k+1} - \varepsilon_k. \tag{2.9}$$
If $t_0 \leq 1/2$ then $(\eta_1^n)_{2t_0}(\vartheta(u_0 + t_0 e)) = (\eta_2^n)_{2t_0}(\sigma(u_0))$, contradicting (2.8). Therefore, $t_0 > 1/2$ and $(\eta_1^n)_1(\vartheta(u_0 + t_0 e)) = \tau\,[(\eta_2^n)_1(\widetilde{\sigma}(u_0 + (2t_0 - 1)e)]$. Inequalities (2.6) y (2.9) yield
$$\Phi_1(\tau\,[(\eta_2^n)_1(\widetilde{\sigma}(u_0 + (2t_0 - 1)e)]) = \Phi_1((\eta_1^n)_1(\vartheta(u_0 + t_0 e))) \geq \zeta_1(1, c_{k+1} - \varepsilon_k)$$

and, since $(\eta_2^n)_1(\widetilde{\sigma}(u_0+(2t_0-1)e) \in B_k^n$, it follows that
$$\sup \Phi_1(\tau(B_k^n)) \geq \zeta_1(1, c_{k+1}-\varepsilon_k).$$
This proves (c). $\square$

We now show that, if the sequence (2.1) is unbounded, the hypothesis of Lemma 2.5 holds for infinitely many k's.

Lemma 2.6. *If the sequence (2.1) is unbounded, then the sequences*
$$\Big(\min_{0\leq t\leq 1}\,(\zeta_1(t,c_{k+1})-\zeta_2(t,c_k))\Big) \quad \text{and} \quad \Big(\min_{0\leq t\leq 1}\,\zeta_2(t,c_k)\Big)$$
are unbounded above.

Proof. The flows ζ_ν satisfy $|s-\zeta_\nu(t,s)| \leq a(\max\{|\theta_\nu(t,s)| : t\in[0,1]\}+\delta)$ for some constant $a>0$ (cf. for example [25]). Therefore,
$$0 \leq c_{k+1}-c_k \leq \zeta_1(t,c_{k+1})-\zeta_2(t,c_k)+a(\max_{0\leq t\leq 1}|\theta_1(t,c_{k+1})|+\max_{0\leq t\leq 1}|\theta_2(t,c_k)|+2\delta).$$
Since the sequence (2.1) is unbounded, our claim follows. $\square$

Proof of Theorem 2.1. By Lemma 2.6, passing to a subsequence if necessary, we may assume that
$$M+1 < \zeta_2(t,c_k) < \zeta_1(t,c_{k+1}) \quad \text{for all} \;\; t\in[0,1],\;\; k\geq 1.$$
Let $A_k^n \subset B_k^n \subset X^n$, $n \geq m_k$, be as in Proposition 2.5, and let
$$\widetilde{c}_k^n := \inf_{\tau\in\Gamma_k^n(A_k^n)}\;\sup_{u\in B_k^n}\Phi_1(\tau(u))$$
Then
$$\zeta_2(1,c_k) < \zeta_1(1,c_{k+1}) \leq \widetilde{c}_k^n \leq \zeta_2(1,\ell_k(c_k+1)) \quad \text{for all } n\geq m_k. \tag{2.10}$$
Define
$$\widetilde{c}_k := \limsup_{n\to\infty}\widetilde{c}_k^n.$$
It follows from assumption (H1) and Proposition 2.6 (b) in [4] that $\widetilde{c}_k$ is a critical value of $\Phi_1 : X \to X$. Since $\widetilde{c}_k \geq \zeta_2(1,c_k)$, Lemma 2.6 implies that the sequence $(\widetilde{c}_k)$ is unbounded. $\square$

We conclude this section with some remarks.

Remarks. (a) If $\dim(X^*\oplus X^-) < \infty$ then assumptions (H6) and (H7) follow easily from assumption (H4).
(b) If $X^* = \{0\}$, assumptions (H2)-(H5) are the same as those of Theorem 2.2 in [8], and our assumption (H1) is stronger than the one given there. But, if $\dim(X^-) = \infty$, the minimax levels c_k as defined by Bolle, Ghoussoub and Tehrani in [8] will all be zero. So their result does not yield critical values in the strongly indefinite situation.
(c) Our definition of c_k allows us to take advantage of the topology of the sublevel sets of the semidefinite approximations Φ_0^n of Φ_0 and, in particular, to apply Morse theory methods (as was done by Bahri-Lions [3] and Tanaka [27]) to estimate the growth of the c_k's and derive conditions for the unboundedness of (2.1), see Lemma 3.9 below.
(d) Assumption (H1) is the obvious extension to paths of functionals of the (PS)*-condition introduced by Bahri and Berestycki [2] and Li and Liu [19]. It yields

critical values $\widetilde{c}_k$ of Φ_1 provided that the minimax values $\widetilde{c}_k^n$ of its approximations Φ_1^n are uniformly bounded. This is where (H7) comes into play.

(e) Assumption (H7) requires some knowledge on the topology of the sublevel sets of the approximations of Φ_0. Note that every ι-equivariant map $\sigma \in C^0(X_k^n, X^n)$ such that $\sigma(u) = u$ for $\|u\|$ large enough, has an ι -equivariant extension $\widetilde{\sigma} \in C^0(X_{k+1}^n, X^n)$ such that $\widetilde{\sigma}(u) = u$ for $\|u\|$ sufficiently large. The key point in assumption (H7) is that

$$\sup \Phi_0(\widetilde{\sigma}(X_{k+1}^n)) \le \ell_k(\sup \Phi_0(\sigma(X_k^n)))$$

for some function ℓ_k which does not depend on n. In fact, in our application the function ℓ_k will be linear and will be also independent of k, see Proposition 3.8 below.

(f) It follows from (2.10) that the critical values $\widetilde{c}_k$ of the perturbed functional Φ_1 satisfy

$$\zeta_2(1, c_k) < \zeta_1(1, c_{k+1}) \le \widetilde{c}_k \le \zeta_2(1, \ell_k(c_k + 1)).$$

This inequality will be used to obtain estimates on the energy of the solutions of problem ($\wp$), see Theorem 3.1.

3. Strongly indefinite elliptic systems

We apply Theorem 2.1 to obtain infinitely many solutions of the elliptic system (1.1). The variational setting is as follows: Let $0 < \lambda_1 \le \dots \le \lambda_j \le \dots$ be the Dirichlet eigenvalues of $-\Delta$ on $H_0^1(\Omega)$ counted with their multiplicity, and let $e_j \in H_0^1(\Omega)$ be the eigenfunction which corresponds to λ_j with $|e_j|_2 = 1$. Consider the Banach space $H_0^1(\Omega) \cap L^q(\Omega)$ equipped with the norm $\|v\|_{(q)} := (|\nabla v|_2^2 + |v|_q^2)^{1/2}$, where $|\cdot|_r$ denotes the usual norm in L^r. Let $V^q(\Omega)$ be the closure of $span\{e_n : n \ge 1\}$ in $H_0^1(\Omega) \cap L^q(\Omega)$ with respect to the $\|v\|_{(q)}$-norm. Then $V^q(\Omega)$ is a Banach space. Since the eigenfunctions satisfy $|e_n|_\infty^2 \le C\lambda_n^{N/2}$ for some positive constant C [10, Chap. IV, Theorem 8], integrating by parts one can easily show that the Fourier coefficients of a function $\varphi \in C_0^{2m}(\Omega)$ decrease as λ_n^{-m}. It follows that $C_0^\infty(\Omega) \subset V^q(\Omega)$.

Let X be the direct sum

$$X := H_0^1(\Omega) \oplus V^q(\Omega)$$

We denote the elements of X by $z = (u, v)$, and their norm by

$$\|z\|_q := (|\nabla u|_2^2 + \|v\|_{(q)}^2)^{1/2} = (|\nabla u|_2^2 + |\nabla v|_2^2 + |v|_q^2)^{1/2}.$$

Set

$$X_k^+ := \operatorname{span}\{e_1, ..., e_k\} \subset H_0^1(\Omega) =: X^+,$$
$$X_n^- := \operatorname{span}\{e_1, ..., e_n\} \subset V^q(\Omega) =: X^-,$$

and $X^* := \{0\}$. The orthogonal projection $H_0^1(\Omega) \to \operatorname{span}\{e_1, \dots, e_n\} \subset H_0^1(\Omega)$ restricts to a continuous operator

$$P_n : V^q(\Omega) \to X_n^-$$

which satisfies $P_n v \to v$ in $V^q(\Omega)$ as $n \to \infty$ for every $v \in V^q(\Omega)$ because, by definition, $\cup_{n\ge1} X_n^-$ is dense in X^-.

Set

$$H(x,z,t) := \frac{1}{p}|u|^p + \frac{1}{q}|v|^q + tf(x,z), \quad x \in \Omega,\ z = (u,v) \in \mathbb{R}^2,$$
$$\Phi(z,t) := \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - |\nabla v|^2\right) dx - \int_\Omega H(x,z,t)dx.$$

Theorem 1.1 is a special case of the following result.

Theorem 3.1. *Let $p \in (2, 2^*)$, $q \in (2,\infty)$, and $f \in C^1(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$. If there exist $d > 0$, $\gamma \in [0,1)$, $\gamma < \min\{(\frac{1}{p} - \frac{1}{2^*})N, \frac{q}{q-1}(\frac{2^*-1}{2^*})\}$ such that*

$$|f(x,z)| + |f_z(x,z)z| \le d(|u|^{\gamma p} + |v|^{\gamma q} + 1),$$
$$|f_z(x,z)| \le d(|u|^{\gamma(p-1)} + |v|^{\gamma(q-1)} + 1)$$

for all $x \in \Omega$, $z = (u,v) \in \mathbb{R}^2$, then problem (1.1) has a sequence of solutions $z_k = (u_k, v_k)$ which satisfy

$$C_1 k^\nu \le \Phi_1(z_k) \le C_2 k^\nu,$$

with $\nu := \frac{2p}{N(p-2)}$, $C_1, C_2 > 0$.

The assumptions of Theorem 3.1 guarantee that the functional Φ is well defined and of class C^1. The critical points of $\Phi_1 := \Phi(\cdot, 1)$ are weak solutions of (1.1). As in section 2 we set

$$X^n := X^+ \oplus X_n^-, \quad X_k^n := X_k^+ \oplus X_n^-,$$

and write $\Phi_t^n : X^n \to \mathbb{R}$ for the functional $\Phi_t^n(z) = \Phi(z,t)$, $z \in X^n$. We now show that Φ satisfies assumptions (H1)-(H7) of Theorem 2.1. In order to prove (H1) we need the following lemma.

Lemma 3.2. *Let $z_k \in X^{n_k}$, $t_k \in [0,1]$ be such that $n_k \to \infty$, $\Phi_{t_k}(z_k) \to c$ and $(\Phi_{t_k}^{n_k})'(z_k) \to 0$ as $k \to \infty$. Then (z_k) is bounded in X.*

Proof. Our assumptions on f yield

$$\frac{1}{2}H_z(x,z,t)z - H(x,z,t) \ge a_1\left(|u|^p + |v|^q\right) - a_2. \tag{3.1}$$

Therefore, for k large enough,

$$|u_k|_p^p + |v_k|_q^q \le a_3(\Phi_{t_k}(z_k) - \frac{1}{2}(\Phi_{t_k}^{n_k})'(z_k)z_k + 1) \le a_4(1 + \|z_k\|_q). \tag{3.2}$$

By assumption, $\eta = q/(q - \gamma(q-1)) < 2^*$. So, using Hölder's inequality, Sobolev's embedding theorem and inequality (3.2), we obtain

$$\int_\Omega |v_k|^{\gamma(q-1)}|u_k| \le |v_k|_q^{\gamma(q-1)}|u_k|_\eta \le a_5(1 + \|z_k\|_q^{1+[\gamma(q-1)/q]}). \tag{3.3}$$

Our assumptions on f and γ, together with (3.2) and (3.3), imply

$$\begin{aligned}\int_\Omega H_u(x,z_k,t_k)u_k &= |u_k|_p^p + t_k\int_\Omega f_u(x,z_k)u_k \\ &\le |u_k|_p^p + d\int_\Omega(|u_k|^{\gamma(p-1)} + |v_k|^{\gamma(q-1)} + 1)|u_k| \\ &\le a_6(1 + \|z_k\|_q^\sigma)\end{aligned}$$

with $1 < \sigma < 2$. They also imply

$$\begin{aligned}
&-\int_\Omega H_v(x,z_k,t_k)v_k \\
&= \int_\Omega H_u(x,z_k,t_k)u_k - \int_\Omega H_z(x,z_k,t_k)z_k \\
&\le a_6(1+\|z_k\|_q^\sigma) - \int_\Omega (|u_k|^p + |v_k|^q) + d\int_\Omega (|u_k|^{\gamma p} + |v_k|^{\gamma q} + 1) \\
&\le a_6(1+\|z_k\|_q^\sigma) - c_2(|u_k|_p^p + |v_k|_q^q) + c_3 \\
&\le a_7(1+\|z_k\|_q^\sigma).
\end{aligned}$$

We conclude that, for k large enough,

$$|\nabla u_k|_2^2 = (\Phi_{t_k}^{n_k})'(z_k)(u_k,0) + \int_\Omega H_u(x,z_k,t_k)u_k \le a_7(1+\|z_k\|_q^\sigma), \tag{3.4}$$

$$|\nabla v_k|_2^2 = -(\Phi_{t_k}^{n_k})'(z_k)(0,v_k) - \int_\Omega H_v(x,z_k,t_k)v_k \le a_7(1+\|z_k\|_q^\sigma). \tag{3.5}$$

Inequalities (3.2), (3.4) and (3.5) yield

$$\|z_k\|_q^2 \le a_8(1+\|z_k\|_q^\sigma)$$

with $\sigma < 2$. Therefore (z_k) must be bounded in X. □

Proposition 3.3. *The function Φ satisfies (H1).*

Proof. Let $z_k \in X^{n_k}$, $t_k \in [0,1]$ be such that $n_k \to \infty$, $t_k \to t$, $\Phi_{t_k}(z_k) \to c$ and $(\Phi_{t_k}^{n_k})'(z_k) \to 0$ as $k \to \infty$. Write $z_k = (u_k, v_k)$. Since (z_k) is bounded in X, $u_k \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $v_k \rightharpoonup v$ weakly in $H_0^1(\Omega) \cap L^q(\Omega)$. Hence $u_k \to u$ strongly in $L^s(\Omega)$ for every $s \in [1, 2^*)$ and, by interpolation, $v_k \to v$ strongly in $L^s(\Omega)$ for every $s \in [1, \max\{2^*, q\})$. Our assumptions and Hölder's inequality yield

$$\Big|\int_\Omega H_u(x,z_k,t_k)(u_k-u)\Big| \le a_1\big(|u_k|_p^{p-1}|u_k-u|_p + |v_k|_q^{\gamma(q-1)}|u_k-u|_\eta + |u_k-u|_1\big),$$

with $\eta = q/(q-\gamma(q-1)) < 2^*$. Hence,

$$\int_\Omega \nabla u_k \nabla(u_k-u) = (\Phi_{t_k}^{n_k})'(z_k)(u_k-u,0) + \int_\Omega H_u(x,z_k,t_k)(u_k-u) \longrightarrow 0$$

as $k \to \infty$; therefore, $u_k \to u$ strongly in $H_0^1(\Omega)$.

The projection $P_n : V^q(\Omega) \to V^q(\Omega)_n$ satisfies $P_n v \to v$ in $V^q(\Omega)$ as $n \to \infty$ for every $v \in V^q(\Omega)$. Thus, $(v_k - P_{n_k}v) \to 0$ in $L^s(\Omega)$ for every $s \in [1, \max\{2^*, q\})$. As above we have

$$\begin{aligned}
&\Big|\int_\Omega t_k f_v(x,z_k)(v_k - P_{n_k}v)\Big| \\
&\le a_1\big(|u_k|_p^{p-1}|v_k - P_{n_k}v|_p + |v_k|_q^{\gamma(q-1)}|v_k - P_{n_k}v|_\eta + |v_k - P_{n_k}v|_1\big),
\end{aligned}$$

with $\eta = q/(q-\gamma(q-1)) < 2^*$. Hence,

$$\int_\Omega t_k f_v(x,z_k)(v_k - P_{n_k}v) \to 0 \quad \text{as } k \to \infty.$$

Therefore,

$$\begin{aligned}|\nabla v_k|_2^2 - |\nabla v|_2^2 + o(1) &= \int_\Omega \nabla v_k \nabla(v_k - P_{n_k} v)\\ &= -(\Phi_{t_k}^{n_k})'(z_k)(0, v_k - P_{n_k} v) - \int_\Omega H_v(x, z_k, t_k)(v_k - P_{n_k} v)\\ &= o(1) - \int_\Omega |v_k|^{q-2} v_k (v_k - P_{n_k} v)\\ &= o(1) + \int_\Omega |v_k|^{q-2} v_k (v - v_k)\\ &\le o(1) + |v|_q^q - |v_k|_q^q,\end{aligned}$$

where $o(1) \to 0$ as $k \to \infty$. It follows that

$$0 \le \liminf |\nabla v_k|_2^2 - |\nabla v|_2^2 \le \limsup |\nabla v_k|_2^2 - |\nabla v|_2^2 \le |v|_q^q - \liminf |v_k|_q^q \le 0.$$

Hence, up to a subsequence, $v_k \to v$ strongly in $V^q(\Omega)$. This proves that $z_k \to z$ strongly in X. In particular, $\Phi'_{t_k}(z_k)\zeta \to \Phi'_t(z)\zeta$ for every $\zeta \in X$. Hence, $\Phi'_t(z)\zeta = 0$ for every $\zeta \in \cup_{n\ge 1} X^n$. Since $\cup_{n\ge1} X^n$ is dense in X, z is a critical point of Φ_t. □

Proposition 3.4. *The function Φ satisfies (H2)-(H6) with $\theta_2(t,s) = A(s^2+1)^{\gamma/2} = -\theta_1(t,s)$, $A > 0$.*

Proof. Our assumptions on f yield

$$|\frac{\partial}{\partial t}\Phi(z,t)| \le \int_\Omega |f(x,z)| \le d_1 \big|\Phi_t(z) - \frac{1}{2}\Phi'_t(z)z + 1\big|^\gamma. \tag{3.6}$$

If $z \in X^n$ and $|\left(\Phi_t^n\right)(z)| \le b$, this inequality implies that

$$|\frac{\partial}{\partial t}\Phi(z,t)| \le d_2(\|\left(\Phi_t^n\right)'(z)\|_{(X^n)'} \|z\|_q + 1).$$

This proves (H2). If $\Phi'_t(z) = 0$ then (3.6) yields

$$|\frac{\partial}{\partial t}\Phi(z,t)| \le d_1 |\Phi_t(z) + 1|^\gamma \le A(\Phi_t(z)^2 + 1)^{\gamma/2}.$$

This proves (H3) with $A(\Phi_t(z)^2+1)^{\gamma/2} = \theta_2(t, \Phi_t(z)) = -\theta_1(t, \Phi_t(z))$. Properties (H4)-(H6) are also easy. □

We now show that (H7) holds with $\ell(t) = \alpha t + \beta$, α, β positive constants independent of k. We split the proof into several lemmas.The first two were proved in [9]. We sketch their proofs for the readers convenience.

Lemma 3.5. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$. Then there exists $a \in \mathbb{R}$ with the following properties:*

- (i) *$(x', a) \in \Omega$ for some $x' \in \mathbb{R}^{N-1}$.*
- (ii) *If $(x', b) \in \Omega$ and $b \ge a$, then $(x', t) \in \Omega$ for all $a \le t \le b$.*

Proof. Let $e_N = (0, \dots, 0, 1) \in \mathbb{R}^N$, let $M = \max\{x \cdot e_N : x \in \overline{\Omega}\}$ and let $K = \{x \in \overline{\Omega} : x \cdot e_N = M\}$. Let $\nu : \partial\Omega \to \mathbb{R}^N$ be the outer unit normal field, and let $\mathcal{O} = \{x \in \partial\Omega : \nu(x) \cdot e_N > 0\}$. Then $\mathcal{O}$ is an open neighborhood of K in $\partial\Omega$ and, since K is compact, there is an $a < M$ such that the set $A = \{x \in \partial\Omega : x \cdot e_N \ge a\} \subset \mathcal{O}$. Thus, for every $(x', t) \in A$ there exists $\varepsilon > 0$ such that $(x', s) \notin \Omega$ if $t < s < t + \varepsilon$ and $(x', s) \in \Omega$ if $t - \varepsilon < s < t$. It follows that, for every $(x', t) \in A$,

$$(\{x'\} \times [a, M]) \cap \partial\Omega = \{(x', t)\} \quad \text{and} \quad \{x'\} \times [a, t) \subset \Omega$$

as claimed. ☐

Set

$$I(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p}\int_\Omega |u|^p,$$
$$J(v) := \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{q}\int_\Omega |v|^q,$$
$$I^\#(u) := 2\int_\Omega |\nabla u|^2 - \frac{1}{p}\int_\Omega |u|^p.$$

Lemma 3.6. *There is an even continuous function $\tau : H_0^1(\Omega) \to [0,\infty)$ with the following properties:*

(i) $I([(1-s)+s\tau(u)]u) \le I(u)$ *for every $u \in H_0^1(\Omega)$, $0 \le s \le 1$.*
(ii) *If* $I^\#(u) \le 0$ *then* $\tau(u) = 1$.
(iii) *If* $2I(u) \le \max_{t\ge 0} I(tu)$ *then* $I^\#(\tau(u)u) \le 0$.
(iv) $I^\#(\tau(u)u) \le \max\{\alpha I(u), 0\}$ *with* $\alpha := 2^{(3p-2)/(p-2)}$.

Proof. Fix $v \in H_0^1(\Omega)$ with $\|v\| = 1$ and define $0 < t_v^- < \widehat{t}_v < t_v^+ < T_v < \infty$ as follows:

$$I(\widehat{t}_v v) = \max_{t\ge 0} I(tv),$$
$$2I(tv) \ge \max_{t\ge 0} I(tu) \Longleftrightarrow t \in [t_v^-, t_v^+],$$
$$2(T_v)^2 = \frac{1}{p}|v|_p^p (T_v)^p.$$

For $t \ge 0$ let $\rho(tv)$ be the piecewise linear function such that $\rho(tv) = 0$ if $0 \le t \le t_v^-$, $\rho(\widehat{t}_v v) = \widehat{t}_v$, $\rho(tv) = T_v$ if $t_v^+ \le t \le T_v$, and $\rho(tv) = t$ if $t \ge T_v$. For $u = tv \in H_0^1(\Omega)$ with $\|v\| = 1$, $t \ge 0$, define

$$\tau(u) = \frac{\rho(tv)}{t}.$$

This function is continuous and satisfies (i), (ii), (iii), (iv). ☐

Lemma 3.7. *There exist $\alpha, \beta > 0$, depending only on Ω and p, with the following property: For every map $\varphi : X_k^n \to X^n$ such that $\varphi(z) = z$ if $\|z\| \ge R$, there exist a map $\psi : X_k^n \times [0,\infty) \to X^n$ and an $R' \ge R$ which satisfy:*

(i) $\psi(z,0) = \varphi(z)$ *for every* $z \in X_k^n$.
(ii) $\Phi_0(\psi(z,s)) \le -1$ *if* $\|z\| \ge R'$ *or* $s \ge R'$.
(iii) $\Phi_0(\psi(z,s)) \le \max\{\alpha\Phi_0(\varphi(z)) + \beta, 0\}$ *for every* $z \in X_k^n$, $s \ge 0$.

Proof. Let a be as in Lemma 3.5. We may assume without loss of generality that $a = 0$. Let $\tau : H_0^1(\Omega) \to \mathbb{R}$ be as in Lemma 3.6. We write $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \equiv \mathbb{R}^N$. For each $u \in H_0^1(\Omega) \subset H^1(\mathbb{R}^N)$ and $0 \le s \le 2$, define

$$u_s(x) := \begin{cases} [(1-s)+s\tau(u)]u(x) & \text{if } 0 \le s \le 1 \\ \tau(u)u(x', sx_N) & \text{if } x_N \ge 0,\ 1 \le s \le 2 \\ \tau(u)u(x', x_N) & \text{if } x_N \le 0,\ 1 \le s \le 2 \end{cases}$$

Then $u_s \in H_0^1(\Omega)$, and

$$\int_\Omega |\nabla u_s|^2 \le s\int_\Omega |\nabla(\tau(u)u)|^2, \quad \int_\Omega |u_s|^p \ge s^{-1}\int_\Omega |\tau(u)u|^p, \quad \text{if } 1 \le s \le 2.$$

Hence, for $1 \le s \le 2$,

$$I(u_s) \le sI(u_s) \le \frac{s^2}{2}\int_\Omega |\nabla(\tau(u)u)|^2 - \frac{1}{p}\int_\Omega |\tau(u)u|^p \le I^{\#}(\tau(u)u). \tag{3.7}$$

Let $\Omega_2 := \{(x', \frac{1}{2}x_N) : x \in \Omega,\ x_N \ge 0\} \cup \{x \in \Omega : x_N \le 0\}$. By Lemma 3.5, $\Omega_2 \subsetneq \Omega$. Thus, we may choose $\omega \in C_c^\infty(\Omega \setminus \Omega_2)$, $\omega \ne 0$, with

$$\int_\Omega |\nabla\omega|^2 = \int_\Omega |\omega|^p. \tag{3.8}$$

Define $\psi = (\psi^+, \psi^-) : X_k^n \times [0,\infty) \to X^n$ by

$$\psi^+(z,s) := \begin{cases} [\varphi^+(z)]_s & \text{if } 0 \le s \le 2 \\ [\varphi^+(z)]_2 + (s-2)\omega & \text{if } 2 \le s \end{cases}$$

$$\psi^-(z,s) := \begin{cases} [(1-s) + \alpha s]\varphi^-(z) & \text{if } 0 \le s \le 1 \\ \alpha\varphi^-(z) & \text{if } 1 \le s \end{cases}$$

with $\alpha := 2^{(3p-2)/(p-2)}$. Then (i) holds. Lemma 3.6, together with (3.7), yields

$$I(\psi^+(z,s)) \le \begin{cases} I(\varphi^+(z)) & \text{if } 0 \le s \le 1 \\ I^{\#}(\tau(\varphi^+(z))\varphi^+(z)) & \text{if } 1 \le s \le 2 \\ I^{\#}(\tau(\varphi^+(z))\varphi^+(z)) + I((s-2)\omega) & \text{if } 2 \le s\,. \end{cases}$$

Indeed, the first inequality follows from Lemma 3.6(i), the second one is a consequence of (3.7), and the third inequality follows from the second one because ω and u_2 have disjoint supports for every $u \in H_0^1(\Omega)$ and, therefore,

$$I([\varphi^+(z)]_2 + (s-2)w) = I([\varphi^+(z)]_2) + I((s-2)w).$$

Clearly,

$$J(\psi^-(z,s)) \ge \begin{cases} J(\varphi^-(z)) & \text{if } 0 \le s \le 1 \\ \alpha J(\varphi^-(z)) & \text{if } 1 \le s\,. \end{cases}$$

Hence, Lemma 3.6 yields

$$\Phi_0(\psi(z,s)) = I(\psi^+(z,s)) - J(\psi^-(z,s)) \le \max\{\alpha\Phi_0(\varphi(z)) + \beta,\ 0\}, \tag{3.9}$$

with $\beta := I(\omega)$. Thus, (iii) holds. Finally, let $R' \ge \max\{2, R\}$ be such that

$$\begin{gathered} I(u) \le -1, \quad \tau(u) = 1, \quad I^{\#}(u) \le -I(\omega) - 1 \quad \text{if } u \in X_k^+,\ \|u\| \ge R', \\ I((s-2)\omega) \le -\max\{I^{\#}(\tau(\varphi^+(z))\varphi^+(z)) : z \in X_k^n\} - 1 \quad \text{if } s \ge R', \\ J(v) \ge \max\{I(\psi^+(z,s)) : z \in X_k^n,\ s \ge 0\} + 1 \quad \text{if } v \in X_n^-,\ \|v\| \ge R'. \end{gathered}$$

Since, for $z = (u,v)$,

$$\max\{\|u\|, \|v\|\} \ge R' \Longrightarrow \|z\| \ge R \Longrightarrow \varphi(z) = z,$$

the previous inequalities yield

$$\begin{gathered} \Phi_0(\psi(z,s)) \le I(\psi^+(z,s)) \le -1 \quad \text{if } z = (u,v),\ \max\{\|u\|, s\} \ge R', \\ \Phi_0(\psi(z,s)) = I(\psi^+(z,s)) - J(\psi^-(z,s)) \le -1 \quad \text{if } z = (u,v),\ \|v\| \ge R'. \end{gathered}$$

This proves (ii). $\square$

Proposition 3.8. *The function Φ satisfies (H7). More precisely, there exist $\alpha, \beta > 0$, depending only on Ω and p, with the following property: For every odd map $\sigma : X_k^n \to X^n$ such that $\sigma(z) = z$ if $\|z\| \geq R$, there exist an odd map $\widetilde{\sigma} : X_{k+1}^n \to X^n$ and an $\widetilde{R} \geq R$ which satisfy:*

(i) $\widetilde{\sigma}(z) = \sigma(z)$ *if* $z \in X_k^n$.
(ii) $\widetilde{\sigma}(z) = z$ *if* $\|z\| \geq \widetilde{R}$.
(iii) $\Phi_0(\widetilde{\sigma}(z)) \leq \alpha\Phi_0(\sigma(\pi(z))) + \beta$ *where* $\pi : X_{k+1}^n \to X_k^n$ *is the orthogonal projection.*

Proof. For $\varphi := \sigma$ and R, let $\psi : X_k^n \times [0, \infty) \to X^n$ and $R' > R$ be as in Lemma 3.7. Fix $e \in X_{k+1}^+$ orthogonal to X_k^+ with $\|e\| = 1$, and extend σ to the half space $W = \{(z, se) : z \in X_k^n,\ s \geq 0\} \subset X_{k+1}^n$ by setting

$$\sigma(z, se) = \psi(z, s) \quad \text{if } z \in X_k^n,\ s \geq 0.$$

Then there exists $R'' \geq R'$ such that $\Phi_0(w) \leq -1$ and $\Phi_0(\sigma(w)) \leq -1$ for all $w \in W$ with $\|w\| \geq R''$. The sublevel set

$$D := \{z \in X^n : \Phi_0(z) \leq -1\}$$

is homotopy equivalent to the unit sphere in X^n. Therefore, it is contractible. Hence, there exists a homotopy

$$\Psi : \{w \in W : \|w\| = R''\} \times [0, 1] \to D$$

such that $\Psi(w, 0) = \sigma(w)$, $\Psi(w, 1) = w$, and $\Psi(z, t) = z$ for $z \in X_k^n$, $t \in [0, 1]$. Let $\widetilde{R} := R'' + 1$ and define

$$\widetilde{\sigma}(w) = \begin{cases} \sigma(w) & \text{if } w \in W,\ \|w\| \leq R'' \\ \frac{\|w\|}{R''}\Psi(R''\frac{w}{\|w\|}, \|w\| - R'') & \text{if } w \in W,\ R'' \leq \|w\| \leq \widetilde{R} \\ w & \text{if } w \in W,\ \widetilde{R} \leq \|w\| \\ -\widetilde{\sigma}(-w) & \text{if } -w \in W \end{cases}$$

Since σ is odd, $\widetilde{\sigma}$ is well defined and it is, by definition, an odd extension of σ to X_{k+1}^n which satisfies $\widetilde{\sigma}(w) = w$ if $\|w\| \geq \widetilde{R}$. Note that $su \in D$ if $u \in D$ and $s \geq 1$. Hence $\Phi_0(\widetilde{\sigma}(w)) \leq -1$ if $\|w\| \geq R''$ and, by Lemma 3.7,

$$\Phi_0(\widetilde{\sigma}(z, se)) = \Phi_0(\psi(z, |s|)) \leq \alpha\Phi_0(\sigma(z)) + \beta \quad \text{if } \|\,(z, se)\,\| \leq R'',$$

as claimed. □

We now give some estimates for the values c_k defined in (2.2). We shall use the semiclassical inequality of Cwickel [13], Lieb [20] and Rosenbljum [24] which states that, if $V \in L^{N/2}(\mathbb{R}^N)$, $N \geq 3$, then the number of negative or zero eigenvalues counted with multiplicity of the Schrödinger operator $-\Delta + V$ on $L^2(\mathbb{R}^N)$ is bounded by

$$C_0 \int_{\mathbb{R}^N} |V^-|^{N/2}$$

for some $C_0 > 0$.

Lemma 3.9. *There exist positive constants B_1, B_2 such that*

$$B_1 k^\nu \leq c_k \leq B_2 k^\nu$$

where $\nu := \frac{2}{N}\frac{p}{p-2}$.

Proof. For every $n, k \geq 1$ there exists $z_k^n \in X^n$ such that

$$\Phi_0^n(z_k^n) \leq c_k^n, \quad (\Phi_0^n)'(z_k^n) = 0, \quad \mu_0(z_k^n) \geq k + n,$$

where $\mu_0(z_k^n)$ denotes the Morse index plus the nullity of z_k^n [27]. Thus $z_k^n = (u_k^n, 0)$, and u_k^n is a critical point of the restriction of Φ_0 to X^+ with Morse index

$$\mu_0(u_k^n) \geq k.$$

The semiclassical inequality of Cwickel [13], Lieb [20], and Rosenbljum [24] gives

$$k \leq \mu_0(u_k^n) \leq C_1 \int_\Omega |u_k^n|^\theta$$

with $\theta = \frac{N}{2}(p-2)$. Using Hölder's inequality we obtain

$$B_1 k^\nu \leq C_2 \Big(\int_\Omega |u_k^n|^\theta \Big)^{p/\theta} \leq \frac{(p-2)}{2p} |u_k^n|_p^p = \Phi_0^n(z_k^n) \leq c_k^n$$

with $\nu = \frac{p}{\theta} = \frac{2}{N}\frac{p}{p-2}$. This yields the first inequality of this lemma. We turn to the second one. Let

$$\Gamma_k^+ := \{\sigma^+ \in C^0(X_k^+, X^+) : \sigma^+ \text{is odd, and } \sigma^+(u) = u \text{ if } \|u\| \text{ is large enough}\}.$$

If $\sigma^+ \in \Gamma_k^+$ and $\sigma \in C^0(X_k^n, X^n)$ is given by $\sigma(u,v) = (\sigma^+(u), v)$, then $\Phi_0(\sigma(z)) \leq 0$ for $\|z\|$ large enough. Arguing as in the proof of Proposition 3.8 we may assume that $\sigma \in \Gamma_k^n$, and that

$$\max_{z \in X_k^n} \Phi_0(\sigma(z)) = \max_{u \in X_k^+} \Phi_0(\sigma^+(u)).$$

Hence, inequality (40) in [3] yield

$$c_k^n \leq \inf_{\sigma^+ \in \Gamma_k^+} \max_{u \in X_k^+} \Phi_0(\sigma^+(u)) \leq B_2 k^\nu.$$

for all n. Our claim follows. □

Proof of Theorem 3.1. We have shown that assumptions (H1)-(H7) of Theorem 2.1 hold. We now show that the sequence (2.1) is unbounded. Assume, by contradiction, there exists a $B > 0$ such that

$$c_{k+1} - c_k \leq B(\theta(c_{k+1}) + \theta(c_k) + 1)$$

where $\theta(s) = -\theta_1(t,s) = \theta_2(t,s) = A(s^2+1)^{\gamma/2}$. As in [23, (10.47)], this implies the existence of a constant $D > 0$ such that

$$c_k \leq D k^{1/(1-\gamma)} \quad \text{for all } k.$$

This contradicts Lemma 3.9 if $\gamma < (\frac{1}{p} - \frac{1}{2^*})N$, as assumed. Thus, Theorem 2.1 yields a sequence of critical values $(\widetilde{c}_k)$ of Φ_1 which satisfy

$$\zeta_2(1, c_k) < \widetilde{c}_k \leq \zeta_2(1, \alpha(c_k+1) + \beta) \tag{3.10}$$

with $a, \beta > 0$ as in Proposition 3.8 (see Remark *(f)* at the end of section 2). For our particular θ, we have that $s \leq \zeta_2(1,s) \leq A_1(s+1)$. Thus, (3.10) and Lemma 3.9 yield

$$C_1 k^\nu \leq \widetilde{c}_k \leq C_2 k^\nu$$

with $\nu := \frac{2p}{N(p-2)}$, as claimed. □

Proof of Theorem 1.1. Assume that f satisfies the assumptions of Theorem 1.1. We show that it satisfies also those of Theorem 3.1, with the same γ. Since $0 \leq \sigma - 1 \leq \frac{q}{p}(\gamma p - 1)$ and $p \leq q$,

$$|f_z(x,u,v)| \leq d(|u|^{\gamma p-1} + |v|^{\gamma q-1} + 1).$$

Using, in addition, Young's inequality and the fact that $\gamma \geq \frac{1}{p}$ we obtain

$$\begin{aligned} |f_z(x,z)z| &\leq d_1(|u|^{\gamma p} + |v|^{\gamma q} + |u|^{\gamma p-1}|v| + |u||v|^{\sigma-1} + 1) \\ &\leq d(|u|^{\gamma p} + |v|^{\gamma q} + 1). \end{aligned}$$

Moreover,

$$\begin{aligned} |f(x,u,v)| - |f(x,0,0)| &\leq |f(x,u,v) - f(x,u,0)| + |f(x,u,0) - f(x,0,0)| \\ &\leq \int_0^{|v|} |f_v(x,u,\xi)|d\xi + \int_0^{|u|} |f_u(x,\xi,0)|d\xi \\ &\leq d_2(|u|^{\gamma p-1}|v| + |u|^{\gamma p} + |v|^{\gamma q} + 1) \\ &\leq d(|u|^{\gamma p} + |v|^{\gamma q} + 1) \end{aligned}$$

Thus, f satisfies the assumptions of Theorem 3.1. □

In particular, if $f(x,u,v) = g(x)u$ and $p < \frac{2N-2}{N-2}$, Theorem 3.1 yields infinitely many solutions of the perturbed elliptic equation

$$\begin{gathered} -\Delta u = |u|^{p-2}u + g(x) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

which is Bahri and Lions's result [3]. The upper estimates for their energy were recently established by Castro and Clapp [9].

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