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FROM DISCRETE BOLTZMANN EQUATION TO COMPRESSIBLE LINEARIZED EULER EQUATIONS

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Abstract. This paper concerns the asymptotic analysis of the linearized Euler limit for a general discrete velocity model of the Boltzmann equation. This is done for any dimension of the physical space, for densities which remain in a suitable small neighbourhood of global Maxwellians. Providing that the initial fluctuations are smooth, the scaled solutions of discrete Boltzmann equation are shown to have fluctuations that locally in time converge weakly to a limit governed by a solution of linearized Euler equations. The weak limit becomes strong when the initial fluctuations converge to appropriate initial data. As applications, the two-dimensional 8-velocity model and the one-dimensional Broadwell model are analyzed in detail.

1. Introduction

This paper shows how a suitable asymptotic analysis of a discrete kinetic theory leads to a macroscopic model of the compressible linearized Euler equations. The analysis is applied to the discrete Boltzmann equation which is a nonlinear mathematical model of the kinetic theory of gases, that describes the evolution of a gas of particles allowed to move in all space with a finite number of velocities. The discrete kinetic theory was systematically developed in the Lecture Notes by Gatignol [20], which provides a detailed analysis of the relevant aspects of the theory: modelling, analysis of thermodynamic equilibrium, and application to fluid-dynamic problems. The interested reader can recover in the book by Gatignol [20] various examples of models. Recent developments which include generalizations to arbitrary number of velocities, e.g. [1, 25], and development of computational schemes, are reported in the book edited by Bellomo and Gatignol [5]. The mathematical literature concerning the analysis of the initial and initial-boundary value problem is reviewed in [6].

The asymptotic theory for small Knudsen numbers for models of the kinetic theory of gases means, as known [29], deals with the analysis of the macroscopic description delivered by the kinetic equations when the distance between particles tends to zero. Then one obtains a macroscopic description from the microscopic one as an alternative to the purely phenomenological derivation. The method applies to

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various kinetic equations. Different scaling generate different macroscopic equations as documented by the formal expansions proposed in [13].

On the other hand, the derivation of fluid dynamical equations by methods of the kinetic theory, is well understood at the formal level, however its full mathematical justifications is still missing. Indeed, the justification of the formal approximation for the classical Boltzmann equation has shown to be difficult considering that many basic regularity questions remain unsolved. Some approaches to overcome these difficulties have emerged in the pertinent literature see [2, 3, 4, 7, 8, 9, 10, 11, 12, 15, 16, 17, 18, 19, 23, 24, 26, 28, 29, 30, 31, 33, 36]. Specifically, a rigorous result about the fluid-dynamical limit for the discrete Boltzmann equation towards the compressible Euler equations provided that the initial fluctuations are smooth was obtained by Caflish and Papanicolaou [16] for the one-dimensional Broadwell model. They proved the validity of the fluid-dynamical approximation for this model up to the first appearance of a shock discontinuity in the corresponding Euler equations. Their method is based on the (assumed) existence of smooth solutions to the fluid equations. On the other hand, it was shown by Inoue, and Nishida [26] that the solution of the Broadwell model will be smooth for a finite time, with analytic initial data and, for $\varepsilon \to 0$, arbitrarily close to the local Maxwellian, where ε is a dimensionless parameter related to the Knudsen number. This solution converges strongly to the solution of the compressible Euler equations. The proof uses an abstract Cauchy-Kowalewski Theorem in the scale of Banach spaces of analytic functions, that is the method proposed by Nirenberg [32] and Ovsjannicov [34]. Recently, the hydrodynamical limit for the nonlinear discrete Boltzmann equation towards the incompressible Navier-Stokes was investigated in [9, 10].

This present work describes the asymptotic trend of the solutions of the discrete Boltzmann equation to the solutions of the linearized compressible Euler equations. This paper consists of 8 Sections. In Section 2, we give a review of the basic concepts in the discrete kinetic theory for later use. Sections 3 and 4 provide the statement of the problem and introduce the associated fluid equations. Precisely, Section 4 deals with the formal scaling that leads from the discrete model to the linearized compressible Euler equations. The asymptotic behavior as $\varepsilon \to 0$ of the solution is investigated. Formally, it is shown that fluctuations of order to $\varphi(\varepsilon)$ ($\varphi(\varepsilon) \to 0$ when $\varepsilon \to 0$) converge to the solution of the linearized Euler equations, which are strictly hyperbolic. A proposition concerning this formal derivation is proposed. A precise statement is given in Section 5. Section 6 contains the proof of uniform existence of the solutions. The proof holds with the scaling of the fluctuation $\varphi(\varepsilon) = O(\varepsilon^p), p \geq \frac{1}{2}$. In Section 7, when the scaling is more restrictive $\varphi(\varepsilon) = O(\varepsilon^p), p \geq 1$, an estimate of the derivative of solutions is obtained, and is used to prove the strong convergence of the solution to the solution of the fluid equations. Applications concerning the two-dimensional 8-velocity model and the one-dimensional Broadwell model are dealt with in Section 8.

2. Preliminaries

Some basic concepts of the discrete kinetic theory are summarized in this Section. Let us denote by m the space dimension and by $v_1, \ldots, v_n$ constants vectors in $\mathbb{R}^m$. As known [20], the so called *discrete velocity Boltzmann equation* can be written as follows:

$$\partial_t F_i + v_i.\nabla_x F_i = Q_i(F, F), \quad i = 1, \ldots, n, \tag{2.1}$$

where $F_i = F_i(t,x)$ represents the mass density of gas particles linked to the constant velocities $v_i = (v_i^1,\dots,v_i^m) \in \mathbb{R}^m$ at time $t \geq 0$ and position $x = (x_1,\dots,x_m) \in \mathbb{R}^m$; Q_i is a quadratic operator related to the binary collisions:

$$Q_i(F,G) = \frac{1}{2\alpha_i}\sum_{jkl}\{(A_{kl}^{ij}(F_kG_l + F_lG_k) - A_{ij}^{kl}(F_iG_j + F_jG_i)\}, \tag{2.2}$$

where α_i are positive constants, and where the terms A_{ij}^{kl} are the so-called *transition rates* referred to the collisions

$$(v_i, v_j) \longleftrightarrow (v_k, v_l)$$

for a collision scheme such that momentum and energy are preserved in the collision.

The transition rates are positive constants which, according to the indistinguishability property of the gas particles and to the reversibility of the collisions, satisfy the following relations

$$A_{lk}^{ij} = A_{kl}^{ij} = A_{kl}^{ji}. \tag{2.3}$$

A detailed computation of the terms A_{ij}^{kl} can be obtained by specifying the velocity discretization and analyzing the related collision mechanics.

A large part of the models for simple monoatomic gases, with binary collisions is described in the lectures notes by Gatignol [20]. In this case, gas particles collide by simple binary collisions which preserve momentum and energy:

$$\begin{aligned} v_i + v_j &= v_l + v_k,\\ v_i^2 + v_j^2 &= v_l^2 + v_k^2. \end{aligned}$$

Moreover the transition rates are linked to the corresponding transition probability densities by the relation

$$A_{ij}^{kl} = S|v_i - v_j|a_{ij}^{kl},$$

where S is the cross section area and a_{ij}^{kl} denotes the transition probability density, which is characterized, as A_{ij}^{kl}, by the properties indicated in (2.3) and, in addition, by the normalization property with respect to one,

$$\sum_k a_{ij}^{kl} = \sum_l a_{ij}^{kl} = 1.$$

A summational invariant is an element $\phi = {}^t(\phi_1,\dots,\phi_n)$ of $\mathbb{R}^n$ such that for all $i,j,k,l = 1,\dots,n$,

$$A_{kl}^{ij}\Big(\frac{\phi_i}{\alpha_i} + \frac{\phi_j}{\alpha_j} - \frac{\phi_k}{\alpha_k} - \frac{\phi_l}{\alpha_l}\Big) = 0\,. \tag{2.4}$$

It is well-known that any one of the following three properties implies the others.

- (i) ψ is a summational invariant
- (ii) $\langle \psi, Q(F,G)\rangle = 0$ for all $F,G \in \mathbb{R}^n$
- (iii) $\langle \psi, Q(F,F)\rangle = 0$ for all $F \in \mathbb{R}^n$.

Here $\langle,\rangle$ denotes the standard inner product in $\mathbb{R}^n$. The set of summational invariants, is denoted by $\mathbb{M}$. Then $0 < \dim\mathbb{M} < n$ because ${}^t(\alpha_1,\dots,\alpha_n) \in \mathbb{M}$ and $\mathbb{M} \neq \mathbb{R}^n$. Let $\dim\mathbb{M} = r$, while $\{\psi^{(1)},\dots,\psi^{(r)}\}$ is a basis of $\mathbb{M}$. For $F \in \mathbb{R}^n$, we put

$$w_k = \langle \psi^{(k)}, F\rangle, \quad k = 1,\dots,r, \tag{2.5}$$

and let $w = {}^t(w_1,\dots w_r)$. The w_k are called hydrodynamical moments or simply moments of F with respect to the basis $\{\psi^{(1)},\dots,\psi^{(r)}\}$.

Let $F = {}^t(F_1,\dots,F_n) \in \mathbb{R}^n$. We write $F > 0$ if $F_i > 0$ for all $i = 1,\dots,n$. Then $F = {}^t(F_1,\dots,F_n) > 0$ is called a Maxwellian if

$$A_{kl}^{ij}(F_iF_j - F_kF_l) = 0, \quad \text{for all} \quad i,j,k,l = 1,\dots,n. \tag{2.6}$$

In particular, $F(t,x) > 0$ is called absolute Maxwellian if it is a locally Maxwellian state and is independent of t and x. Any of the following three properties implies the others, provided that $F_i > 0$ for $i = 1,\dots,n$

- (i) F is a Maxwellian
- (ii) ${}^t(\alpha_1 \log F_1,\dots,\alpha_n \log F_n) \in \mathbb{M}$,
- (iii) $Q(F,F) = 0$.

Let us denote by $\mathbb{N}$ the set of all Maxwellians ($\mathbb{N}$ is a r-dimensional open manifold in $\mathbb{R}^n$), and let $F \in \mathbb{N}$. Then, the following expression holds

$$\alpha_i \log F_i = \sum_{l=1}^{r} u_l \psi_i^{(l)}, \quad i = 1,\dots,n. \tag{2.7}$$

Here $u_l \in \mathbb{R}$, $l = 1,\dots,r$, and $\psi_i^{(l)}$ denote the i-th component of $\psi^{(l)}$. Let us put $u = {}^t(u_1,\dots,u_r)$, then it is easily to show that the application $F \to u$ is a one-to one map. The domain of this mapping is $\mathbb{N}$. Let us call u the standard coordinates of F with respect to the basis $\{\psi^{(1)},\dots,\psi^{(r)}\}$. The range of the above mapping $F \to u$ coincides with $\mathbb{R}^r$. If $F = M(u)$ denotes the inverse mapping $u \mapsto F$, then one has $M(u) = {}^t(M_1(u),\dots,M_n(u))$ with

$$M_i(u) = \exp\Big(\frac{1}{\alpha_i}\sum_{l=1}^{r} u_l \psi_i^{(l)}\Big), \quad i = 1,\dots,n. \tag{2.8}$$

Therefore, the moments w of a Maxwellian $F = M(u)$ with respect to the basis $\{\psi^{(1)},\dots,\psi^{(r)}\}$ can be regarded as functions of u. Then $w(u) = {}^t(w_1(u),\dots,w_r(u))$ with

$$w_k(u) = \langle \psi^{(k)}, M(u)\rangle, \quad k = 1,\dots,r. \tag{2.9}$$

A direct computation yields $\frac{\partial}{\partial u_l} M(u) = \Lambda_{M(u)}\psi^{(l)}$, $l = 1,\dots r$. Here

$$\Lambda_M = \operatorname{diag}\Big(\frac{M_1}{\alpha_1},\dots,\frac{M_n}{\alpha_n}\Big), \tag{2.10}$$

for $M = {}^t(M_1,\dots,M_n)$. Let the functional matrix $D_u w(u)$ be defined by

$$D_u w(u) = (\langle \psi^k, \Lambda_{M(u)}\psi^{(l)}\rangle)_{1\le k,l\le r}.$$

Since each component of $M(u)$ is positive, $D_u w(u)$ is real symmetric and positive definite for $u \in \mathbb{R}^r$. Hence, in view of the fact that $\mathbb{R}^r$ is a convex set, one conclude that the mapping $u \to w$ is one-to-one.

Let us now denote by Ω the range of this mapping, then Ω is a convex open set in $\mathbb{R}^r$ and the mapping $u \to w$ defined by (2.9) is a diffeomorphism from $\mathbb{R}^r$ onto Ω. This result is due to Gatignol [20, 21]. Consequently, any Maxwellian F can be expressed uniquely as $F = M(u(w))$ by using the moments w of F, once a basis of $\mathbb{M}$ is fixed. Here $u = u(w)$ denotes the inverse mapping $w \to u$. It needs to be remarked that the moments are called macroscopic variables also.

Also we introduce the linearized collision operator. This is obtained if we linearize (2.1) by putting $F(t,x) = M + \Lambda_M^{1/2} f$ with Λ_M given as (2.10). The precise definition is as follows. Let $M = {}^t(M_1,\dots,M_n)$ and let $M_i > 0$ for $i = 1,\dots,n$. Let L_M be an $n\times n$ matrix such that

$$L_M f = -2\Lambda_M^{-\frac{1}{2}} Q(M, \Lambda_M^{1/2} f), \quad \text{for any } f \in \mathbb{R}^n. \tag{2.11}$$

Then, if M is a Maxwellian, L_M is called the linearized collision operator. The following result concerning L_M is well-known: If M is a Maxwellian, L is real symmetric and positive semi-definite, then null space is

$$N(L_M) = \Lambda_M^{1/2}\mathbb{M}.$$

The interested reader is referred to [20, 21, 27, 28], for the topics dealt with in this Section.

3. The hydrodynamical limit

The aim of this Section is to write down the general form of the compressible Euler equation and their linearized equations around the constant state.

The compressible Euler system. We introduce a small parameter $\varepsilon > 0$ and write the system (2.1) in the form

$$\partial_t F_\varepsilon + \sum_{j=1}^m V^j \partial_{x_j} F_\varepsilon = \frac{1}{\varepsilon} Q(F_\varepsilon, F_\varepsilon), \tag{3.1}$$

where $F_\varepsilon = {}^t(F_1,\dots,F_n), Q = {}^t(Q_1,\dots,Q_n)$, $V^j = \operatorname{diag}(v_j^1,\dots,v_j^n)$ and the parameter ε denotes the Knudsen number.

Taking the inner product of (3.1) and $\psi^{(k)}, k = 1,\dots,r$, yields

$$\frac{\partial}{\partial t} w_k + \sum_{j=1}^m \frac{\partial}{\partial x_j}\langle \psi^{(k)}, V^j F\rangle = 0, \tag{3.2}$$

where w_k is given by (2.5).

The number density F_ε is relaxed, as $\varepsilon \to 0$, to a local equilibrium distribution. Suppose that F_ε has a limit F and letting ε tend to 0 in (3.1), yields

$$Q(F, F) = 0. \tag{3.3}$$

It follows from (3.3) that F is a Maxwellian. Hence, recalling the arguments in the preceding Section, one sees that $F = M(u(w))$. Setting $F = M(u(w))$ in (3.2) gives the Euler equations in the form

$$\frac{\partial w_k}{\partial t} + \sum_{j=1}^m \frac{\partial}{\partial x_j}\langle \psi^{(k)}, V^j M(u(w))\rangle = 0, \quad k = 1,\dots r. \tag{3.4}$$

Let us now substitute $w = w(u)$ into (3.4) and change the dependent variable from w to u. This manipulation yields

$$A^0(u)\frac{\partial u}{\partial t} + \sum_{j=1}^m A^j(u)\frac{\partial u}{\partial x_j} = 0, \tag{3.5}$$

where

$$A^0(u) = D_u w(u) = (\langle \psi^{(k)}, \Lambda_{M(u)}\psi^{(l)}\rangle)_{1\le k,l\le r}, \tag{3.6}$$

$$A^j(u) = (\langle \psi^{(k)}, \Lambda_{M(u)} V^j \psi^{(l)}\rangle)_{1\le k,l\le r, \quad j=1,\dots m}. \tag{3.7}$$

The property that $\Lambda_{M(u)}$ and V^j $(j = 1, \dots, m)$ commute with each other, have been used to derive (3.5). Moreover the following properties are easily checked

- (i) $A^0(u)$ is real symmetric and positive definite for $u \in \mathbb{R}^r$.
- (ii) $A^j(u), j = 1, .., m$, are real symmetric for $u \in \mathbb{R}^r$.

Then we conclude that the Euler equation (3.4) can be rewritten as the symmetric hyperbolic system (3.5)-(3.7).

The linearized compressible Euler system. Assume that the fluctuations of density are of the order of ε, and introduce the change of functions

$$u = u_0 + \varepsilon U + O(\varepsilon^2), \tag{3.8}$$

where u_0 is a constant in t and x. In order to derive the equation for U, one needs some preliminary analysis.

Lemma 3.1. *Let u be as defined in* (3.8). *One has*

$$\Lambda_{M(u)} = B_0 + \varepsilon B_1 + O(\varepsilon^2), \tag{3.9}$$

where

$$B_0 = \operatorname{diag}\big(\frac{M_i(u_0)}{\alpha_i}\big), \quad B_1 = \operatorname{diag}\big(\frac{M_i(u_0)}{\alpha_i^2} \sum_{l=1}^{r} U_l \psi_i^{(l)}\big). \tag{3.10}$$

Proof. By substituting (3.8) into (2.8) and using Taylor formula, yields

$$\begin{aligned} M_i(u) &= M_i(u_0) exp(\frac{\varepsilon}{\alpha_i} \sum_{l=1}^{r} U_l \psi_i^{(l)} + O(\varepsilon^2)) \\ &= M_i(u_0)(1 + \frac{\varepsilon}{\alpha_i} \sum_{l=1}^{r} U_l \psi_i^{(l)} + O(\varepsilon^2)) \end{aligned}$$

which, when we substitute it into (2.10), yields (3.9)-(3.10). □

Let $\{e^{(i)}, i = 1, \dots, r\}$ denote an orthonormal basis for $N(L_{M(u_0)})$, and $\psi^{(i)}$ denotes the image of $\{e^{(i)}\}$ $(\psi^{(i)} = \Lambda_{M(u_0)}^{-1/2} e^{(i)})$. Then $\{\psi^{(i)}\}$, $i = 1, \dots, r$ is an orthonormal basis of $\mathbb{M}$. One has

Lemma 3.2. *Let u be as* (3.8)*, then one has*

$$A^0(u) = H_0 + \varepsilon H_1 + O(\varepsilon^2), \quad A^j(u) = K_0^j + \varepsilon K_1^j + O(\varepsilon^2), \tag{3.11}$$

$$(H_0)_{l,k} = \delta_{l,k}, \quad (K_0^j)_{l,k} = \langle V^j e^{(l)}, e^{(k)} \rangle, \quad 1 \le l, k \le r, \tag{3.12}$$

$$(H_1)_{l,k} = \langle \psi^{(k)}, B_1 \psi^{(l)} \rangle, \quad (K_1^j)_{l,k} = \langle \psi^{(k)}, B_1 V^j \psi^{(l)} \rangle, \quad 1 \le l, k \le r, \tag{3.13}$$

Proof. By (3.6), (3.7), and (3.9)-(3.10), it is easy to get (3.11) and (3.13) with $(H_0)_{l,k} = \langle \psi^{(k)}, B_0 \psi^{(l)} \rangle$ and $(K_0^j)_{l,k} = \langle \psi^{(k)}, B_0 V^j \psi^{(l)} \rangle$. One obtains

$$(H_0)_{l,k} = \langle \psi^{(k)}, B_0 \psi^{(l)} \rangle = \sum_i \frac{M_i(u_0)}{\alpha_i} \psi_i^{(k)} \psi_i^{(l)} = \sum_i e_i^{(k)} e_i^{(l)} = (\delta_{k,l}).$$

In the same way, one has

$$(K_0^j)_{l,k} = \langle \psi^{(k)}, B_0 V^j \psi^{(l)} \rangle = \sum_i \frac{M_i(u_0)}{\alpha_i} V_i^j \psi_i^{(k)} \psi_i^{(l)} = \sum_i V_i^j e_i^{(k)} e_i^{(l)}.$$

This completes the proof. □

Now, by inserting (3.8) into system (3.5), using Lemma 3.2, and comparing terms of equal order in ε, the following linear system for U is obtained:

$$\frac{\partial U}{\partial t} + \sum_{j=1}^{m} C^j \frac{\partial U}{\partial x_j} = 0, \tag{3.14}$$

where the C^j are given by

$$C^j = (\langle V^j e^{(l)}, e^{(k)} \rangle)_{1 \le l,k \le r}. \tag{3.15}$$

It is easy to see that C^j, $j = 1, \dots m$ are real symmetric. So we conclude that the linearized system (3.14)-(3.15) is hyperbolic.

4. Formal derivation of linearized Euler system from discrete velocity models

The linearized Euler equations (3.14)-(3.15) can be formally derived from the discrete Boltzmann equation through a scaling in which the density F is close to the absolute Maxwellian M. More precisely, let us consider families of solutions parameterized by the Knudsen number as follows:

$$F_\varepsilon(0) = M + \varphi(\varepsilon) {\Lambda_M}^{1/2} f_\varepsilon(0), \quad F_\varepsilon = M + \varphi(\varepsilon) {\Lambda_M}^{1/2} f_\varepsilon, \tag{4.1}$$

where the fluctuations f_ε and $f_\varepsilon(0)$ are bounded while $\varphi(\varepsilon)$ satisfies

$$\varphi(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{4.2}$$

Consider now a family of formal solutions F_ε to the initial-value problem for the scaled discrete Boltzmann equation

$$\begin{gathered} \partial_t F_\varepsilon + \sum_{j=1}^{m} V^j \partial_{x_j} F_\varepsilon = \frac{1}{\varepsilon} Q(F_\varepsilon, F_\varepsilon), \quad t > 0,\ x \in \mathbb{R}^m, \\ F_\varepsilon(0, x) = F_0(x), \end{gathered} \tag{4.3}$$

whose fluctuations f_ε are given by (4.1) for some $\varphi(\varepsilon)$ that vanishes with ε as in (4.2).

The derivation is developed in two steps: The first step defines the form of the limiting function f. Note that by (4.3) the fluctuations f_ε satisfy

$$\begin{gathered} \varepsilon(\partial_t f_\varepsilon + \sum_{j=1}^{m} V^j \partial_{x_j} f_\varepsilon) + L f_\varepsilon = \varphi(\varepsilon) \Gamma(f_\varepsilon, f_\varepsilon), \\ f_\varepsilon(0, x) = f_0(x), \end{gathered} \tag{4.4}$$

where the operator L is given by (2.11) and

$$\Gamma(f, g) = \Lambda_M^{-\frac{1}{2}} Q(\Lambda_M^{1/2} f, \Lambda_M^{1/2} g). \tag{4.5}$$

It is easily to see that the range of Γ is a subset of $N(L)^\perp$.

Suppose f_ε has a limit f and let $\varepsilon \to 0$ in (4.4), one finds that $Lf = 0$. We can then conclude that f has the form

$$f = \sum_{i=1}^{r} U_i e^{(i)}, \tag{4.6}$$

for some $U_i = U_i(t, x), i = 1, \dots, r$.

The second step shows that the evolution of U_i is governed by the linearized Euler equation (3.14)-(3.15). Observe that the fluctuations f_ε formally satisfy the local conservation laws

$$\langle \partial_t f_\varepsilon, e^{(i)}\rangle + \langle \sum_{j=1}^{m} V^j \partial_{x_j} f_\varepsilon, e^{(i)}\rangle = 0, \quad i = 1, \dots, r. \tag{4.7}$$

By letting $\varepsilon \to 0$ in (4.7) and using the Maxwellian form of f given by (4.6), one finds that U solves the local conservation laws of the linearized Euler equations (3.14)-(3.15). By the formal continuity in time of the density in (4.7), one finds that

$$U(0) = \lim_{\varepsilon\to 0} \langle e, f_\varepsilon(0)\rangle, \tag{4.8}$$

provided that the limits on the right-hand side exist in the distributional sense. Here $U = (U_1, \dots, U_r)$ and $e = (e^{(1)}, \dots, e^{(r)})$. The above formal derivation can be stated more precisely as follows:

Proposition 4.1 (Formal Linearized Euler Theorem)**.** *Let F_ε be a family of distribution solutions of the scaled discrete Boltzmann initial-value problem* (4.3) *with initial data F_0. Let F_ε and $F_\varepsilon(0)$ have fluctuations f_ε and $f_\varepsilon(0)$ given by* (4.1) *that are bounded families for some $\varphi(\varepsilon)$ that vanishes with ε as in* (4.2)*. Also assume that:*

1. *The local conservation laws* (4.7) *are also satisfied in the distributional sense for every f_ε.*
2. *The family f_ε converges in the distributional sense as $\varepsilon \to 0$ to f. Assume, in addition, that $Lf_\varepsilon \to Lf$, that the moments $\langle f_\varepsilon, e^{(i)}\rangle$ converge to the corresponding moments $\langle f, e^{(i)}\rangle$ as $\varepsilon \to 0$.*
3. *The family $f_\varepsilon(0)$ satisfies* (4.8) *in the distributional sense.*

Then f is the unique local Maxwellian (4.6) *determined by the solution U of the linearized Euler equation* (3.14)-(3.15) *with the initial data $U(0)$ obtained from* (4.8)*.*

The above approach will be fully justified in the next Section.

5. Main result

The main result of this Section are an existence Theorem that holds for all $\varepsilon > 0$ and a proof of the validity of the fluid-dynamical approximation (3.14)-(3.15). To state our result precisely, some function spaces need to be introduced.

Let $C(\Omega, X)$ and $L^\infty(\Omega, X)$ denote the spaces of the continuous and bounded functions on $\Omega \subset \mathbb{R}$ with values in a Banach space X, respectively.

Let H^l denote the $L^2(\mathbb{R}^m)$- Sobolev space of order l, with the norm $|.|_l$.

Let

$$\varphi(\varepsilon) = O(\varepsilon^p), \quad p \geq \frac{1}{2}.$$

One gets the following:

Case $p > 1/2$

Theorem 5.1. *Let $l \geq \frac{m}{2} + 1$. Then there exists a_0 such that for any $\varepsilon > 0$ and for any $f_0 \in H^l(\mathbb{R}^m)$ with $\|f_0\|_l \leq a_0$, there exist positive constants T_ε and k, such that the initial value problem* (4.4) *has a unique solution $f_\varepsilon \in L^\infty([0, T_\varepsilon], H^l(\mathbb{R}^m)) \cap C([0, T_\varepsilon], H^{l-1}(\mathbb{R}^m))$ satisfying*

$$\|f_\varepsilon(t)\|_l \leq k, \quad for \quad t \in [0, T_\varepsilon]. \tag{5.1}$$

We remark that T_ε approaches $+\infty$, as $\varepsilon \to 0$ and $T_\varepsilon = O(\frac{1}{\varepsilon^{2p-1}})$.

Case $p = 1/2$

Theorem 5.2. *Let $l \geq \frac{m}{2}+1$. If $f_0 \in H^l(\mathbb{R}^m)$, then there exists a positive constants T and k (depending only on $\|f_0\|_l$) such that the initial value problem* (4.4) *has a unique solution $f_\varepsilon \in L^\infty([0,T], H^l(\mathbb{R}^m)) \cap C([0,T], H^{l-1}(\mathbb{R}^m))$ satisfying*

$$\|f_\varepsilon(t)\|_l \leq k, \quad \textit{for} \quad t \in [0,T]. \tag{5.2}$$

Theorem 5.3. *Let f_ε be as in Theorem 5.1 or Theorem 5.2. Then, as $\varepsilon \to 0$, $f_\varepsilon \to f$ weakly $\star$ in $L^\infty([0,T], H^l(\mathbb{R}^m))$ for any $T > 0$, and the limit has the form*

$$f = \sum_{i=1}^{r} U_i e^{(i)}, \tag{5.3}$$

where $U = (U_1, \dots, U_r)$ satisfies

$$\begin{gathered} \frac{\partial U}{\partial t} + \sum_{j=1}^{m} C^j \frac{\partial U}{\partial x_j} = 0, \\ U(t=0) = \langle f_0, e\rangle, \end{gathered} \tag{5.4}$$

where C^j, $j = 1\dots, m$ are given by (3.15).

This Theorem shows that discrete velocity models can be approximated locally in time as $\varepsilon \to 0$ by the linearized Euler equations (5.4).

When the scaling assumptions on $\varphi(\varepsilon)$ is more restrictive, $\varphi(\varepsilon) = O(\varepsilon^p)$, $p \geq 1$ and the initial datum satisfies

$$\begin{gathered} f_\varepsilon(0) = h_\varepsilon + \varepsilon k_\varepsilon \quad \text{where} \quad h_\varepsilon \in N(L), \quad \text{and} \quad k_\varepsilon \in H^l(\mathbb{R}^m), \\ \lim_{\varepsilon\to 0} \|h_\varepsilon - h\|_{l-1} = 0, \end{gathered} \tag{5.5}$$

then strong convergence is obtained.

Theorem 5.4. *Assume* (5.5)*, and let f_ε be as in Theorem 5.1 or 5.2. Then, as $\varepsilon \to 0$, $f_\varepsilon \to f$ weakly $\star$ in $L^\infty([0,T], H^l(\mathbb{R}^m))$ and strongly in $C([0,T], H^{l-1}(\mathbb{R}^m))$ for any $T > 0$, and the limit has the form* (5.3)*, where $U = (U_1, \dots, U_r)$ satisfies*

$$\begin{gathered} \frac{\partial U}{\partial t} + \sum_{j=1}^{m} C^j \frac{\partial U}{\partial x_j} = 0, \\ U(t=0) = \langle h, e\rangle. \end{gathered} \tag{5.6}$$

Let $(U^\varepsilon) = \langle e, f_\varepsilon\rangle$. Since $\{e^{(i)}, i = 1, \dots r\}$ forms an orthogonal system, U in (5.3) is given by $U = \langle e, f\rangle$. One gets the following result.

Theorem 5.5. *Assume that* (5.5) *holds. Then, as $\varepsilon \to 0$, $U^\varepsilon \to U$ weakly $\star$ in $L^\infty([0,T], H^l(\mathbb{R}^m))$ and strongly in $C([0,T], H^{l-1}(\mathbb{R}^m))$ for any $T > 0$, and the limit U satisfies the Linearized Euler system* (5.6).

Remark. (i) The use of the spaces H^l is necessary in our proof because the nonlinear term Γ defined by (4.5) is bounded for l high enough.

(ii) Let $h_\varepsilon = \Lambda_M^{1/2} t_{(\alpha_1,\alpha_2,\dots,\alpha_n)} w_\varepsilon$, one gets $Lh_\varepsilon = 0$, while an example of assumptions (5.5) is given by

$$\begin{gathered} f_\varepsilon = \Lambda_M^{1/2} t_{(\alpha_1,\alpha_2,\dots,\alpha_n)} w_\varepsilon + \varepsilon k_\varepsilon, \\ \lim_{\varepsilon\to 0} \|w_\varepsilon - w\|_{l-1} = 0. \end{gathered}$$

Acoustic fluid dynamical limit (linearized Euler equations) for the classical Boltzmann equation has been dealt with in [23] for any periodic spatial domain of two or more dimensions. Indeed it was shown that the scaled families of DiPerna-Lions renormalized solutions have fluctuations that globally in time converge weakly to a unique limit governed by a solution of Acoustic equations provided that the fluid moments of their initial fluctuations converge to appropriate L^2 initial data and the scaling of the fluctuations with respect to Knudsen number is essentially optimal. Moreover, the limit becomes strong when the initial fluctuations converge entropically to appropriate L^2 initial data. The proof uses the averaging lemma (cf. [22]). The averaging lemma is valid for continuous solutions and has no counterpart for discrete velocity models (except in one space dimension (cf. Tartar [35]). Recently, asymptotic limit for kinetic models towards the non linearized compressible Euler equations or towards the Acoustic equations when the Knudsen number ε tends to zero has been dealt with in [12]. The existence and uniqueness theorem, in these case, is proven for analytic initial fluctuations on the time interval independent of the small parameter ε. As ε tends to zero, the solution of kinetics models converges strongly to the Maxwellian whose fluid-dynamics parameters solve the Euler and the Acoustic systems.

Here we establish a so-called linearized Euler limit (5.4) for the discrete Boltzmann equation in any space dimension. Equation (4.4) is solved by using the principle of contraction mappings, by means of the iteration scheme related to Lemma 6.3 and 6.4. The convergence of the scheme to a solution of equation (4.4) is proved by using suitable properties of the operators L and Γ.

The strong convergence of the solution of equation (4.4) as $\varepsilon \to 0$ is proved by the uniform estimate and the equicontinuity in $t \in [0,T]$ of the solution with respect to $\varepsilon \in (0,1)$ (Lemma 7.1) provided that the initial fluctuation is smooth, and closed to an $N(L)$ element which converges to appropriate initial data.

6. Uniform existence

This Section deals with the proof of the local existence of solutions to (4.4). Some preliminary estimates are necessary for the proof.

Estimates.

Lemma 6.1. *Let $p > 1/2$ and let $z(t,x)$ be a given function of t and x such that $\|z(t)\|_l \le k$, and let $g(t,x)$ satisfy the linear system*

$$\begin{gathered}\partial_t g + \sum_{j=1}^{m} V^j \partial_{x_j} g + \frac{1}{\varepsilon} L g = \frac{\varphi(\varepsilon)}{\varepsilon}\Gamma(z,g), \\ g(0,x) = g_0(x).\end{gathered} \tag{6.1}$$

Then there exist suitable constants a_0, T_ε such that for any $g_0 \in H^l$ with $\|g_0\|_l \le a_0$, a constant k can be chosen such that

$$\sup_{0\le t\le T_\varepsilon} \|g(t)\|_l \le k. \tag{6.2}$$

Proof. The Fourier transform of (6.1) yields

$$\partial_t \hat{g} + \sum_{j=1}^{m} V^j i\zeta_j \hat{g} + \frac{1}{\varepsilon} L\hat{g} = \frac{\varphi(\varepsilon)}{\varepsilon}\hat{\Gamma}(z,g). \tag{6.3}$$

Taking the inner product (in C^m) of (6.3) with $\hat{g}$, and considering that $\sum_{j=1}^m V^j\zeta_j$ and L are real, symmetric, shows that the real part of (6.3) can be written as follows

$$\frac{\partial_t|\hat{g}|^2}{2} + \frac{1}{\varepsilon}\langle L\hat{g},\hat{g}\rangle = \frac{\varphi(\varepsilon)}{\varepsilon} Re\langle\hat{\Gamma}(z,g),\hat{g}\rangle, \tag{6.4}$$

where $\langle , \rangle$ denotes the standard inner product in C^m.

Let $P^\perp$ be the orthogonal projection operator onto $N(L)^\perp$. Noting that L is positive semi-definite, and as the range of Γ is a subset of $N(L)^\perp$, (6.4) gives the estimate

$$\begin{aligned}\frac{\partial_t|\hat{g}|^2}{2} + \frac{C_1}{\varepsilon}|P^\perp\hat{g}|^2 &\le \varphi(\varepsilon)\frac{|\hat{\Gamma}(z,g)||P^\perp\hat{g}|}{\varepsilon}\\ &\le \frac{\varphi(\varepsilon)^2}{2C_1\varepsilon}|\hat{\Gamma}(z,g)|^2 + \frac{C_1}{2\varepsilon}|P^\perp\hat{g}|^2,\end{aligned}$$

which implies in particular that

$$\frac{\partial_t|\hat{g}|^2}{2} \le C\varepsilon^{2p-1}|\hat{\Gamma}(z,g)|^2. \tag{6.5}$$

Therefore, multiplying (6.5) by $(1+|\zeta|^2)^l$, and integrating over $[0,t]\times\mathbb{R}^m_\zeta$, gives, from Plancherel's Theorem, the inequality

$$\|g\|_l^2 \le \|g_0\|_l^2 + C\varepsilon^{2p-1}k^2T\sup_{t\in[0,T]}\|g(t)\|_l^2. \tag{6.6}$$

Let α be fixed and let T be such that

$$C\varepsilon^{2p-1}T \le \alpha.$$

Then using (6.6),

$$\sup_{t\in[0,T]}\|g(t)\|_l \le \|g_0\|_l + k\sqrt{\alpha}\sup_{t\in[0,T]}\|g(t)\|_l,$$

which implies that if

$$k\sqrt{\alpha} < 1, \tag{6.7}$$

then

$$\sup_{t\in[0,T]}\|g(t)\|_l \le \frac{\|g_0\|_l}{1-k\sqrt{\alpha}}\,. \tag{6.8}$$

Let

$$\|g_0\|_l \le \frac{1}{4\sqrt{\alpha}} = a_0, \quad \text{and} \quad k = \frac{1-\sqrt{1-4\sqrt{\alpha}\|g_0\|_l}}{2\sqrt{\alpha}},$$

then it is easy to show that k is the smaller root of the quadratic equation

$$\sqrt{\alpha}k^2 - k + \|g_0\|_l = 0$$

which shows, since $2k\sqrt{\alpha} = 1-\sqrt{1-4\sqrt{\alpha}\|g_0\|_l}$ that (6.7) is satisfied. Then the desired estimate (6.2) is an immediate consequence of (6.8). Thus the proof of Lemma 6.1 is completed. □

Lemma 6.2. *Let $p = 1/2$ and let $z(t,x)$ be a given function of t and x such that, $\|z(t)\|_l \le k$, and let $g(t,x)$ satisfy the linear system*

$$\begin{gathered}\partial_t g + \sum_{j=1}^m V^j\partial_{x_j}g + \frac{1}{\varepsilon}Lg = \frac{\varphi(\varepsilon)}{\varepsilon}\Gamma(z,g),\\ g(0,x) = g_0(x).\end{gathered}$$

Then there exist suitable constants a_0, T such that for any $g_0 \in H^l$, a constant k can be chosen such that

$$\sup_{0\le t\le T} \|g(t)\|_l \le k. \tag{6.9}$$

Proof. In view of (6.6), one has

$$\|g\|_l^2 \le \|g_0\|_l^2 + Ck^2T \sup_{t\in[0,T]} \|g(t)\|_l^2.$$

Therefore, if

$$T < T_0 = \frac{1}{16C\|g_0\|_l^2} \quad \text{and} \quad k = \frac{1-\sqrt{1-4\sqrt{CT}\|g_0\|_l}}{2\sqrt{CT}},$$

one gets the desired estimate (6.9). □

Equation (4.4) can be solved using Lemma 6.1 or Lemma 6.2 and the principle of contraction mappings. The iteration scheme $\{f_\varepsilon^N\}$ is as follows: $f_\varepsilon^0 = f_0$ and

$$\begin{gathered}\partial_t f_\varepsilon^{N+1} + \sum_{j=1}^m V^j \partial_{xj} f_\varepsilon^{N+1} + \frac{1}{\varepsilon} L f_\varepsilon^{N+1} = \frac{\varphi(\varepsilon)}{\varepsilon}\Gamma(f_\varepsilon^N, f_\varepsilon^{N+1}),\\ f_\varepsilon^{N+1}(0,x) = f_0(x), \quad N = 0,1,2\dots.\end{gathered} \tag{6.10}$$

Lemma 6.3. *Let $p > 1/2$ and $f_0 \in H^l$ such that $\|f_0\|_l \le a_0$. Then suitable constants T_ε, k and β ($\beta < 1$) exist such that for any $\varepsilon > 0$ and for any $t \in [0, T_\varepsilon]$ the following estimates are satisfied:*

$$\|f_\varepsilon^{N+1}\|_l \le k, \tag{6.11}$$

$$\|f_\varepsilon^{N+1} - f_\varepsilon^N\|_l \le C_0 \beta^{\frac{n}{2}}. \tag{6.12}$$

Proof. Since $\|f^0\|_l = \|f_0\|_l \le k$, then (6.11) follows thanks to Lemma 6.1. Let $R_\varepsilon^N = f_\varepsilon^{N+1} - f_\varepsilon^N$. Therefore, R_ε^N satisfies

$$\partial_t R_\varepsilon^N + \sum_{j=1}^m V^j \partial_{xj} R_\varepsilon^N + \frac{1}{\varepsilon} L R_\varepsilon^N = \frac{\varphi(\varepsilon)}{\varepsilon}(\Gamma(R_\varepsilon^{N-1}, f_\varepsilon^{N+1}) + \Gamma(f_\varepsilon^{N-1}, R_\varepsilon^N)).$$

Applying the technique used in the proof of Lemma 6.1, one obtains:

$$\partial_t |\hat{R}_\varepsilon^N|^2 \le C\frac{\varphi(\varepsilon)^2}{\varepsilon}(|\hat{\Gamma}(R_\varepsilon^{N-1}, f_\varepsilon^{N+1})|^2 + |\hat{\Gamma}(f_\varepsilon^{N-1}, R_\varepsilon^N)|^2).$$

Multiplying by $(1+|\zeta|^2)^l$, integrating over $[0,T]\times\mathbb{R}_\zeta^n$. One can use again Plancherel's Theorem with (6.11) to deduce that

$$\begin{aligned}\|R_\varepsilon^N\|_l^2 &\le C\varepsilon^{2p-1}k^2T(\|R_\varepsilon^{N-1}\|_l^2 + \|R_\varepsilon^N\|_l^2)\\ &\le \alpha k^2(\|R_\varepsilon^{N-1}\|_l^2 + \|R_\varepsilon^N\|_l^2).\end{aligned}$$

Since $k^2\alpha < 1$, it follows that

$$\|R_\varepsilon^N\|_l^2 \le \frac{k^2\alpha}{1-k^2\alpha}\|R_\varepsilon^{N-1}\|_l^2. \tag{6.13}$$

Put

$$\beta = \frac{k^2\alpha}{1-k^2\alpha}. \tag{6.14}$$

Then (6.12) follows from (6.13). □

Lemma 6.4. *Let $p = 1/2$ and let $f_0 \in H^l$. Then suitable constants T, k and λ ($\lambda < 1$) exist such that for any $\varepsilon > 0$ and for any $t \in [0,T]$, the following estimates are satisfied:*

$$\|f_\varepsilon^{N+1}\|_l \le k,$$
$$\|f_\varepsilon^{N+1} - f_\varepsilon^N\|_l \le C_0 \lambda^{\frac{n}{2}}.$$

The proof of this lemma follows similar arguments to those in Lemma 6.3.

Proof of Theorems 5.1 and 5.2. In view of Lemma 6.3 or 6.4, Estimates (6.11), (6.12) imply that for each $\varepsilon > 0$, $\{f_\varepsilon^N\}$ is a Cauchy sequence in $L^\infty([0,T], H^l)$. Let us denote its limit by $f_\varepsilon(t)$, and note that it satisfies the estimate (5.1); i.e., this limit is in $L^\infty([0,T], H^l)$.

It can be shown from (6.10) that $\partial_t f_\varepsilon^{N+1}$ can be expressed in terms of sequences converging in $L^\infty([0,T], H^{l-1})$ as $N \to +\infty$. The limit is

$$\theta_\varepsilon = -\sum_{j=1}^m V^j \partial_{x_j} f_\varepsilon - \frac{1}{\varepsilon} L f_\varepsilon + \frac{\varphi(\varepsilon)}{\varepsilon} \Gamma(f_\varepsilon, f_\varepsilon).$$

Now let $\Psi(t,x)$ be a C^∞ function of compact support in $[0,T] \times K$. We have just seen that

$$\int_{[0,T]\times K} \langle \psi(t,x), \partial_t f_\varepsilon^{N+1} \rangle dt dx \to \int_{[0,T]\times K} \langle \psi(t,x), \theta_\varepsilon(t,x) \rangle dt\, dx,$$

as $N \to +\infty$. However

$$\begin{aligned} \int_{[0,T]\times K} \langle \psi(t,x), \partial_t f_\varepsilon^{N+1} \rangle dt dx &= -\int_{[0,T]\times K} \langle \partial_t \psi(t,x), f_\varepsilon^{N+1} \rangle dt dx \\ &\to -\int_{[0,T]\times K} \langle \partial_t \psi(t,x), f_\varepsilon \rangle dt dx, \end{aligned}$$

as $N \to +\infty$. Therefore, $\theta_\varepsilon(t)$ is identified with the distributional derivative in t of f_ε. It follows that f_ε satisfies (4.4), and $\frac{\partial f_\varepsilon}{\partial t} \in L^\infty([0,T], H^{l-1})$; hence $f_\varepsilon \in C([0,T], H^{l-1})$.

7. Strong convergence

The uniform bound (5.1) and additional estimates are needed for proving Theorem 5.4. Under the assumption (5.5), it is possible to find a uniform bound for $\frac{\partial f_\varepsilon}{\partial t}$. The uniform equicontinuity in t is given by the following lemma.

Lemma 7.1. *Let $\varphi(\varepsilon) = O(\varepsilon^p)$, $p \ge 1$, $l > (m/2) + 1$ and assume that assumption (5.5) holds. Then*

$$\|\frac{\partial f_\varepsilon}{\partial t}\|_{l-1} \le C \exp(Ck^2 T), \quad \forall t \in [0,T],\ \varepsilon \in (0,1), \tag{7.1}$$

where the constant C does not depend on ε.

Proof. Following the arguments in the proof of Lemma 6.1, we have

$$\frac{\partial_t |\hat{\partial_t f_\varepsilon}|^2}{2} + \frac{C_1}{\varepsilon} |P^\perp \hat{\partial_t f_\varepsilon}|^2 \le \frac{\varphi(\varepsilon)^2}{2C_1 \varepsilon} |\hat{\Gamma}(\partial_t f_\varepsilon, f_\varepsilon)|^2 + \frac{C_1}{2\varepsilon} |P^\perp \hat{\partial_t f_\varepsilon}|^2. \tag{7.2}$$

Multiplying this inequality by $(1 + |\zeta|^2)^{l-1}$, and integrating over $\mathbb{R}^m_\zeta$, yields

$$\partial_t \|\partial_t f_\varepsilon\|_{l-1}^2 \le C \varepsilon^{2p-1} \|\partial_t f_\varepsilon\|_{l-1}^2 \|f_\varepsilon\|_{l-1}^2. \tag{7.3}$$

Moreover using Gronwall's inequality, allows to rewrite (7.3) as follows

$$\|\partial_t f_\varepsilon\|_{l-1} \leq \exp(Ck^2T)\|\partial_t f_\varepsilon(0)\|_{l-1}.$$

Then, using (4.4) to express $\partial_t f_\varepsilon(0)$ in terms of the initial data yields:

$$\|\partial_t f_\varepsilon\|_{l-1} \leq e^{Ck^2T}(\|\sum_{j=1}^{m} V^j \partial_{x_j} f_0\|_{l-1} + \frac{1}{\varepsilon}\|Lf_0\|_{l-1} + \frac{\varphi(\varepsilon)}{\varepsilon}\|\Gamma(f_0, f_0)\|_{l-1}).$$

Taking into account assumption (5.5),

$$\begin{aligned}\|\partial_t f_\varepsilon\|_{l-1} &\leq Cexp(Ck^2T)(\|\partial_x h_\varepsilon\|_{l-1} + \varepsilon\|\partial_x k_\varepsilon\|_{l-1} + \|Lk_\varepsilon\|_{l-1} \\ &\quad + \varepsilon^{p-1}\|h_\varepsilon\|_l^2 + \varepsilon^p\|h_\varepsilon\|_l\|k_\varepsilon\|_l + \varepsilon^{p+1}\|k_\varepsilon\|_l^2) \\ &\leq C\exp(Ck^2T).\end{aligned}$$

Thus the proof of (7.1) is complete. □

From Lemma 7.1 one can conclude the following: The solution f_ε is uniformly bounded in $C([0,T], H^{l-1})$, $\varepsilon > 0$, and t in any compact subset of the interval $[0,T]$. Moreover f_ε satisfies the bound (7.1). Therefore, by the Ascoli-Arzela Lemma a convergent subsequence f_{ε_j} $(\varepsilon_j \to 0)$ can be chosen such that

$$f_{\varepsilon_j} \to f \quad \text{in} \quad C([0,T], H^{l-1}),$$

and the limit function satisfies the bound (5.1).

8. Examples

In this section, we study the asymptotic behaviour of the two-dimensional 8-velocity model and the one-dimensional Broadwell model.

The two-dimensional 8-velocity model. This Sub-section deals with the presentation of a two dimensional model with 8 velocities for which Condition (2.3) is satisfied. The velocities v_i, $i = 1, \dots, 8$ of the model we are

$$\begin{gathered}v_1 = (v,0), \quad v_2 = (0,v), \quad v_3 = -v_1, \quad v_4 = -v_2, \\ v_5 = (v,v), \quad v_6 = (-v,v), \quad v_7 = -v_5, \quad v_8 = -v_6,\end{gathered}$$

where v is a positive constant. Note that $|v_j| = v$ $(j = 1,..,4)$ and $|v_j| = \sqrt{2}v$ $(j = 5, \dots, 8)$. The above model is characterized by six non-trivial collisions:

Type 1 $(v_1, v_3) \to (v_2, v_4)$,
Type 2 $(v_5, v_7) \to (v_6, v_8)$,
Type 3 $(v_1, v_6) \to (v_3, v_5), (v_1, v_7) \to (v_3, v_8)$ etc.

Assume that for each of the above types the values of A_{kl}^{ij} are given respectively by

$$A_{13}^{24} = \frac{\sigma_1}{2}, \quad A_{57}^{68} = \frac{\sigma_2}{2}, \quad A_{16}^{35} = A_{17}^{38} = A_{27}^{46} = A_{28}^{45} = \frac{\sigma_3}{2},$$

where σ_1, σ_2 and σ_3 are positive constants.

Then letting $(\alpha_1, \dots, \alpha_8) = (1, \dots, 1)$, yields

$$\partial_t F_i + v_i.\nabla_x F_i = \frac{1}{\varepsilon} Q_i(F,F), \quad i = 1, \dots, 8, \tag{8.1}$$

where $Q_i(F,F)$ are given explicitly by

$$
\begin{gathered}
Q_1(F,F) = \sigma_1(F_2F_4 - F_1F_3) + \sigma_3\{(F_3F_5 - F_1F_6) + (F_3F_8 - F_1F_7)\},\\
\dots\\
Q_5(F,F) = \sigma_2(F_6F_8 - F_5F_7) + \sigma_3\{(F_1F_6 - F_3F_5) + (F_2F_8 - F_4F_5)\},
\end{gathered}
$$

and so on. Let $F = t_{(F_1,\dots,F_8)}, Q(F,F) = t_{(Q_1(F,F),\dots,Q_8(F,F))}$ and

$$
V^1 = v\,\mathrm{diag}(1,0,-1,0,1,-1,-1,1), V^2 = v\,\mathrm{diag}(0,1,0,-1,1,1,-1,-1).
$$

Then (8.1) can be written in the form

$$
\frac{\partial F_\varepsilon}{\partial t} + \sum_{j=1}^{2} V^j \frac{\partial F_\varepsilon}{\partial x_j} = \frac{1}{\varepsilon} Q(F_\varepsilon, F_\varepsilon). \tag{8.2}
$$

If $\mathbb{M}$ is the set of summationnal invariants, then it is easy to see that $\dim \mathbb{M} = 4$. Therefore the orthonormal basis for $\mathbb{M}$ is given by $\psi^{(i)}$, $i = 1,\dots,4$,

$$
\begin{gathered}
\psi^{(1)} = \frac{\sqrt{2}}{4} t_{(1,1,1,1,1,1,1,1)}, \quad \psi^{(2)} = \frac{\sqrt{6}}{6} t_{(1,0,-1,0,1,-1,-1,1)},\\
\psi^{(3)} = \frac{\sqrt{6}}{6} t_{(0,1,0,-1,1,1,-1,-1)}, \quad \psi^{(4)} = \frac{\sqrt{2}}{4} t_{(1,1,1,1,-1,-1,-1,-1)}.
\end{gathered}
$$

On the other hand a locally Maxwellian state is a vector $M = t_{(M_1,\dots,M_8)} > 0$ which satisfies (2.6) and $t_{(\log M_1,\dots,\log M_8)} = \sum_{i=1}^4 \beta_i \psi^{(i)}$ for some $(\beta_1,\dots,\beta_4) \in \mathbb{R}^4$. Then putting $M_0 = \exp(\frac{\sqrt{2}}{4}(\beta_1 + \beta_4) + \frac{\sqrt{6}}{6}(\beta_2 + \beta_3))$, $a = \exp(\frac{\sqrt{2}}{4}\beta_4)$, $b = \exp(\frac{\sqrt{6}}{6}\beta_3)$, and $c = \exp(\frac{\sqrt{6}}{6}\beta_2)$, yields

$$
M = M_0 t_{(b,c,bc^2,b^2c,a^2,a^2c^2,a^2b^2c^2,a^2b^2)}.
$$

For simplicity we deal here with the case where $\beta_2 = \beta_3 = 0$ (i.e., $b = c = 1$ and $M_0 = M_1$). Let $M > 0$ be an absolute Maxwellian state with the simple form: $M = M_1 t_{(1,1,1,1,a^2,a^2,a^2,a^2)}$, where $M_1 > 0$ and $a = (\frac{M_5}{M_1})^{1/2} > 0$ are constants. In this case one has $\Lambda = M_1\,\mathrm{diag}(1,1,1,1,a^2,a^2,a^2,a^2)$. Substituting

$$
F_\varepsilon(t,x) = M + \varphi(\varepsilon)\Lambda^{1/2} f_\varepsilon(t,x),
$$

into (8.2) yields the following system for f_ε:

$$
\begin{gathered}
\partial_t f_\varepsilon + V^1 \partial_{x_1} f_\varepsilon + V^2 \partial_{x_2} f_\varepsilon + \frac{L f_\varepsilon}{\varepsilon} = \frac{\varphi(\varepsilon)}{\varepsilon} \Gamma(f_\varepsilon, f_\varepsilon),\\
f_\varepsilon(t=0,x) = f_0(x).
\end{gathered} \tag{8.3}
$$

Since $N(L) = \Lambda^{1/2}\mathbb{M}$, a simple calculation generates the orthonormal basis $\{e^{(i)}, i = 1,\dots,4\}$ for $N(L)$:

$$
\begin{gathered}
e^{(1)} = \frac{1}{2b_1} t_{(1,1,1,1,a,a,a,a)}, \quad e^{(2)} = \frac{\sqrt{2}}{2b_2} t_{(1,0,-1,0,a,-a,-a,a)},\\
e^{(3)} = \frac{\sqrt{2}}{2b_2} t_{(0,1,0,-1,a,a,-a,-a)}, \quad e^{(4)} = \frac{1}{2b_1} t_{(a,a,a,a,-1,-1,-1,-1)},
\end{gathered}
$$

where $b_1 = (1+a)^{1/2}$ and $b_2 = (1+2a^2)^{1/2}$.

To determine the linearized Euler equation for this model, we evaluate the terms $C^1 = \langle V^1 e^{(l)}, e^{(k)}\rangle_{1\le l,k\le 4}$ and $C^2 = \langle V^2 e^{(l)}, e^{(k)}\rangle\rangle_{1\le l,k\le 4}$, one gets

$$C^1 = \begin{pmatrix} 0 & \frac{v}{\sqrt{2}b_1b_2}(1+a^2) & 0 & 0 \\ \frac{v}{\sqrt{2}b_1b_2}(1+a^2) & 0 & 0 & -\frac{v}{\sqrt{2}b_1b_2}a \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{v}{\sqrt{2}b_1b_2}a & 0 & 0 \end{pmatrix}, \tag{8.4}$$

$$C^2 = \begin{pmatrix} 0 & \frac{v}{\sqrt{2}b_1b_2}(1+2a^2) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{v}{\sqrt{2}b_1b_2}(1+2a^2) & 0 & 0 & -\frac{v}{\sqrt{2}b_1b_2}a \\ 0 & 0 & -\frac{v}{\sqrt{2}b_1b_2}a & 0 \end{pmatrix}. \tag{8.5}$$

Then applying the results of Section 5, one gets the existence of a local solution for (8.3) satisfying the uniform estimate

$$\|f_\varepsilon(t)\|_l \le k, \quad for \quad t\in[0,T_\varepsilon]. \tag{8.6}$$

As $\varepsilon \to 0$, $f_\varepsilon \to f$ weakly $\star$ in $L^\infty([0,T],H^l)$ for any $T>0$, and the limit has the form $f=\sum_{i=1}^4 U_i e^{(i)}$, where $U=(U_1,\dots,U_4)$ satisfies

$$\begin{gathered} \frac{\partial U}{\partial t} + C^1\frac{\partial U}{\partial x_1} + C^2\frac{\partial U}{\partial x_2} = 0, \\ U(t=0) = \langle f_0, e\rangle, \end{gathered}$$

where C^j, $j=1,2$ are given by (8.4)-(8.5).

The one-dimensional Broadwell model. A simple mathematical model of gas kinetics was proposed by Broadwell [14]. It describes an idealization of a discrete velocity gas of particles in one dimension subject to a simple binary collision mechanism. This model, which describes a gas as consisting of particles with essentially only three speeds, is simple enough to be mathematically tractable, however it contains enough physics to generate interesting kinetic and fluid equations.

The above discrete model of the Boltzmann equation, which describes the particle density function in phase space, reads

$$\begin{gathered} \frac{\partial F_\varepsilon}{\partial t} + V\frac{\partial F_\varepsilon}{\partial x} = \frac{1}{\varepsilon}Q(F_\varepsilon,F_\varepsilon), \\ F_\varepsilon(0,x) = F_0(x), \end{gathered} \tag{8.7}$$

where the density function is $F_\varepsilon = t_{(F_1^\varepsilon,F_2^\varepsilon,F_3^\varepsilon)}$. The scalar functions $F_1^\varepsilon, F_2^\varepsilon, F_3^\varepsilon$ represent the number densities for particles moving in the positive x-direction, the negative x-direction, and the positive or negative y-or z-directions, respectively. All particles move with speed c. The matrix V is given by: $V=\mathrm{diag}(c,0,-c)$ and the collision operator Q is

$$Q(f,g) = \frac{1}{2}(2f_2g_2 - (f_1g_3+f_3g_1))t_{(1,-\frac{1}{2},1)}.$$

An absolute Maxwellian state for (8.7) is a vector $M = t_{(M_1,M_2,M_3)} > 0$ satisfying $M_2^2 - M_1M_3 = 0$. Therefore, it has the expression $M = M_1 t_{(1,a,a^2)}$ where $M_1>0$ and $a=\frac{M_2}{M_1}>0$ are constants. Let $\Lambda = M_1 diag(1,\frac{a}{4},a^2)$. We substitute $F_\varepsilon(t,x) =$

$M + \varphi(\varepsilon)\Lambda^{1/2} f_\varepsilon(t, x)$ into (8.7). Then, the following system for f_ε is obtained:

$$\begin{gathered} \partial_t f_\varepsilon + V\partial_x f_\varepsilon + \frac{Lf_\varepsilon}{\varepsilon} = \frac{\varphi(\varepsilon)}{\varepsilon}\Gamma(f_\varepsilon, f_\varepsilon), \\ f_\varepsilon(t = 0, x) = f_0(x). \end{gathered} \tag{8.8}$$

Simple calculations provide the orthonormal $N(L)$ basis $\{e^{(i)}, i = 1, 2\}$ for $N(L)$:

$$e^{(1)} = \frac{1}{b_1} t_{(1,2a^{1/2},a)}, \quad e^{(2)} = \frac{1}{b_1 b_2} t_{(a^{1/2}(2+a),-(1-a^2),-a^{1/2}(1+2a))},$$

with $b_1 = (1 + 4a + a^2)^{1/2}$ and $b_2 = (1 + a + a^2)^{1/2}$.

The results of Section 5 can be applied to deduce the local existence of solution of (8.8), satisfying (8.6) and, as $\varepsilon \to 0$, $f_\varepsilon \to f$ weakly $\star$ in $L^\infty([0, T], H^l)$ for any $T > 0$, moreover the limit can be written as follows

$$f = U_1 e^{(1)} + U_2 e^{(2)},$$

where $U = (U_1, U_2)$ satisfies

$$\begin{gathered} \frac{\partial}{\partial t}\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} + C\frac{\partial}{\partial x}\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ U(t = 0) = (\langle f_0, e^{(1)}\rangle, \langle f_0, e^{(2)}\rangle), \end{gathered}$$

where

$$C = \begin{pmatrix} \frac{c(1-a^2)}{b_1^2} & \frac{2ca^{1/2}b_2}{b_1^2} \\ \frac{2ca^{1/2}b_2}{b_1^2} & \frac{3ca(1-a^2)}{b_1^2 b_2^2} \end{pmatrix}.$$

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