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LOW REGULARITY SOLUTIONS FOR DIRAC-KLEIN-GORDON EQUATIONS IN ONE SPACE DIMENSION

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ABSTRACT. We establish the existence of local and global solutions for Dirac-Klein-Gordon equations in one space dimension. This is done using a null form estimate and a fixed point argument.

1. Introduction and Main Results

In this paper, we study the Cauchy problem for the Dirac-Klein-Gordon system. The unknown quantities are a spinor field $\psi : \mathbb{R} \times \mathbb{R}^1 \mapsto \mathbb{C}^4$ and a scalar field $\phi : \mathbb{R} \times \mathbb{R}^1 \mapsto \mathbb{R}$. The evolution equations for these fields are

$$\begin{gathered}\mathcal{D}\psi = \phi\psi, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^1; \\ \Box\phi = \overline{\psi}\psi; \\ \psi(0,x) := \psi_0(x), \quad \phi(0,x) = \phi_0(x), \quad \phi_{,t}(0,x) = \phi_1(x),\end{gathered} \tag{1.1}$$

where $\mathcal{D}$ is the Dirac operator, $\mathcal{D} := -i\gamma^\mu\partial_\mu$, $\mu = 0,1$, and γ^μ are the Dirac matrices, the wave operator $\Box = -\partial_{tt} + \partial_{xx}$,and $\overline{\psi} = \psi^\dagger\gamma^0$, and $\dagger$ is the complex conjugate transpose.

The purpose of this work is to demonstrate a variant null form estimate, by employing the solution representations in Fourier transform of the DKG equations. We will take advantage of the null form structure depicted in the nonlinear term $\overline{\psi}\psi$, which has been observed by [11] and [3]. We interpret the null form in a way that is different from that given in Bournaveas' paper [3]. Equipping with this estimate, we can lower the regularity of the spinor field.

For the DKG system, there are many conserved quantities which are not positive definite, such as the energy. However the known positive conserved quantity is the law of conservation of charge,

$$\int |\psi(t)|^2\,dx = \text{constant} \tag{1.2}$$

which is applicable to lead to the global existence result, once the local existence result is established, see [3] and [7].

In 1973, Chadam showed that the Cauchy problem for the DKG equations has a global unique solution for $\psi_0 \in H^1$, $\phi_0 \in H^1$, $\phi_1 \in L^2$, see [4]. In 1993, Zheng

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proved that there exists a global weak solution to the Cauchy problem of a modified DKG equations, based on the technique of compensated compactness, with $\psi_0 \in L^2$, $\phi_0 \in H^1$, $\phi_1 \in L^2$, see [14]. In 2000, Bournaveas derived a new proof of a global existence for the DKG equations, via a null form estimate, if $\psi_0 \in L^2$, $\phi_0 \in H^1$, $\phi_1 \in L^2$, see [1]. In 2002, Fang gave a direct proof for (1.1), based on a variant null form estimate, which is straight forward, and the result is parallel to Bournaveas', see [7]

The outline of this paper is as follows. First we derive some solutions representations in Fourier transform, depending on various purposes. Next we prove some a priori estimates of solutions for Dirac equation and for wave equation. Then we show a local and global results for (1.1), employing the null form estimate together with other estimates derived previously, and a fixed point argument. Finally we show the null form estimate.

The main result in this work is as follows.

Theorem 1.1 (Local Existence)**.** *Let* $0 < \epsilon \le \frac{1}{4}$ *and* $0 < \delta \le 2\epsilon$*. If the initial data of* (1.1) $\psi_0 \in H^{-\frac{1}{4}+\epsilon}$, $\phi_0 \in H^{\frac{1}{2}+\delta}$, $\phi_1 \in H^{-\frac{1}{2}+\delta}$, *then there is a unique local solution for* (1.1)*.*

Theorem 1.2 (Global Existence)**.** *Let* $\delta > 0$*. If the initial data of* (1.1) $\psi_0 \in L^2$, $\phi_0 \in H^{\frac{1}{2}+\delta}$, $\phi_1 \in H^{-\frac{1}{2}+\delta}$, *then there is a unique global solution for* (1.1)*.*

Remarks. 1. The DKG equations follow from the Lagrangian

$$\int_{\mathbb{R}^{1+1}} \big\{ |\nabla\phi|^2 - |\phi_t|^2 - \overline{\psi}\mathcal{D}\psi - \phi\overline{\psi}\psi \big\} dx\, dt. \tag{1.3}$$

2. The Dirac-Klein-Gordon system must be

$$\mathcal{D}\psi = \phi\psi; \Box\phi + m^2\phi = \overline{\psi}\psi, \tag{1.4}$$

3. $\widehat{\mathcal{D}}^2 = \widehat{\Box} I$, where I is the 4×4 identity matrix.
4. $\overline{\psi}\psi = \psi^\dagger\gamma^0\psi = |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2$, where ψ_j are the component functions of the vector function ψ, which take values in $\mathbb{C}$.

The case $\delta = 0$ is critical in the following sense. Assuming that the initial data (ϕ_0, ϕ_1) are in $H^{\frac{1}{2}} \times H^{-1/2}$ does not imply that $\phi(t, \cdot)$ is bounded. In fact, it is a BMO function. One of the motivations for proving the existence of global solution with low regularity, is based on an observation made by Grillakis, which is that the initial data of (1.1): $\psi_0 \in L^2$, $\phi_0 \in H^{\frac{1}{2}}$, $\phi_1 \in H^{-\frac{1}{2}}$, is a right space for the existence of an invariant measure, see [1] and [12], resulted from the DKG equations.

2. Solution Representation

In what follows, we denote by (t, x) the time-space variables and by (τ, ξ) the dual variables with respect to the Fourier transform. We will use $\alpha = \frac{1}{4} - \epsilon$ in this paper. We will also often skip the constant in the inequalities. For convenience, we also denote the multipliers by

$$\begin{aligned}\widehat{E}(\tau,\xi) &= |\tau|+|\xi|+1\\ \widehat{S}(\tau,\xi) &= \big||\tau|-|\xi|\big|+1\\ \widehat{W}(\tau,\xi) &= \tau^2-|\xi|^2\\ \widehat{D}(\tau,\xi) &= \gamma^0\tau+\gamma^1\xi\\ \widehat{M}(\xi) &= |\xi|+1\,.\end{aligned}$$

We use $\widehat{W}$ and $\widehat{D}$ as the symbols of the wave and Dirac operators respectively. Consider the Dirac equation,

$$\begin{gathered}\mathcal{D}\psi = G, \quad (t,x)\in\mathbb{R}^1\times\mathbb{R}^1,\\ \psi(0)=\psi_0.\end{gathered} \tag{2.1}$$

First by taking the Fourier transform on (2.1) over the space variable and solving the resulting ODE, we can formally write down the solution as follows.

$$\begin{aligned}\widetilde{\psi}(t,\xi) &= \frac{e^{it|\xi|}}{2|\xi|}\widehat{D}(|\xi|,\xi)\gamma^0\widehat{\psi}_0(\xi)+\frac{e^{-it|\xi|}}{2|\xi|}\widehat{D}(|\xi|,-\xi)\gamma^0\widehat{\psi}_0(\xi)\\ &\quad+\int_0^t\frac{e^{i(t-s)|\xi|}}{2|\xi|}\widehat{D}(|\xi|,\xi)i\widetilde{G}(s,\xi)\,ds+\int_0^t\frac{e^{-i(t-s)|\xi|}}{2|\xi|}\widehat{D}(|\xi|,-\xi)i\widetilde{G}(s,\xi)\,ds.\end{aligned} \tag{2.2}$$

Rewriting the inhomogeneous terms in (2.2) gives

$$\begin{aligned}\widetilde{\psi}(t,\xi) &= \big[\frac{e^{it|\xi|}}{2|\xi|}\widehat{D}(|\xi|,\xi)+\frac{e^{-it|\xi|}}{2|\xi|}\widehat{D}(|\xi|,-\xi)\big]\gamma^0\widehat{\psi}_0(\xi)\\ &\quad+\int\big[\frac{e^{it\tau}-e^{it|\xi|}}{2|\xi|(\tau-|\xi|)}\widehat{D}(|\xi|,\xi)+\frac{e^{it\tau}-e^{-it|\xi|}}{2|\xi|(\tau+|\xi|)}\widehat{D}(|\xi|,-\xi)\big]\widehat{G}(\tau,\xi)d\tau.\end{aligned} \tag{2.3}$$

Now we split the function $\widehat{G}$ into several parts in the following manner. Consider $\widehat{a}(\tau)$ a cut-off function equals 1 if $|\tau|\le\frac{1}{2}$ and equals 0 if $|\tau|\ge 1$, $\widehat{a}_6(\tau)=\widehat{a}(\frac{\tau}{6})$, and denote by $h(\tau)$ the Heaviside function. For simplicity, let us write

$$\begin{aligned}\widehat{G}_\pm(\tau,\xi) &:= h(\pm\tau)\widehat{a}(\tau\pm|\xi|)\widehat{G}(\tau,\xi),\\ \widehat{G}_f(\tau,\xi) &:= \widehat{G}(\tau,\xi)-\big(\widehat{G}_+(\tau,\xi)+\widehat{G}_-(\tau,\xi)\big),\\ \widehat{D}_\pm &:= \widehat{D}(|\xi|,\pm\xi).\end{aligned}$$

Note that $\widehat{G}_\pm$ are supported in the regions $\{(\tau,\xi):\pm\tau>0,|\tau\mp|\xi||\le 1\}$ respectively. Using the decomposition of the forcing term $\widehat{G}=\widehat{G}_f+\widehat{G}_++\widehat{G}_-$, the inhomogeneous term in (2.3) can be written as

$$\begin{aligned}&\int\big[\frac{e^{it\tau}-e^{it|\xi|}}{2|\xi|(\tau-|\xi|)}\widehat{D}(|\xi|,\xi)+\frac{e^{it\tau}-e^{-it|\xi|}}{2|\xi|(\tau+|\xi|)}\widehat{D}(|\xi|,-\xi)\big]\widehat{G}_f(\tau,\xi)d\tau\\ &=\int e^{it\tau}\frac{\widehat{D}(\tau,\xi)}{\tau^2-|\xi|^2}\widehat{G}_f d\tau-e^{it|\xi|}\frac{\widehat{D}_+}{2|\xi|}\int\frac{\widehat{G}_f}{\tau-|\xi|}d\tau-e^{-it|\xi|}\frac{\widehat{D}_-}{2|\xi|}\int\frac{\widehat{G}_f}{\tau+|\xi|}d\tau,\end{aligned} \tag{2.4}$$

$$
\begin{aligned}
&\int \frac{e^{it\tau}-e^{it|\xi|}}{2|\xi|(\tau-|\xi|)}\widehat{D}_+(\widehat{G}_+ + \widehat{G}_-)d\tau \\
&= e^{it|\xi|}\frac{\widehat{D}_+}{2|\xi|}\int \frac{e^{it(\tau-|\xi|)}-1}{\tau-|\xi|}(\widehat{G}_+ + \widehat{a}(\tau)\widehat{G}_-)d\tau \\
&\quad + \int e^{it\tau}\frac{(1-\widehat{a}(\tau))\widehat{D}_+\widehat{G}_-}{2|\xi|(\tau-|\xi|)}d\tau - e^{it|\xi|}\frac{\widehat{D}_+}{2|\xi|}\int \frac{(1-\widehat{a}(\tau))\widehat{G}_-}{\tau-|\xi|}d\tau,
\end{aligned} \tag{2.5}
$$

$$
\begin{aligned}
&\int \frac{e^{it\tau}-e^{-it|\xi|}}{2|\xi|(\tau+|\xi|)}\widehat{D}_-(\widehat{G}_+ + \widehat{G}_-)d\tau \\
&= e^{-it|\xi|}\frac{\widehat{D}_-}{2|\xi|}\int \frac{e^{it(\tau+|\xi|)}-1}{\tau+|\xi|}(\widehat{a}(\tau)\widehat{G}_+ + \widehat{G}_-)d\tau \\
&\quad + \int e^{it\tau}\frac{(1-\widehat{a}(\tau))\widehat{D}_-\widehat{G}_+}{2|\xi|(\tau+|\xi|)}d\tau - e^{-it|\xi|}\frac{\widehat{D}_-}{2|\xi|}\int \frac{(1-\widehat{a}(\tau))\widehat{G}_+}{\tau+|\xi|}d\tau,
\end{aligned} \tag{2.6}
$$

Combining (2.3)-(2.6), we can give a formula for $\widehat{\psi}$, namely

$$
\widehat{\psi}(\tau,\xi) = \sum_{k=0}^{\infty}\big(\delta_+^{(k)}(\tau,\xi)\widehat{A}_{+,k}(\xi) + \delta_-^{(k)}(\tau,\xi)\widehat{A}_{-,k}(\xi)\big) + \widehat{K}(\tau,\xi), \tag{2.7}
$$

where $\delta_\pm(\tau,\xi)$ are the delta functions supported on $\{\tau = \pm|\xi|\}$ respectively, $\delta^{(k)}$ mean derivatives of the delta function, and

$$
\widehat{K}(\tau,\xi) := \frac{\widehat{D}(\tau,\xi)}{\widehat{W}(\tau,\xi)}\widehat{G}_f + \frac{(1-\widehat{a}_6(\tau))\widehat{D}_+\widehat{G}_-}{2|\xi|(\tau-|\xi|)} + \frac{(1-\widehat{a}_6)\widehat{D}_-\widehat{G}_+}{2|\xi|(\tau+|\xi|)}, \tag{2.8}
$$

$$
\widehat{A}_{\pm,0}(\xi) := \frac{\widehat{D}_\pm}{2|\xi|}\big[\gamma^0\widehat{\psi}_0 - \int \frac{\widehat{G}_f + (1-\widehat{a}_6(\lambda))\widehat{G}_\mp}{\lambda\mp|\xi|}d\lambda\big], \tag{2.9}
$$

$$
\widehat{A}_{\pm,k}(\xi) := \frac{\widehat{D}_\pm(-1)^k}{2|\xi|k!}\int(\lambda\mp|\xi|)^{k-1}\big[\widehat{G}_\pm + \widehat{a}_6(\lambda)\widehat{G}_\mp\big]d\lambda. \tag{2.10}
$$

Now we split ψ in a different manner. Consider the cut-off function $\widehat{b}(\tau)$ equals 1 if $|\tau| < R$, and equals 0 if $|\tau| > 2R$. Let $\widehat{b}(\tau) + \widehat{c}(\tau) = 1$. Applying (2.3), we can give the following formula for $\widehat{\psi}$.

$$
\widehat{\psi}(\tau,\xi) = \sum_{k=0}^{\infty}\big(\delta_+^{(k)}(\tau,\xi)\widehat{U}_{+,k}(\xi) + \delta_-^{(k)}(\tau,\xi)\widehat{U}_{-,k}(\xi)\big) + \widehat{U}(\tau,\xi), \tag{2.11}
$$

where

$$
\begin{gathered}
\widehat{U}_{\pm,0}(\xi) := \frac{\widehat{D}_\pm}{2|\xi|}\big[\gamma^0\widehat{\psi}_0 - \int \frac{\widehat{c}(\lambda\mp|\xi|)}{\lambda\mp|\xi|}\widehat{G}d\lambda\big], \\
\widehat{U}_{\pm,k}(\xi) := \frac{\widehat{D}_\pm(-1)^k}{2|\xi|k!}\int(\lambda\mp|\xi|)^{k-1}\widehat{b}(\lambda\mp|\xi|)\widehat{G}d\lambda, \\
\widehat{U}(\tau,\xi) := \big[\frac{\widehat{D}_+\widehat{c}(\tau-|\xi|)}{2|\xi|(\tau-|\xi|)} + \frac{\widehat{D}_-\widehat{c}(\tau+|\xi|)}{2|\xi|(\tau+|\xi|)}\big]\widehat{G}(\tau,\xi).
\end{gathered} \tag{2.12}
$$

Consider the wave equation,

$$
\begin{gathered}
\Box\phi = F, \quad (t,x)\in\mathbb{R}^1\times\mathbb{R}^1, \\
\phi(0) = \phi_0, \quad \phi_t(0) = \phi_1.
\end{gathered} \tag{2.13}
$$

Taking Fourier transform on (2.13) and solving the resulting ODE gives

$$\widetilde{\phi}(t,\xi) = \cos t|\xi|\widehat{\phi}_0(\xi) + \frac{\sin t|\xi|}{|\xi|}\widehat{\phi}_1(\xi) + \int_0^t \frac{\sin(t-s)|\xi|}{|\xi|}\widetilde{F}(s,\xi)ds. \tag{2.14}$$

Thus we can rewrite it as follows.

$$\begin{aligned}\widetilde{\phi}(t,\xi) &= \frac{e^{it|\xi|}+e^{-it|\xi|}}{2}\widehat{\phi}_0(\xi) + \frac{e^{it|\xi|}-e^{-it|\xi|}}{i2|\xi|}\widehat{\phi}_1(\xi) \\ &\quad + \frac{-1}{2|\xi|}\int \frac{e^{it\tau}-e^{it|\xi|}}{\tau-|\xi|}\widehat{F}(\tau,\xi)d\tau + \frac{1}{2|\xi|}\int \frac{e^{it\tau}-e^{-it|\xi|}}{\tau+|\xi|}\widehat{F}(\tau,\xi)d\tau.\end{aligned} \tag{2.15}$$

For convenience, we define the following

$$\begin{aligned}\widehat{v}_{\pm,0}(\xi) &:= \frac{1}{2}\widehat{\phi}_0(\xi) \pm \frac{1}{i2|\xi|}\widehat{\phi}_1(\xi) \pm \frac{1}{2|\xi|}\int \frac{\widehat{c}(\lambda \mp |\xi|)}{\lambda \mp |\xi|}\widehat{F}(\lambda,\xi)d\lambda, \\ \widehat{v}_{\pm,k}(\xi) &:= \mp\frac{(-1)^k}{2|\xi|k!}\int (\lambda \mp |\xi|)^{k-1}\widehat{b}(\lambda \mp |\xi|)\widehat{F}(\lambda,\xi)d\lambda, \\ \widehat{v}(\tau,\xi) &:= \frac{-1}{2|\xi|}\Big[\frac{\widehat{c}(\tau-|\xi|)}{\tau-|\xi|} - \frac{\widehat{c}(\tau+|\xi|)}{\tau+|\xi|}\Big]\widehat{F}(\tau,\xi).\end{aligned} \tag{2.16}$$

Combining (2.15) and (2.16), and invoking the cut-off function, we have

$$\widehat{\phi}(\tau,\xi) = \sum_{k=0}^{\infty}\big(\widehat{V}_{+,k}(\tau,\xi) + \widehat{V}_{-,k}(\tau,\xi)\big) + \widehat{V}(\tau,\xi) + \sum_{k=0}^{\infty}\widehat{N}_k(\tau,\xi) + \widehat{N}(\tau,\xi), \tag{2.17}$$

where

$$\begin{gathered}\widehat{V}_{\pm,k}(\tau,\xi) = \big(1-\widehat{a}(\xi)\big)\delta_{\pm}^{(k)}(\tau,\xi)\widehat{v}_{\pm,k}(\xi), \\ \widehat{V}(\tau,\xi) = \big(1-\widehat{a}(\xi)\big)\widehat{v}(\tau,\xi), \\ \widehat{N}_k(\tau,\xi) = \widehat{a}(\xi)\big[\delta_{+}^{(k)}(\tau,\xi)\widehat{v}_{+,k}(\xi) + \delta_{-}^{(k)}(\tau,\xi)\widehat{v}_{-,k}(\xi)\big], \\ \widehat{N}(\tau,\xi) = \widehat{a}(\xi)\widehat{v}(\tau,\xi).\end{gathered}$$

Remark. We need to localize the solutions for Dirac equation and wave equation due to the presence of the delta function.

3. Estimates

To localize the solution in time, let $\varphi(t)$ be a cut-off function such that $\varphi(t)$ equals 1 if $|t| \le 1/2$, and equals 0 if $|t| > 1$, and $\varphi_T(t) = \varphi(t/T)$. Notice that, for an arbitrary function $f(t,x)$, we have

$$\|\widehat{\varphi}_T * \widehat{f}\|_{L^2} = \|\varphi_T f\|_{L^2} \le \|\varphi_T\|_{L^\infty}\|f\|_{L^2}. \tag{3.1}$$

For the Dirac equation (2.1), using (2.11), we define

$$\widehat{\Psi}_T(\tau,\xi) = \widehat{\varphi}_T * \sum_{k=0}^{\infty}\big(\delta_{+}^{(k)}\widehat{U}_{+,k} + \delta_{-}^{(k)}\widehat{U}_{-,k}\big)(\tau,\xi) + \widehat{U}(\tau,\xi), \tag{3.2}$$

Lemma 3.1. *Let $\epsilon > 0$ and $TR \sim 1$. If $\psi_0 \in H^{-\alpha}$, then we have*

$$\big\|\widehat{S}^{\frac{1}{2}}\widehat{M}^{-\alpha}\widehat{\Psi}_T\big\|_{L^2(\mathbb{R}^1\times\mathbb{R}^1)} \le C\big(\|\psi_0\|_{H^{-\alpha}} + T^{\epsilon}\big\|\frac{\widehat{G}}{\widehat{M}^{\alpha}\widehat{S}^{\frac{1}{2}-\epsilon}}\big\|_{L^2}\big). \tag{3.3}$$

We will only outline the proof. For more details, please see [8].

Proof. Applying formulae (2.11), we can derive the following bounds:

$$\big\|\widehat{S}^{\frac{1}{2}}\widehat{M}^{-\alpha}\widehat{\varphi}_T*\big(\delta_\pm\widehat{U}_{\pm,0}\big)\big\|_{L^2}\le C\big(\|\psi_0\|_{H^{-\alpha}}+T^\epsilon\big\|\frac{\widehat{G}}{\widehat{M}^\alpha\widehat{S}^{\frac{1}{2}-\epsilon}}\big\|_{L^2}\big),$$

$$\big\|\widehat{S}^{\frac{1}{2}}\widehat{M}^{-\alpha}\widehat{\varphi}_T*\big(\delta_\pm^{(k)}\widehat{U}_{\pm,k}\big)\big\|_{L^2}\le C\frac{(4RT)^k}{k!}T^\epsilon\big\|\frac{\widehat{G}}{\widehat{M}^\alpha\widehat{S}^{\frac{1}{2}-\epsilon}}\big\|_{L^2},$$

$$\big\|\widehat{S}^{\frac{1}{2}}\widehat{M}^{-\alpha}\widehat{U}\big\|_{L^2(\mathbb{R}^1\times\mathbb{R}^1)}\le CT^\epsilon\big\|\frac{\widehat{G}}{\widehat{M}^\alpha\widehat{S}^{\frac{1}{2}-\epsilon}}\big\|_{L^2},$$

Combining these estimates, we have (3.3). □

Consider two Dirac equations,

$$\begin{gathered}\mathcal{D}\psi_j=G_j,\quad j=1,2,\\ \psi_j(0)=\psi_{0j}.\end{gathered}\tag{3.4}$$

For the solutions of this system, we have the following key estimate whose proof will be presented in the last section.

Lemma 3.2 (Null Form Estimate)**.** *Let $\epsilon>0$, and ψ_1, ψ_2 be the solutions for (3.4). If $\psi_{0j}\in H^{-\alpha}$, we have*

$$\big\|\frac{\widehat{(\varphi_T\overline{\psi}_1\psi_2)}}{\widehat{M}^\alpha\widehat{S}^{2\alpha}}\big\|_{L^2}\quad\le C(T)\big(\|\psi_{01}\|_{H^{-\alpha}}+\big\|\frac{\widehat{G}_1}{\widehat{M}^\alpha\widehat{S}^{\frac{1}{4}}}\big\|_{L^2}\big)\big(\|\psi_{02}\|_{H^{-\alpha}}+\big\|\frac{\widehat{G}_2}{\widehat{M}^\alpha\widehat{S}^{\frac{1}{4}}}\big\|_{L^2}\big).\tag{3.5}$$

For the wave equation (2.13), we define

$$\widehat{\Phi}_T(\tau,\xi)=\widehat{\varphi}_T*\sum_{k=0}^\infty(\widehat{V}_{+,k}+\widehat{V}_{-,k})(\tau,\xi)+\widehat{V}(\tau,\xi)+\widehat{\varphi}_T*\sum_{k=0}^\infty\widehat{N}_k(\tau,\xi)+\widehat{N}(\tau,\xi).\tag{3.6}$$

Thus we have the following estimate.

Lemma 3.3. *Let $\epsilon>0$, $\delta\ge0$, $TR\sim1$, and ϕ be the solution of (2.13). If $\phi_0\in H^{\frac{1}{2}+\delta}$ and $\phi_1\in H^{-\frac{1}{2}+\delta}$, then*

$$\big\|\widehat{S}^{\frac{1}{2}}\widehat{M}^{\frac{1}{2}+\delta}\widehat{\Phi}_T\big\|_{L^2}\le C\big(\|\phi_0\|_{H^{\frac{1}{2}+\delta}}+\|\phi_1\|_{H^{-\frac{1}{2}+\delta}}+T^\epsilon\big\|\frac{\widehat{F}}{\widehat{M}^{\frac{1}{2}-\delta}\widehat{S}^{\frac{1}{2}-\epsilon}}\big\|_{L^2}\big).\tag{3.7}$$

Proof. Applying formula (2.17), we can derive the following bounds:

$$\big\|\widehat{S}^{\frac{1}{2}}\widehat{M}^{\frac{1}{2}+\delta}\widehat{\varphi}_T*\widehat{V}_{\pm,0}\big\|_{L^2}\le C\big(\|\phi_0\|_{H^{\frac{1}{2}+\delta}}+\|\phi_1\|_{H^{-\frac{1}{2}+\delta}}+T^\epsilon\big\|\frac{\widehat{F}}{\widehat{M}^{\frac{1}{2}-\delta}\widehat{S}^{\frac{1}{2}-\epsilon}}\big\|_{L^2}\big),$$

$$\big\|\widehat{S}^{\frac{1}{2}}\widehat{M}^{\frac{1}{2}+\delta}\widehat{\varphi}_T*\widehat{N}_0\big\|_{L^2}\le C\big(\|\phi_0\|_{H^{\frac{1}{2}+\delta}}+\|\phi_1\|_{H^{-\frac{1}{2}+\delta}}+T^\epsilon\big\|\frac{\widehat{F}}{\widehat{M}^{\frac{1}{2}-\delta}\widehat{S}^{\frac{1}{2}-\epsilon}}\big\|_{L^2}\big),$$

$$\big\|\widehat{S}^{\frac{1}{2}}\widehat{M}^{\frac{1}{2}+\delta}\widehat{\varphi}_T*\widehat{V}_{\pm,k}\big\|_{L^2}\le C\frac{(4RT)^k}{k!}T^\epsilon\big\|\frac{\widehat{F}}{\widehat{M}^{\frac{1}{2}-\delta}\widehat{S}^{\frac{1}{2}-\epsilon}}\big\|_{L^2},$$

$$\big\|\widehat{S}^{\frac{1}{2}}\widehat{M}^{\frac{1}{2}+\delta}\widehat{\varphi}_T*\widehat{N}_k\big\|_{L^2}\le C\frac{(4RT)^k}{k!}T^\epsilon\big\|\frac{\widehat{F}}{\widehat{M}^{\frac{1}{2}-\delta}\widehat{S}^{\frac{1}{2}-\epsilon}}\big\|_{L^2},$$

$$\big\|\widehat{S}^{\frac{1}{2}}\widehat{M}^{\frac{1}{2}+\delta}\widehat{V}\big\|_{L^2}\le CT^\epsilon\big\|\frac{\widehat{F}}{\widehat{M}^{\frac{1}{2}-\delta}\widehat{S}^{\frac{1}{2}-\epsilon}}\big\|_{L^2},$$

$$\big\|\widehat{S}^{\frac{1}{2}}\widehat{M}^{\frac{1}{2}+\delta}\widehat{N}\big\|_{L^2}\le CT^\epsilon\big\|\frac{\widehat{F}}{\widehat{M}^{\frac{1}{2}-\delta}\widehat{S}^{\frac{1}{2}-\epsilon}}\big\|_{L^2}.$$

Combining the above inequalitites, we complete the proof. □

We will also need some technical lemmas.

Lemma 3.4 (Hardy-Littlewood-Polya)**.** *Let $r = 2 - \frac{1}{p} - \frac{1}{q}$. Then we have*

$$\int_{\mathbb{R}^1\times\mathbb{R}^1} \frac{f(s)g(t)}{|s-t|^r} dsdt \le C\|f\|_{L^p}\|g\|_{L^q}. \tag{3.8}$$

Lemma 3.5. *Let $f(t,x)$ and $g(t,x)$ be any functions such that $f \in L^q(L^2(\mathbb{R}^n))$ and $\widehat{S}^\beta \widehat{g} \in L^2(L^2(\mathbb{R}^n))$. Assume that $\epsilon > 0$, $\frac{1}{q} = 1-\epsilon$, $\frac{1}{r} = \frac{1}{2} - \beta$, and $2 \le r < \infty$. Then we have*

$$\Big\|\frac{\widehat{f}}{\widehat{S}^{\frac{1}{2}-\epsilon}}\Big\|_{L^2(L^2(\mathbb{R}^n))} \le C\|f\|_{L^q(L^2(\mathbb{R}^n))}, \tag{3.9}$$

$$\|g\|_{L^r(L^2(\mathbb{R}^n))} \le C\|\widehat{S}^\beta\widehat{g}\|_{L^2(L^2(\mathbb{R}^n))}. \tag{3.10}$$

Proof. The proofs for (3.9) and (3.10) are analogous. Therefore, we will only prove the case of (3.10). Taking the inverse Fourier transform in the time variable over the identity

$$\widehat{g} = \frac{1}{\widehat{S}^\beta}\widehat{S}^\beta\widehat{g} \tag{3.11}$$

gives

$$\widetilde{g}(t,\xi) = \int \frac{e^{\pm i(t-s)|\xi|}}{|t-s|^{1-\beta}} \mathcal{F}_\tau^{-1}(\widehat{S}^\beta\widehat{g})(s,\xi)\, ds. \tag{3.12}$$

Then we use duality and Hardy-Littlewood-Polya inequality to compute

$$\begin{aligned}
\big|\langle g,\varphi\rangle\big| &= \big|\langle \widetilde{g},\widetilde{\varphi}\rangle\big| = \Big|\iint\int \frac{e^{\pm i(t-s)|\xi|}}{|t-s|^{1-\beta}}\mathcal{F}_\tau^{-1}(\widehat{S}^\beta\widehat{g})(s,\xi)ds\overline{\widetilde{\varphi}}(t,\xi)dtd\xi\Big| \\
&\le \int \frac{\|\mathcal{F}_\tau^{-1}(\widehat{S}^\beta\widehat{g})(s)\|_{L^2}\|\widetilde{\varphi}(t)\|_{L^2}}{|t-s|^{1-\beta}} dsdt \\
&\le C\|\mathcal{F}_\tau^{-1}(\widehat{S}^\beta\widehat{g})\|_{L^2}\|\widetilde{\varphi}\|_{L^{r'}(L^2)} = C\|\widehat{S}^\beta\widehat{g}\|_{L^2}\|\varphi\|_{L^{r'}(L^2)}.
\end{aligned} \tag{3.13}$$

This completes the proof of (3.10). □

4. Local and Global Existence

Now we are ready to prove the local existence for the (DKG) equations.

Proof of Theorem 1.1. Consider the DKG equations

$$\begin{aligned}
\mathcal{D}\psi &= \varphi_T\phi\psi, \\
\Box\phi &= \varphi_T\overline{\psi}\psi,
\end{aligned} \tag{4.1}$$

and the map $\mathcal{T}(\psi,\phi) = (\Psi_T,\Phi_T)$. We want to show that $\mathcal{T}$ is a contraction under the norm

$$N(\psi,\phi) = \big\|\widehat{S}^{\frac{1}{2}}\widehat{M}^{-\alpha}\widehat{\psi}\big\|_{L^2} + \big\|\widehat{S}^{\frac{1}{2}}\widehat{M}^{\frac{1}{2}+\delta}\widehat{\phi}\big\|_{L^2}. \tag{4.2}$$

For convenience, we define

$$J(0) = \|\phi_0\|_{H^{\frac{1}{2}+\delta}} + \|\phi_1\|_{H^{-\frac{1}{2}+\delta}} + \|\psi_0\|^2_{H^{-\alpha}} + 1. \tag{4.3}$$

First we apply (3.7) and (3.5) to compute

$$\big\|\widehat{S}^{\frac12}\widehat{M}^{\frac12+\delta}\widehat{\Phi}_T\big\|_{L^2} \le C\big(J(0)+T^{\epsilon}\big\|\frac{\widehat{\varphi_T\overline{\psi}\psi}}{\widehat{M}^{\frac12-\delta}\widehat{S}^{\frac12-\epsilon}}\big\|_{L^2}\big) \le C\big(J(0)+T^{\epsilon}\big\|\frac{\widehat{\varphi_T\phi\psi}}{\widehat{M}^{\alpha}\widehat{S}^{\frac14}}\big\|^2_{L^2}\big). \tag{4.4}$$

To bound the term above, we first compute

$$\begin{aligned}\big\|\widehat{M}^{-\alpha}\widetilde{\phi\psi}(t)\big\|_{L^2} \sim \|G_\alpha*(\phi\psi)(t)\|_{L^2} &\le \|\phi(t)\|_{L^\infty}\|G_\alpha*\psi(t)\|_{L^2}\\ &\le \|\phi(t)\|_{H^{\frac12+\delta}}\|\psi(t)\|_{H^{-\alpha}},\end{aligned} \tag{4.5}$$

where $G_\alpha(x)$ is an L^1-function such that

$$\widehat{G_\alpha}(\xi)\sim(1+|\xi|)^{-\alpha}, \tag{4.6}$$

see [13]. Then we invoke (3.9), (3.10) and obtain

$$\begin{aligned}\big\|\frac{\widehat{\varphi_T\phi\psi}}{\widehat{M}^{\alpha}\widehat{S}^{\frac14}}\big\|_{L^2} &\le C\big\|\frac{\widetilde{\varphi_T\phi\psi}}{\widehat{M}^{\alpha}}\big\|_{L^q(L^2)}\\ &\le \|\phi\|_{L^{2q}(H^{\frac12+\delta})}\|\psi\|_{L^{2q}(H^{-\alpha})}\\ &\le \big\|\widehat{S}^{\frac12}\widehat{M}^{\frac12+\delta}\widehat{\phi}\big\|_{L^2}\big\|\widehat{S}^{\frac12}\widehat{M}^{-\alpha}\widehat{\psi}\big\|_{L^2}.\end{aligned} \tag{4.7}$$

Thus we get

$$\big\|\frac{\widehat{\varphi_T\phi\psi}}{\widehat{M}^{\alpha}\widehat{S}^{\frac12-\epsilon}}\big\|_{L^2} \le CN^2(\psi,\phi). \tag{4.8}$$

Next we want to bound the term involving $\widehat{\Psi}_T$. The estimate (3.3) implies

$$\big\|\widehat{S}^{\frac12}\widehat{M}^{-\alpha}\widehat{\Psi}_T\big\|_{L^2(\mathbb{R}^1\times\mathbb{R}^1)} \le C\big(\|\psi_0\|_{H^{-\alpha}}+T^{\epsilon}\big\|\frac{\widehat{\varphi_T\phi\psi}}{\widehat{M}^{\alpha}\widehat{S}^{\frac14}}\big\|_{L^2}\big). \tag{4.9}$$

Hence, using (4.3), (4.8), and (4.5), we have

$$N(\mathcal{T}(\psi,\phi)) \le C\big(J(0)+T^{\epsilon}N^4(\psi,\phi)\big). \tag{4.10}$$

Choosing sufficiently large L, for suitable T, we have

$$N(\psi,\phi)\le L \Longrightarrow N\big(\mathcal{T}(\psi,\phi)\big)\le L, \tag{4.11}$$

provided that $C(J(0)+T^{\epsilon}L^4)\le L$.

Now we consider the difference $\mathcal{T}(\psi,\phi)-\mathcal{T}(\psi',\phi')$. Base on the observations

$$\begin{aligned}\overline{\psi}\psi-\overline{\psi'}\psi' &= \frac12(\overline{\psi-\psi'})(\psi+\psi')+\frac12(\overline{\psi+\psi'})(\psi-\psi'),\\ \phi\psi-\phi'\psi' &= \frac12(\phi-\phi')(\psi+\psi')+\frac12(\phi+\phi')(\psi-\psi'),\end{aligned} \tag{4.12}$$

Using (3.7), (3.5), (4.12), and (4.8), we first calculate

$$\begin{aligned}
&\big\|\widehat{S}^{\frac12}\widehat{M}^{\frac12+\delta}\mathcal{F}(\Phi_T-\Phi'_T)\big\|_{L^2}\\
&\le CT^\epsilon\big(\|\frac{\mathcal{F}((\overline{\psi-\psi'})(\psi+\psi'))}{\widehat{M}^{\frac12-\delta}\widehat{S}^{\frac12-\epsilon}}\|_{L^2}+\|\frac{\mathcal{F}((\overline{\psi+\psi'})(\psi-\psi'))}{\widehat{M}^{\frac12-\delta}\widehat{S}^{\frac12-\epsilon}}\|_{L^2}\big)\\
&\le CT^\epsilon\big(\|\frac{\mathcal{F}((\phi-\phi')(\psi+\psi'))}{\widehat{M}^{\alpha}\widehat{S}^{\frac14}}\|_{L^2}+\|\frac{\mathcal{F}((\phi+\phi')(\psi-\psi'))}{\widehat{M}^{\alpha}\widehat{S}^{\frac14}}\|_{L^2}\big)\\
&\quad\times\big(J(0)+\|\frac{\mathcal{F}(\phi\psi+\phi'\psi')}{\widehat{M}^{\alpha}\widehat{S}^{\frac14}}\|_{L^2}\big)\\
&\le CT^\epsilon\big(\|\widehat{S}^{\frac12}\widehat{M}^{\frac12+\delta}\widehat{\phi-\phi'}\|_{L^2}+\|\widehat{S}^{\frac12}\widehat{M}^{-\alpha}\widehat{\psi-\psi'}\|_{L^2}\big)L(J(0)+L^2)\\
&\le CT^\epsilon L^3\big(\|\widehat{S}^{\frac12}\widehat{M}^{\frac12+\delta}\widehat{\phi-\phi'}\|_{L^2}+\|\widehat{S}^{\frac12}\widehat{M}^{-\alpha}\widehat{\psi-\psi'}\|_{L^2}\big)
\end{aligned}\tag{4.13}$$

Analogously, we get

$$\big\|\widehat{S}^{\frac12}\widehat{M}^{-\alpha}(\widehat{\Psi_T-\Psi'_T})\big\|_{L^2}\le CT^\epsilon L\big(\|\widehat{S}^{\frac12}\widehat{M}^{-\alpha}\widehat{\psi-\psi'}\|_{L^2}+\|\widehat{S}^{\frac12}\widehat{M}^{\frac12+\delta}\widehat{\phi-\phi'}\|_{L^2}\big).\tag{4.14}$$

Combining (4.13)and (4.14), we have

$$N\big(\mathcal{T}(\psi-\psi',\phi-\phi')\big)\le CT^\epsilon L^3N(\psi-\psi',\phi-\phi').\tag{4.15}$$

Therefore for suitable T, we obtain

$$N\big(\mathcal{T}(\psi-\psi',\phi-\phi')\big)\le\frac12N(\psi-\psi',\phi-\phi'),\tag{4.16}$$

provided that $CT^\epsilon L^3\le\frac12$. We can conclude that the map $\mathcal{T}$ is indeed a contraction with respect to the norm N, thus it has a unique fixed point. □

We now prove existence of a gobal solution.

Proof of Theorem 1.2. ¿From the law of conservation of charge, we have

$$\sup_{[0,T]}\|\psi(t)\|_{L^2}=\|\psi_0\|_{L^2}.\tag{4.17}$$

To bound ϕ we apply the formula (2.14),

$$2\phi(t,x)=(\phi_0(x+t)+\phi_0(x-t))+\int_{x-t}^{x+t}\phi_1(y)dy+\int_0^t\int_{x-t+s}^{x+t-s}\overline{\psi}\psi(s,y)dyds.\tag{4.18}$$

First we write $\phi=\phi_L+\phi_N$, the homogeneous and inhomogeneous parts of the solution, then we obtain

$$\|\phi_L(t)\|_{L^\infty}\le\|\phi_L(t)\|_{H^{\frac12+\delta}}\le\|\phi_0\|_{H^{\frac12+\delta}}+\|\phi_1\|_{H^{-\frac12+\delta}}\le J(0),\tag{4.19}$$

and

$$\|\phi_N(t)\|_{L^\infty}\le\int_0^t\int_{x-t+s}^{x+t-s}\big|\overline{\psi}\psi(s,y)\big|dy\,ds\le CT\|\psi_0\|_{L^2}^2.\tag{4.20}$$

Combining (4.19) and (4.20), we obtain

$$\|\phi(t)\|_{L^\infty}\le C(T,J(0)).\tag{4.21}$$

Taking Fourier transform of the solution $\phi(t)$, we have

$$\widetilde{\phi}(t,\xi)=\cos t|\xi|\widehat{\phi}_0(\xi)+\frac{\sin t|\xi|}{|\xi|}\widehat{\phi}_1(\xi)+\int_0^t\frac{\sin{(t-s)|\xi|}}{|\xi|}\widetilde{\varphi_T\overline{\psi}\psi}(s,\xi)ds.\tag{4.22}$$

Then we invoke (3.5), (3.9), (for $\epsilon = \frac{1}{4}$), and (4.20) to compute

$$\begin{aligned}
\|\phi(t)\|_{H^{\frac{1}{2}+\delta}} &\le \|\phi_0\|_{H^{\frac{1}{2}+\delta}} + \|\phi_1\|_{H^{-\frac{1}{2}+\delta}} + \int_0^t \|\varphi_T\overline{\psi}\psi(s)\|_{H^{-\frac{1}{2}+\delta}} ds \\
&\le J(0) + T^{\frac{1}{2}} \Big\| \frac{\widehat{\varphi_T\overline{\psi}\psi}}{\widehat{M}^{\frac{1}{2}-\delta}} \Big\|_{L^2} \\
&\le J(0) + T^{\frac{1}{2}} \|\widehat{\varphi_T\overline{\psi}\psi}\|_{L^2} \\
&\le J(0) + T^{\frac{1}{2}} \Big\| \frac{\widehat{\varphi_T\phi\psi}}{\widehat{S}^{\frac{1}{4}}} \Big\|_{L^2}^2 \\
&\le J(0) + T^{\rho} \|\varphi_T\phi\psi\|_{L^2} \\
&\le J(0) + T^{\rho} \int_0^T \|\phi(t)\|_{L^\infty}^2 \|\psi(t)\|_{L^2}^2 dt \\
&\le C\big(T, J(0)\big),
\end{aligned} \tag{4.23}$$

where ρ is some positive number. The calculation for $\|\phi_t(t)\|_{H^{-\frac{1}{2}+\delta}}$ is analogous. Thus the above bounds ensure us to proceed the construction of solution beyond T. $\square$

5. Null Form Estimate

In this section, we demonstrate the key estimate in Lemma 3.2: Let $\epsilon > 0$ and ψ_1, ψ_2 be the solutions for the Dirac equations (3.4). If the initial data $\psi_{0j} \in H^{-\frac{1}{4}+\epsilon}$, $j = 1, 2$, then we have

$$\Big\| \frac{\widehat{\varphi_T\overline{\psi}_1\psi_2}}{\widehat{M}^{\alpha}\widehat{S}^{2\alpha}} \Big\|_{L^2} \le C(T)\big(\|\psi_{01}\|_{H^{-\alpha}} + \Big\|\frac{\widehat{G}_1}{\widehat{M}^{\alpha}\widehat{S}^{\frac{1}{4}}}\Big\|_{L^2}\big)\big(\|\psi_{02}\|_{H^{-\alpha}} + \Big\|\frac{\widehat{G}_2}{\widehat{M}^{\alpha}\widehat{S}^{\frac{1}{4}}}\Big\|_{L^2}\big). \tag{5.1}$$

The proof of this estimate is based on the duality argument and it will be given in a number of steps. Without loss of generality, we assume that $\psi_1 = \psi_2$, and prove: if ψ is a solution of the Dirac equation (2.1), then

$$\Big\| \frac{\widehat{\varphi_T\overline{\psi}\psi}}{\widehat{M}^{\alpha}\widehat{S}^{2\alpha}} \Big\|_{L^2} \le C(T)\big(\|\psi_0\|_{H^{-\alpha}} + \Big\|\frac{\widehat{G}}{\widehat{M}^{\alpha}\widehat{S}^{\frac{1}{4}}}\Big\|_{L^2}\big)^2. \tag{5.2}$$

Recall the notation:

$$\begin{gathered}
\widehat{E}(\tau,\xi) := |\tau| + |\xi| + 1, \quad \widehat{S}(\tau,\xi) := \big||\tau| - |\xi|\big| + 1, \\
\widehat{W}(\tau,\xi) := \tau^2 - |\xi|^2, \quad \widehat{D}(\tau,\xi) := \gamma^0\tau + \gamma^1\xi, \\
\widehat{D}_+ := \widehat{D}(|\xi|, +\xi), \quad \widehat{D}_- := \widehat{D}(|\xi|, -\xi),
\end{gathered}$$

The formula for $\widehat{\psi}$, as in (2.7), for the Dirac equation (2.1) is given by

$$\widehat{\psi}(\tau,\xi) = \sum_{k=0}^{\infty} \big(\delta_+^{(k)}(\tau,\xi)\widehat{A}_{+,k}(\xi) + \delta_-^{(k)}(\tau,\xi)\widehat{A}_{-,k}(\xi)\big) + \widehat{K}(\tau,\xi), \tag{5.3}$$

where $\delta_\pm(\tau,\xi)$ are the delta functions supported on $\{\tau = \pm|\xi|\}$ respectively, $\delta^{(k)}$ mean derivatives of the delta function, and

$$\widehat{K}(\tau,\xi) := \frac{\widehat{D}(\tau,\xi)}{\widehat{W}(\tau,\xi)}\widehat{G}_f + \frac{(1-\widehat{a}_6(\tau))\widehat{D}_+\widehat{G}_-}{2|\xi|(\tau-|\xi|)} + \frac{(1-\widehat{a}_6)\widehat{D}_-\widehat{G}_+}{2|\xi|(\tau+|\xi|)}, \tag{5.4}$$

$$\widehat{A}_{\pm,0}(\xi) := \frac{\widehat{D}_\pm}{2|\xi|}\big[\gamma^0\widehat{\psi}_0 - \int \frac{\widehat{G}_f + (1-\widehat{a}_6(\lambda))\widehat{G}_\mp}{\lambda \mp |\xi|}d\lambda\big], \tag{5.5}$$

$$\widehat{A}_{\pm,k}(\xi) := \frac{\widehat{D}_\pm(-1)^k}{2|\xi|k!}\int(\lambda\mp|\xi|)^{k-1}\big[\widehat{G}_\pm + \widehat{a}_6(\lambda)\widehat{G}_\mp\big]d\lambda. \tag{5.6}$$

Moreover we write

$$\widehat{A}_{\pm,k}(\xi) := \frac{\widehat{D}_\pm}{2|\xi|}\widehat{f}_{\pm,k}(\xi), \tag{5.7}$$

and set $\widehat{K} = \widehat{K}_1 + \widehat{K}_2$, where

$$\widehat{K}_1 := \frac{\widehat{D}(\tau,\xi)}{\widehat{W}(\tau,\xi)}\widehat{G}_f; \quad \widehat{K}_2 := \frac{b_1\widehat{D}_+\widehat{G}_- + b_2\widehat{D}_-\widehat{G}_+}{\widehat{E}\widehat{S}}, \tag{5.8}$$

and b_1, b_2 are bounded functions. The Fourier transform of the quadratic expression, $\widehat{\overline{\psi}\psi} = \widehat{\overline{\psi}} * \widehat{\psi}$, can be written as the sum of the following terms.

$$\sum_{k,l}\big(\delta_\mp^{(k)}\widehat{\overline{A}}_{\pm,k}\big) * \big(\delta_\pm^{(l)}\widehat{A}_{\pm,l}\big), \tag{5.9}$$

$$\sum_{k,l}\big(\delta_\mp^{(k)}\widehat{\overline{A}}_{\pm,k}\big) * \big(\delta_\mp^{(l)}\widehat{A}_{\mp,l}\big), \tag{5.10}$$

$$\sum_k\big(\delta_\mp^{(k)}\widehat{\overline{A}}_{\pm,k}\big) * \big(\widehat{K}_1 + \widehat{K}_2\big) + \big(\widehat{\overline{K}}_1 + \widehat{\overline{K}}_2\big) * \sum_k\big(\delta_\pm^{(k)}\widehat{A}_{\pm,k}\big),\ labele4.8c \tag{5.11}$$

$$\widehat{\overline{K}}_1 * \widehat{K}_1 + \widehat{\overline{K}}_1 * \widehat{K}_2 + \widehat{\overline{K}}_2 * \widehat{K}_1 + \widehat{\overline{K}}_2 * \widehat{K}_2. \tag{5.12}$$

Note that

$$\widehat{A^\dagger_{\pm,k}}(\xi) = \widehat{A}^\dagger_{\pm,k}(-\xi), \quad \widehat{f^+_{\pm,k}}(\xi) = \widehat{f}^\dagger_{\pm,k}(-\xi), \tag{5.13}$$

$$\widehat{\overline{A}_{\pm,k}}(\xi) = \widehat{f}^\dagger_{\pm,k}(-\xi)\frac{\widehat{D}_\pm}{|\xi|}\gamma^0, \quad \widehat{\overline{K}}(\tau,\xi) = \widehat{K}^\dagger(-\tau,-\xi)\gamma^0, \tag{5.14}$$

and

$$\widehat{\overline{\psi}}(\tau,\xi) = \sum_{k=0}^\infty\big(\delta_-^{(k)}(\tau,\xi)\widehat{\overline{A}}_{+,k}(\xi) + \delta_+^{(k)}(\tau,\xi)\widehat{\overline{A}}_{-,k}(\xi)\big) + \widehat{\overline{K}}(\tau,\xi), \tag{5.15}$$

Lemma 5.1. *Let $\alpha < 1/4$. Then the following estimate holds*

$$\begin{aligned}&\Big\|\frac{\widehat{\varphi}_T * \big(\delta_\mp^{(k)}\widehat{f^\dagger_{\pm,k}}\frac{\widehat{D}_\pm}{|\xi|}\gamma^0\big) * \big(\delta_\mp^{(l)}\frac{\widehat{D}_\mp}{|\xi|}\widehat{f}_{\mp,l}\big)}{\widehat{M}^\alpha\widehat{S}^{2\alpha}}\Big\|_{L^2}\\ &\le C(k+l+1)T^{k+l-\frac12}\|f_{\pm,k}\|_{H^{-\alpha}}\|f_{\mp,l}\|_{H^{-\alpha}}.\end{aligned}$$

Proof. Let

$$\widehat{Z}_{\pm,k} \equiv \delta_\pm^{(k)}\frac{\widehat{D}_\pm}{2|\xi|}\widehat{f}_{\pm,k} = \delta_\pm^{(k)}\widehat{A}_{\pm,k}. \tag{5.16}$$

Using duality, we demonstrate the case $(-,+)$, while the case $(+,-)$ is being similar. We first compute the fractional term

$$\frac{\widehat{D}(|\xi|,-\xi)\gamma^0\widehat{D}(|\eta|,\eta)}{|\xi||\eta|}=\begin{cases}0, & \text{if } \xi\eta>0,\\ 2\gamma^0\pm 2\gamma^1, & \text{if } \xi\eta<0.\end{cases}$$

Throughout elementary analysis we have the bound:

$$\frac{(1+|\xi|)^\alpha(1+|\eta|)^\alpha}{(1+|\xi+\eta|)^\alpha\big(\big||\xi|+|\eta|\big|-|\xi+\eta|\big|+1\big)^{2\alpha}}\le C, \tag{5.17}$$

for $\xi\eta<0$. Thus

$$\begin{aligned}&\big|\left\langle \varphi_T\overline{Z}_{-,k}Z_{+,l},g\right\rangle\big|\\ =&\big|\int \widehat{f}^\dagger_{-,k}(-\xi)\frac{\widehat{D}(|\xi|,-\xi)\gamma^0\widehat{D}(|\eta|,\eta)}{|\xi||\eta|}\widehat{f}_{+,l}(\eta)\overline{\widehat{t^{k+l}\varphi_T g}}(|\xi|+|\eta|,\xi+\eta)d\xi d\eta\big|\\ &\le C\|f_{-,k}\|_{H^{-\alpha}}\|f_{+,l}\|_{H^{-\alpha}}\|\widehat{M}^\alpha\widehat{S}^{2\alpha}\widehat{t^{k+l}\varphi_T g}\|_{L^2},\end{aligned}$$

and through some computations, we have

$$\|\widehat{M}^\alpha\widehat{S}^{2\alpha}\widehat{t^{k+l}\varphi_T g}\|_{L^2}\le C(k+l+1)T^{k+l-\frac12}\|\widehat{M}^\alpha\widehat{S}^{2\alpha}\widehat{g}\|_{L^2}. \tag{5.18}$$

This completes the proof. $\square$

Lemma 5.2. *Let $\alpha<1/4$. The following estimate holds*

$$\begin{aligned}&\Big\|\frac{\widehat{\varphi}_T*\big(\delta^{(k)}_{\mp}\widehat{f^\dagger_{\pm,k}}\frac{\widehat{D}_\pm}{|\xi|}\gamma^0\big)*\big(\delta^{(l)}_{\pm}\frac{\widehat{D}_\pm}{|\xi|}\widehat{f}_{\pm,l}\big)}{\widehat{M}^\alpha\widehat{S}^{2\alpha}}\Big\|_{L^2}\\ &\le C(k+l+1)T^{k+l-\frac12}\|f_{\pm,k}\|_{H^{-\alpha}}\|f_{\pm,l}\|_{H^{-\alpha}}.\end{aligned}$$

Proof. Using duality, we demonstrate the case $(+,+)$, while the case $(-,-)$ is being similar. We first compute the fractional term

$$\frac{\widehat{D}(|\xi|,\xi)\gamma^0\widehat{D}(|\eta|,\eta)}{|\xi||\eta|}=\begin{cases}0, & \text{if } \xi\eta<0,\\ 2\gamma^0\mp 2\gamma^1, & \text{if } \xi\eta>0.\end{cases}$$

Throughout elementary analysis we have the bound:

$$\frac{(1+|\xi|)^\alpha(1+|\eta|)^\alpha}{(1+|\xi+\eta|)^\alpha\big(\big|-|\xi|+|\eta|\big|-|\xi+\eta|\big|+1\big)^{2\alpha}}\le C \tag{5.19}$$

for $\xi\eta>0$. Thus

$$\begin{aligned}&\big|\left\langle \varphi_T\overline{Z}_{+,k}Z_{+,l},g\right\rangle\big|\\ &=\big|\int \widehat{f}^\dagger_{+,k}(-\xi)\frac{\widehat{D}(|\xi|,\xi)\gamma^0\widehat{D}(|\eta|,\eta)}{|\xi||\eta|}\widehat{f}_{+,l}(\eta)\overline{\widehat{t^{k+l}\varphi_T g}}(-|\xi|+|\eta|,\xi+\eta)d\xi d\eta\big|\\ &\le C\|f_{+,k}\|_{H^{-\alpha}}\|f_{+,l}\|_{H^{-\alpha}}\|\widehat{M}^\alpha\widehat{S}^{2\alpha}\widehat{t^{k+l}\varphi_T g}\|_{L^2}.\end{aligned}$$

This together with (5.18) complete the proof. $\square$

Lemma 5.3. *With the notation above, the following two estimates hold*

$$\|f_{\pm,0}\|_{H^{-\alpha}} \le C\big(\|\psi_0\|_{H^{-\alpha}} + \big\|\frac{\widehat{G}}{\widehat{M}^\alpha \widehat{S}^{\frac14}}\big\|_{L^2}\big), \tag{5.20}$$

$$\|f_{\pm,k}\|_{H^{-\alpha}} \le C\frac{1}{k!}\|\frac{\widehat{G}_\pm}{\widehat{M}^\alpha \widehat{S}^{\frac14}}\|_{L^2}. \tag{5.21}$$

The proof for the Lemma 5.3 is straight forward so that we skip it. Notice that, in the (5.21), $\widehat{S} \sim 1$ on the support of $\widehat{G}_\pm$.

Lemma 5.4. *With the notation above, the following estimate holds*

$$\big\|\frac{\widehat{\varphi}_T * \widehat{\widehat{K}}_1 * \widehat{K}_1}{\widehat{M}^\alpha \widehat{S}^{2\alpha}}\big\|_{L^2} \le C\big\|\frac{\widehat{G}_f}{\widehat{M}^\alpha \widehat{S}^{\frac14}}\big\|^2_{L^2}. \tag{5.22}$$

Proof. For simplicity, we write $\widehat{G} := \widehat{G}_f$ and $\widehat{K} := \widehat{K}_1$. We use dyadic decomposition to handle this case. Assume that

$$\widehat{G} = \sum_{k=1}^{\infty} \widehat{G}_{\pm,\pm,k}, \tag{5.23}$$

where $\widehat{G}_{\pm,\pm,k}(\tau,\xi)$ is supported in one of the following four types of regions:

$$\begin{aligned}
\Sigma_{+,+} &:= \{(\tau,\xi) : \tau > 0, +2^{k-1} < \tau - |\xi| < +2^{k+1}\},\\
\Sigma_{+,-} &:= \{(\tau,\xi) : \tau > 0, -2^{k+1} < \tau - |\xi| < -2^{k-1}\},\\
\Sigma_{-,+} &:= \{(\tau,\xi) : \tau < 0, +2^{k-1} < \tau + |\xi| < +2^{k+1}\},\\
\Sigma_{-,-} &:= \{(\tau,\xi) : \tau < 0, -2^{k+1} < \tau + |\xi| < -2^{k-1}\}.
\end{aligned} \tag{5.24}$$

The decomposition of $\widehat{G}$ induces a decomposition for $\widehat{K}$, namely

$$\widehat{K}_{\pm,\pm,k} = \frac{\widehat{D}}{\widehat{W}}\widehat{G}_{\pm,\pm,k}. \tag{5.25}$$

To compute the convolution in (5.22),

$$\begin{aligned}
\widehat{\widehat{K}}_{\pm,\pm,k} * \widehat{K}_{\pm,\pm,l}(-\tau,-\xi) &= \int \widehat{\widehat{K}}_{\pm,\pm,k}(-\tau-\sigma,-\xi-\eta)\widehat{K}_{\pm,\pm,l}(\sigma,\eta)d\sigma d\eta\\
&= \int \widehat{K}^\dagger_{\pm,\pm,k}(\tau+\sigma,\xi+\eta)\gamma^0\widehat{K}_{\pm,\pm,l}(\sigma,\eta)d\sigma d\eta,
\end{aligned} \tag{5.26}$$

we have 16 cases resulted from (5.24) and (5.26) as follows.

$$\begin{gathered}
\{(\tau,\sigma,\xi,\eta) : \tau+\sigma > 0, \sigma > 0, \tau+\sigma-|\xi+\eta| \sim \pm 2^k, \sigma - |\eta| \sim \pm 2^l\}\\
\{(\tau,\sigma,\xi,\eta) : \tau+\sigma < 0, \sigma < 0, \tau+\sigma+|\xi+\eta| \sim \pm 2^k, \sigma + |\eta| \sim \pm 2^l\}\\
\{(\tau,\sigma,\xi,\eta) : \tau+\sigma < 0, \sigma > 0, \tau+\sigma+|\xi+\eta| \sim \pm 2^k, \sigma - |\eta| \sim \pm 2^l\}\\
\{(\tau,\sigma,\xi,\eta) : \tau+\sigma > 0, \sigma < 0, \tau+\sigma-|\xi+\eta| \sim \pm 2^k, \sigma + |\eta| \sim \pm 2^l\}
\end{gathered}$$

We label them as $\Sigma_{k,l}[(\pm,\pm);(\pm,\pm)]$, and denote by $\Sigma_{k,l}$ without specifying which one precisely. We also use $\widehat{K}_k$ for abbreviation of $\widehat{K}_{\pm,\pm,k}$ and $\widehat{G}_k$ for $\widehat{G}_{\pm,\pm,k}$.

Let g be an arbitrary function. We first compute

$$\big[\gamma^0(\tau+\sigma)-\gamma^1(\xi+\eta)\big]\gamma^0\big[\gamma^0\tau+\gamma^1\eta\big] = \gamma^0\big[(\tau+\sigma)\sigma-(\xi+\eta)\eta\big]+\gamma^1\big[(\tau+\sigma)\eta-\sigma(\xi+\eta)\big].$$

Thus, we have

$$\begin{aligned}\Big|\Big\langle \widehat{\overline{K}}_k * \widehat{K}_l, \widehat{g}\Big\rangle\Big| &= \Big|\int \widehat{G}_k^{\dagger}(\tau+\sigma,\xi+\eta)\frac{\gamma^0(\tau+\sigma)-\gamma^1(\xi+\eta)}{(\tau+\sigma)^2-(\xi+\eta)^2}\gamma^0\frac{\gamma^0\sigma+\gamma^1\eta}{\sigma^2-\eta^2}\widehat{G}_l(\sigma,\eta) \\ &\quad \times \overline{\widehat{g}}(-\tau,-\xi)d\sigma\, d\eta\, d\tau\, d\xi\Big| \\ &\le C\|\frac{\widehat{G}_k}{\widehat{M}^\alpha}\|_{L^2}\|\frac{\widehat{G}_l}{\widehat{M}^\alpha}\|_{L^2}\Big(\int I_{k,l}(\tau,\xi)\big|\widehat{g}(-\tau,-\xi)\big|^2 d\tau\, d\xi\Big)^{1/2},\end{aligned}$$

where

$$I_{k,l}(\tau,\xi) := \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)Q(\tau,\sigma,\xi,\eta)}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)}d\sigma d\eta, \tag{5.27}$$

$$Q(\tau,\sigma,\xi,\eta) := \big[(\tau+\sigma)\sigma-(\xi+\eta)\eta\big]^2+\big[(\tau+\sigma)\eta-\sigma(\xi+\eta)\big]^2, \tag{5.28}$$

and $D_{k,l}(\tau,\xi)$ is a slice of $\Sigma_{k,l}$ for fixed (τ,ξ); i.e.,

$$D_{k,l}(\tau,\xi) := \{(\sigma,\eta) : (\tau,\sigma,\xi,\eta)\in\Sigma_{k,l}\}. \tag{5.29}$$

We need to sort the cases into two sets,

$$\Sigma_{k,l}[(\pm,\cdot);(\pm,\cdot)] \quad\text{and}\quad \Sigma_{k,l}[(\pm,\cdot);(\mp,\cdot)], \tag{5.30}$$

due to the fact that the computation for the 8 cases in each set is similar. For simplicity, we will assume $k\ge l$, while the other case is similar.

Cases H. We have the following estimate

$$\|\frac{\widehat{\overline{K}}_{+,\cdot,k} * \widehat{K}_{+,\cdot,l}}{\widehat{M}^\alpha \widehat{S}^{2\alpha}}\|_{L^2} \le \frac{C}{2^{\frac{l}{2}}}\frac{1}{2^{(\frac12-\alpha)k}}\|\frac{\widehat{G}_{+,\cdot,k}}{\widehat{M}^\alpha}\|_{L^2}\|\frac{\widehat{G}_{+,\cdot,l}}{\widehat{M}^\alpha}\|_{L^2}, \tag{5.31}$$

$$\|\frac{\widehat{\overline{K}}_{-,\cdot,k} * \widehat{K}_{-,\cdot,l}}{\widehat{M}^\alpha \widehat{S}^{2\alpha}}\|_{L^2} \le \frac{C}{2^{\frac{l}{2}}}\frac{1}{2^{(\frac12-\alpha)k}}\|\frac{\widehat{G}_{-,\cdot,k}}{\widehat{M}^\alpha}\|_{L^2}\|\frac{\widehat{G}_{-,\cdot,l}}{\widehat{M}^\alpha}\|_{L^2}, \tag{5.32}$$

In these cases, we have $(\tau+\sigma)\sigma>0$. Throughout some algebraic manipulation, the expression Q can be written as

$$\begin{aligned}2Q &= (\tau+\sigma-|\xi+\eta|)^2(\sigma+|\eta|)^2+(\tau+\sigma+|\xi+\eta|)^2(\sigma-|\eta|)^2 \\ &\quad +8(\tau+\sigma)\sigma\big[|\xi+\eta||\eta|-(\xi+\eta)\eta\big].\end{aligned} \tag{5.33}$$

Take the case of

$$\widehat{\overline{K}}_{+,+,k} * \widehat{K}_{+,+,l}, \tag{5.34}$$

as an example and in which $D_{k,l} = \{(\eta,\sigma) : \tau+\sigma-|\xi+\eta|\sim 2^k, \sigma-|\eta|\sim 2^l, (\tau,\sigma,\xi,\eta)\in\Sigma_{k,l}[(+,+);(+,+)]\}$. In this case $\tau+\sigma>0$ and $\sigma>0$. In the $\eta\sigma$-plane, this is the region of the intersection of two forward cones. One has the thickness of 2^k and the translation of $(-\xi,-\tau)$, while the other has thickness of 2^l. It is bounded mostly, except for the extreme case which is when one cone moves along the other cone such that the intersection region is unbounded.

For the first part, we distinguish three cases: $|\xi+\eta| \le |\eta|$, $|\xi+\eta| \ge |\eta|$, and the extreme case. For the first two cases, we have

$$\begin{aligned}
I^1_{k,l}(\tau,\xi) &:= \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)(\tau+\sigma-|\xi+\eta|)^2(\sigma+|\eta|)^2}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)} d\sigma\, d\eta \\
&= \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)}{(\tau+\sigma+|\xi+\eta|)^2(\sigma-|\eta|)^2} d\sigma d\eta \\
&\sim \frac{1}{2^l}\int_{\widetilde{D}_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)}{(2^k+|\xi+\eta|)^2} d\eta \\
&\le \frac{1}{2^l}\int_{\widetilde{D}_{k,l}} \frac{1}{(2^k+|\xi+\eta|)^{2-2\alpha}} d\eta \widehat{M}^{2\alpha}(\xi) \\
&\le \frac{C}{2^{(1-2\alpha)k+l}} \widehat{M}^{2\alpha}(\xi)\widehat{S}^{4\alpha}(\tau,\xi).
\end{aligned} \tag{5.35}$$

For the extreme case, we obtain

$$\begin{aligned}
I^1_{k,l}(\tau,\xi) &\sim \frac{1}{2^l}\int_{\widetilde{D}_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)}{(2^k+|\xi+\eta|)^2} d\eta \\
&\le \frac{1}{2^l}\int_{\widetilde{D}_{k,l}} \frac{1}{(2^k+|\xi+\eta|)^{2-4\alpha}} d\eta \\
&\le \frac{C}{2^{(1-2\alpha)k+l}} \widehat{M}^{2\alpha}(\xi)\widehat{S}^{4\alpha}(\tau,\xi).
\end{aligned} \tag{5.36}$$

For the second part, again we distinguish three cases: $|\xi+\eta| \le |\eta|$, $|\xi+\eta| \ge |\eta|$, and the extreme case. For the first two cases, we get

$$\begin{aligned}
I^2_{k,l}(\tau,\xi) &:= \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)(\tau+\sigma+|\xi+\eta|)^2(\sigma-|\eta|)^2}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)} d\sigma\, d\eta \\
&= \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)}{(\tau+\sigma-|\xi+\eta|)^2(\sigma+|\eta|)^2} d\sigma d\eta \\
&\sim \frac{1}{2^{2k-l}}\int_{\widetilde{D}_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)}{(2^l+|\eta|)^2} d\eta \\
&\le \frac{1}{2^{2k-l}}\int_{\widetilde{D}_{k,l}} \frac{1}{(2^l+|\eta|)^{2-2\alpha}} d\eta \widehat{M}^{2\alpha}(\xi) \\
&\le \frac{C}{2^{k+(1-2\alpha)l}} \widehat{M}^{2\alpha}(\xi)\widehat{S}^{4\alpha}(\tau,\xi).
\end{aligned} \tag{5.37}$$

For the extreme case, we have

$$\begin{aligned}
I^2_{k,l}(\tau,\xi) &\sim \frac{1}{2^{2k-l}}\int_{\widetilde{D}_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)}{(2^l+|\eta|)^2} d\eta \\
&\le \frac{1}{2^{2k-l}}\int_{\widetilde{D}_{k,l}} \frac{1}{(2^l+|\eta|)^{2-4\alpha}} d\eta \widehat{M}^{2\alpha}(\xi) \\
&\le \frac{C}{2^{k+(1-2\alpha)l}} \widehat{M}^{2\alpha}(\xi)\widehat{S}^{4\alpha}(\tau,\xi).
\end{aligned} \tag{5.38}$$

For the third part, we get

$$
\begin{aligned}
I^3_{k,l}(\tau,\xi) &:= \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)(\tau+\sigma)\sigma\big[|\xi+\eta||\eta|-(\xi+\eta)\eta\big]}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)} d\sigma d\eta \\
&\le \frac{C}{2^{2k+2l}} \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)(\tau+\sigma)\sigma|\xi+\eta||\eta|}{(\tau+\sigma+|\xi+\eta|)^2(\sigma+|\eta|)^2} d\sigma d\eta \\
&\le \frac{C}{2^{2k+2l}} \int_{D_{k,l}} \widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta) d\sigma d\eta \\
&\le \frac{C}{2^{(1-2\alpha)k+l}} \widehat{M}^{2\alpha}(\xi)\widehat{S}^{4\alpha}(\tau,\xi).
\end{aligned}
\tag{5.39}
$$

The extreme case will not cause trouble since $\xi+\eta$ and η are of the same sign except on a bounded region, i.e. $\big[|\xi+\eta||\eta|-(\xi+\eta)\eta\big]=0$ except on a bounded region. Let us denote the small region by R.

$$
\begin{aligned}
I^3_{k,l}(\tau,\xi) &\le \frac{C}{2^{2k+2l}} \int_R \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)(\tau+\sigma)\sigma|\xi+\eta||\eta|}{(\tau+\sigma+|\xi+\eta|)^2(\sigma+|\eta|)^2} d\sigma d\eta \\
&\le \frac{C}{2^{2k+2l}} \int_R \widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta) d\sigma d\eta \\
&\le \frac{C}{2^{2k+2l}} \int_R d\sigma d\eta 2^{2\alpha k}\widehat{M}^{2\alpha}(\xi) \\
&\le \frac{C}{2^{(1-2\alpha)k+l}} \widehat{M}^{2\alpha}(\xi)\widehat{S}^{4\alpha}(\tau,\xi).
\end{aligned}
\tag{5.40}
$$

Cases E We have the following estimate

$$
\Big\|\frac{\widehat{\widehat{K}}_{+,\cdot,k} * \widehat{K}_{-,\cdot,l}}{\widehat{M}^\alpha\widehat{S}^{2\alpha}}\Big\|_{L^2} \le \frac{C}{2^{\frac{l}{2}}}\frac{1}{2^{(\frac12-\alpha)k}}\Big\|\frac{\widehat{G}_{+,\cdot,k}}{\widehat{M}^\alpha}\Big\|_{L^2}\Big\|\frac{\widehat{G}_{-,\cdot,l}}{\widehat{M}^\alpha}\Big\|_{L^2}, \tag{5.41}
$$

$$
\Big\|\frac{\widehat{\widehat{K}}_{-,\cdot,k} * \widehat{K}_{+,\cdot,l}}{\widehat{M}^\alpha\widehat{S}^{2\alpha}}\Big\|_{L^2} \le \frac{C}{2^{\frac{l}{2}}}\frac{1}{2^{(\frac12-\alpha)k}}\Big\|\frac{\widehat{G}_{-,\cdot,k}}{\widehat{M}^\alpha}\Big\|_{L^2}\Big\|\frac{\widehat{G}_{+,\cdot,l}}{\widehat{M}^\alpha}\Big\|_{L^2}, \tag{5.42}
$$

In these cases, we have $(\tau+\sigma)\sigma<0$. Throughout some algebraic manipulation, the expression Q can be written as

$$
\begin{aligned}
2Q &= (\tau+\sigma+|\xi+\eta|)^2(\sigma+|\eta|)^2 + (\tau+\sigma-|\xi+\eta|)^2(\sigma-|\eta|)^2 \\
&\quad - 8(\tau+\sigma)\sigma\big[|\xi+\eta||\eta|+(\xi+\eta)\eta\big].
\end{aligned}
$$

Take the case of

$$
\widehat{\widehat{K}}_{-,+,k} * \widehat{K}_{+,+,l}, \tag{5.43}
$$

as an example and in which $D_{k,l}=\{(\eta,\sigma):\tau+\sigma+|\xi+\eta|\sim 2^k,\sigma-|\eta|\sim 2^l,(\tau,\sigma,\xi,\eta)\in\Sigma_{k,l}[(-,+);(+,+)]\}$, In this case $\tau+\sigma<0$ and $\sigma>0$. In $\eta\sigma$-plane, this is the region of the intersection of a forward cone with a truncated backward cone. One has the thickness of 2^k and the translation of $(-\xi,-\tau)$, while the other has thickness of 2^l. It is bounded for all cases. We still have the extreme case which is when one cone moves along the other cone, though the region of intersection can be as large as possible, nevertheless it is bounded.

Again for the first part, we can estimate

$$\begin{aligned} I^1_{k,l}(\tau,\xi) &:= \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)(\tau+\sigma+|\xi+\eta|)^2(\sigma+|\eta|)^2}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)} d\sigma d\eta \\ &= \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)}{(\tau+\sigma-|\xi+\eta|)^2(\sigma-|\eta|)^2} d\sigma d\eta \\ &\le \frac{1}{2^{2l}} \int_{D_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}}{(\tau+\sigma-|\xi+\eta|)^2} d\sigma d\eta. \end{aligned} \tag{5.44}$$

To estimate the above integral, we separate the cases for $|\xi+\eta| \ge |\eta|$, $|\xi+\eta| \le |\eta|$, and the extreme case. Throughout some calculations, in each case, we have

$$I^1_{k,l}(\tau,\xi) \le \frac{1}{2^l}\frac{1}{2^{(1-2\alpha)k}} \widehat{M}^{2\alpha}\widehat{S}^{4\alpha}. \tag{5.45}$$

For the second part, we derive

$$\begin{aligned} I^2_{k,l}(\tau,\xi) &:= \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)(\tau+\sigma-|\xi+\eta|)^2(\sigma-|\eta|)^2}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)} d\sigma d\eta \\ &= \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)}{(\tau+\sigma+|\xi+\eta|)^2(\sigma+|\eta|)^2} d\sigma d\eta \\ &\le \frac{C}{2^{2k}} \int_{D_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}}{(\sigma+|\eta|)^2} d\sigma d\eta \\ &\le \frac{C2^l}{2^{2k}} \int_{\widetilde{D}_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}}{(2^l+|\eta|)^{2+2\alpha}} d\eta. \end{aligned} \tag{5.46}$$

To estimate the above integral, we separate the cases for $|\xi+\eta| \ge |\eta|$, $|\xi+\eta| \le |\eta|$, and the extreme case. Throughout some calculations, in each case, we have

$$I^2_{k,l}(\tau,\xi) \le \frac{1}{2^l}\frac{1}{2^{(1-2\alpha)k}} \widehat{M}^{2\alpha}\widehat{S}^{4\alpha}. \tag{5.47}$$

For the third part, we have

$$\begin{aligned} I^3_{k,l}(\tau,\xi) &:= \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)|\tau+\sigma|\sigma\big[|\xi+\eta||\eta|+(\xi+\eta)\eta\big]}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)} d\sigma d\eta \\ &\le \frac{C}{2^{2k+2l}} \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)|\tau+\sigma|\sigma|\xi+\eta||\eta|}{(\tau+\sigma-|\xi+\eta|)^2(\sigma+|\eta|)^2} d\sigma d\eta \\ &\le \frac{C}{2^{2k+2l}} \int_{D_{k,l}} (|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha} d\sigma d\eta. \end{aligned} \tag{5.48}$$

To estimate the above integral, we separate the cases for $|\xi+\eta| \ge |\eta|$, $|\xi+\eta| \le |\eta|$, and the extreme case. Notice that for the extreme case, we have $|\xi+\eta||\eta|+(\xi+\eta)\eta = 0$ except on a small part of the region of the intersection. Throughout some calculations, in each case, we have

$$I^3_{k,l}(\tau,\xi) \le \frac{1}{2^l}\frac{1}{2^{(1-2\alpha)k}} \widehat{M}^{2\alpha}\widehat{S}^{4\alpha}. \tag{5.49}$$

Now we return to the proof of (5.22). Combine the above, we get

$$\begin{aligned}\big|\langle \overline{K}_k K_l, g\rangle\big| \le& C\Big\|\frac{\widehat{G}_k}{\widehat{M}^\alpha}\Big\|_{L^2}\Big\|\frac{\widehat{G}_l}{\widehat{M}^\alpha}\Big\|_{L^2}\Big(\int I_{k,l}(\tau,\xi)\big|\widehat{g}(-\tau,-\xi)\big|^2 d\tau d\xi\Big)^{1/2}\\ &\le \frac{C}{2^{\frac{l}{2}}}\frac{1}{2^{(\frac12-\alpha)k}}\Big\|\frac{\widehat{G}_k}{\widehat{M}^\alpha}\Big\|_{L^2}\Big\|\frac{\widehat{G}_l}{\widehat{M}^\alpha}\Big\|_{L^2}\|\widehat{M}^\alpha\widehat{S}^{2\alpha}\widehat{g}\|_{L^2}\\ &\le \frac{C}{2^{\frac14 l}}\frac{1}{2^{\epsilon k}}\Big\|\frac{\widehat{G}_k}{\widehat{M}^\alpha\widehat{S}^{\frac14}}\Big\|_{L^2}\Big\|\frac{\widehat{G}_l}{\widehat{M}^\alpha\widehat{S}^{\frac14}}\Big\|_{L^2}\|\widehat{M}^\alpha\widehat{S}^{2\alpha}\widehat{g}\|_{L^2}.\end{aligned}\tag{5.50}$$

Finally, we have

$$\begin{aligned}\Big\|\frac{\widehat{\overline{K}}*\widehat{K}}{\widehat{M}^\alpha\widehat{S}^{2\alpha}}\Big\|_{L^2} &\le \sum_{k,l}\Big\|\frac{\widehat{\overline{K}}_k*\widehat{K}_l}{\widehat{M}^\alpha\widehat{S}^{2\alpha}}\Big\|_{L^2}\\ &\le \sum_{k,l}\frac{C}{2^{\epsilon k+\frac14 l}}\Big\|\frac{\widehat{G}_k}{\widehat{M}^\alpha\widehat{S}^{\frac14}}\Big\|_{L^2}\Big\|\frac{\widehat{G}_l}{\widehat{M}^\alpha\widehat{S}^{\frac14}}\Big\|_{L^2} \le C\Big\|\frac{\widehat{G}}{\widehat{M}^\alpha\widehat{S}^{\frac14}}\Big\|^2_{L^2}.\end{aligned}\tag{5.51}$$

This completes the proof. $\square$

The estimates for the remaining cases are given in the following Lemma.

Lemma 5.5. *For $j = 1,2$ and $k = 0,1,2,\cdots$. The following estimates hold*

$$\Big\|\frac{\widehat{\varphi}_T*\big(\delta^{(k)}_{\mp}\widehat{f^\dagger_{\pm,k}\frac{\widehat{D}_\pm}{|\xi|}\gamma^0}\big)*\big(\widehat{K}_j\big)}{\widehat{M}^\alpha\widehat{S}^{2\alpha}}\Big\|_{L^2} \le C(k+1)T^{k-\frac12}\|f_{\pm,k}\|_{H^{-\alpha}}\Big\|\frac{\widehat{G}}{\widehat{M}^\alpha\widehat{S}^{\frac14}}\Big\|_{L^2},$$

$$\Big\|\frac{\widehat{\varphi}_T*\widehat{\overline{K}}_j*\big(\delta^{(k)}_{\pm}\frac{\widehat{D}_\pm}{|\xi|}\widehat{f}_{\pm,k}\big)}{\widehat{M}^\alpha\widehat{S}^{2\alpha}}\Big\|_{L^2} \le C(k+1)T^{k-\frac12}\|f_{\pm,k}\|_{H^{-\alpha}}\Big\|\frac{\widehat{G}}{\widehat{M}^\alpha\widehat{S}^{\frac14}}\Big\|_{L^2},$$

$$\Big\|\frac{\widehat{\varphi}_T*\widehat{\overline{K}}_1*\widehat{K}_2}{\widehat{M}^\alpha\widehat{S}^{2\alpha}}\Big\|_{L^2} \le C\Big\|\frac{\widehat{G}}{\widehat{M}^\alpha\widehat{S}^{\frac14}}\Big\|^2_{L^2},$$

$$\Big\|\frac{\widehat{\varphi}_T*\widehat{\overline{K}}_2*\widehat{K}_j}{\widehat{M}^\alpha\widehat{S}^{2\alpha}}\Big\|_{L^2} \le C\Big\|\frac{\widehat{G}}{\widehat{M}^\alpha\widehat{S}^{\frac14}}\Big\|^2_{L^2},$$

The proof of this lemma is a repetition of the arguments presented in Lemmas 5.1, 5.2, and 5.4, so that we omit it.

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