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ATTRACTORS OF ASYMPTOTICALLY PERIODIC MULTIVALUED DYNAMICAL SYSTEMS GOVERNED BY TIME-DEPENDENT SUBDIFFERENTIALS

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Abstract. We study a nonlinear evolution equation associated with time-dependent subdifferential in a separable Hilbert space. In particular, we consider an asymptotically periodic system, which means that time-dependent terms converge to time-periodic terms as time approaches infinity. Then we consider the large-time behavior of solutions without uniqueness. In such a situation the corresponding dynamical systems are multivalued. In fact, we discuss the stability of multivalued semiflows from the view-point of attractors. Namely, the main object of this paper is to construct a global attractor for asymptotically periodic multivalued dynamical systems, and to discuss the relationship to one for the limiting periodic systems.

1. Introduction

We consider non-autonomous systems, in a real separable Hilbert space H, of the form

$$v'(t) + \partial\varphi^t(v(t)) + G(t, v(t)) \ni f(t) \quad \text{in } H, \quad t > s\ (\geq 0), \tag{1.1}$$

where $v' = \frac{dv}{dt}$, $\partial\varphi^t$ is a subdifferential of time-dependent proper lower semicontinuous (l.s.c.) convex function φ^t on H, $G(t,\cdot)$ is a multivalued perturbation small relative to φ^t, and f is a forcing term.

In the case when $G(t,\cdot) \equiv 0$, many mathematicians studied the existence-uniqueness and the asymptotic behavior of solutions, the time periodic problem and the almost periodic case for (1.1) (cf. [7], [8], [13], [14], [15], [16], [18], [23], [24]).

For the multivalued nonmonotone perturbation $G(t,\cdot)$, Ôtani has already shown the existence of solution for (1.1) in [21]. The large-time behavior of solutions for (1.1) was discussed in [28] from the view-point of attractors. For the time periodic case, assuming the periodicity conditions with same period T_0, $0 < T_0 < +\infty$, i.e.

$$\varphi^t = \varphi^{t+T_0}, \quad G(t,\cdot) = G(t+T_0,\cdot), \quad f(t) = f(t+T_0), \qquad \forall t \in R_+ := [0,\infty),$$

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the existence of periodic solution for (1.1) was proved in [22]. Moreover, the periodic stability was discussed in [29]. In fact, the author showed the existence and characterization of time-periodic global attractors for (1.1) in [29].

In this paper, for a given positive number $T_0 > 0$, we treat the case when φ^t, $G(t,\cdot)$ and $f(t)$ are asymptotically T_0-periodic in time. Namely we assume that

$$\varphi^t - \varphi_p^t \to 0, \quad G(t,\cdot) - G_p(t,\cdot) \to 0, \quad f(t) - f_p(t) \to 0 \tag{1.2}$$

in appropriate senses as $t \to +\infty$, where $\varphi_p^t = \varphi_p^{t+T_0}$, $G_p(t,\cdot) = G_p(t+T_0,\cdot)$ and $f_p(t) = f_p(t+T_0)$ for any $t \in R_+$. By the asymptotically T_0-periodic condition (1.2), we have the limiting T_0-periodic system for (1.1) of the form:

$$u'(t) + \partial\varphi_p^t(u(t)) + G_p(t, u(t)) \ni f_p(t) \quad \text{in } H, \quad t > s \quad (\geq 0). \tag{1.3}$$

In the case when $G(t,\cdot)$ and $G_p(t,\cdot)$ are single-valued, the asymptotically T_0-periodic problem has already been discussed in [11]. To guarantee the uniqueness of solutions for the Cauchy problem of (1.1) and (1.3), they assumed some conditions on φ^t, φ_p^t, $G(t,\cdot)$ and $G_p(t,\cdot)$. Then, they discussed the asymptotically T_0-periodic stability for (1.1) from the view-point of attractors (cf. [11]). The main object of this paper is to develop the result obtained in [11] in order to consider the large-time behavior of solution for (1.1) without uniqueness. Namely, we would like to construct the attractor for the asymptotically T_0-periodic multivalued flows associated with (1.1). Moreover we shall discuss the relationship to the T_0-periodic attractor for (1.3) obtained in [29].

In the next Section 2, we recall the known results for the Cauchy problem of (1.1). In Section 3 we consider the limiting T_0-periodic problem (1.3) and recall the abstract results obtained in [29]. In Section 4, we introduce the notion of a metric topology on the family $\{\varphi^t; t \geq 0\}$ which was constructed in [16]. And we present and prove the main results in this paper. In proving main results, we generalize the results obtained in [11] and [30]. In the final section we apply our abstract results to the parabolic variational inequality with asymptotically T_0-periodic double obstacles. Then we can discuss the asymptotic stability for the asymptotically T_0-periodic double obstacle problem without uniqueness of solutions.

Notation. Throughout this paper, let H be a (real) separable Hilbert space with norm $|\cdot|_H$ and inner product $(\cdot,\cdot)_H$. For a proper l.s.c. convex function φ on H we use the notation $D(\varphi)$, $\partial\varphi$ and $D(\partial\varphi)$ to indicate the effective domain, subdifferential and its domain of φ, respectively; for their precise definitions and basic properties see [4].

For two non-empty sets A and B in H, we define the so-called Hausdorff semi-distance

$$\operatorname{dist}_H(A,B) := \sup_{x\in A} \inf_{y\in B} |x-y|_H.$$

2. Preliminaries

In this section, we recall the known results for a nonlinear evolution equation in H of the form:

$$u'(t) + \partial\varphi^t(u(t)) + G(t, u(t)) \ni f(t) \quad \text{in } H, \quad t \in J, \tag{2.1}$$

where J is an interval in R_+, $\partial\varphi^t$ is the subdifferential of a time-dependent proper l.s.c. and convex function φ^t on H, $G(t,\cdot)$ is a multivalued operator from a subset $D(G(t,\cdot)) \subset H$ into H for each $t \in R_+$ and f is a given function in $L^2_{\text{loc}}(J;H)$.

We begin by defining a solution for (2.1).

Definition 2.1. (i) For a compact interval $J := [t_0, t_1] \subset R_+$ and $f \in L^2(J; H)$, a function $u : J \to H$ is called a solution of (2.1) on J, if $u \in C(J; H) \cap W^{1,2}_{\mathrm{loc}}((t_0, t_1]; H)$, $\varphi^{(\cdot)}(u(\cdot)) \in L^1(J)$, $u(t) \in D(\partial\varphi^t)$ for a.e. $t \in J$, and if there exists a function $g \in L^2_{\mathrm{loc}}(J; H)$ such that $g(t) \in G(t, u(t))$ for a.e. $t \in J$ and

$$f(t) - g(t) - u'(t) \in \partial\varphi^t(u(t)), \quad \text{a.e. } t \in J.$$

(ii) For any interval J in R_+ and $f \in L^2_{\mathrm{loc}}(J; H)$, a function $u : J \to H$ is called a solution of (2.1) on J, if it is a solution of (2.1) on every compact subinterval of J in the sense of (i).
(iii) Let J be any interval in R_+ with initial time $s \in R_+$. For $f \in L^2_{\mathrm{loc}}(J; H)$, a function $u : J \to H$ is called a solution of the Cauchy problem for (2.1) on J with given initial value $u_0 \in H$, if it is a solution of (2.1) on J satisfying $u(s) = u_0$.

For the rest of this paper, let $\{a_r\} := \{a_r; r \geq 0\}$ and $\{b_r\} := \{b_r; r \geq 0\}$ be families of real functions in $W^{1,2}_{\mathrm{loc}}(R_+)$ and $W^{1,1}_{\mathrm{loc}}(R_+)$, respectively, such that

$$\sup_{t\in R_+} |a'_r|_{L^2(t,t+1)} + \sup_{t\in R_+} |b'_r|_{L^1(t,t+1)} < +\infty \qquad \text{for each } r \geq 0.$$

Now we define the class $\Phi(\{a_r\}, \{b_r\})$ of time-dependent convex function φ^t.

Definition 2.2. A function $\{\varphi^t\}$ belongs to $\Phi(\{a_r\}, \{b_r\})$ if φ^t is a proper l.s.c. convex function on H and satisfies the following three properties:

(Φ1) For each $r > 0$, $s, t \in R_+$ and $z \in D(\varphi^s)$ with $|z|_H \leq r$, there exists $\tilde{z} \in D(\varphi^t)$ such that

$$|\tilde{z} - z|_H \leq |a_r(t) - a_r(s)|(1 + |\varphi^s(z)|^{\frac{1}{2}}),$$
$$\varphi^t(\tilde{z}) - \varphi^s(z) \leq |b_r(t) - b_r(s)|(1 + |\varphi^s(z)|).$$

(Φ2) There exists a positive constant C_1 such that

$$\varphi^t(z) \geq C_1|z|^2_H, \quad \forall t \in R_+,\ \forall z \in D(\varphi^t).$$

(Φ3) For each $k > 0$ and $t \in R_+$, the level set $\{z \in H; \varphi^t(z) \leq k\}$ is compact in H.

Next, we introduce the class $\mathcal{G}(\{\varphi^t\})$ of time-dependent multivalued perturbation $G(t, \cdot)$ associated with $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$.

Definition 2.3. An operator $\{G(t, \cdot)\}$ belongs to $\mathcal{G}(\{\varphi^t\})$ if $G(t, \cdot)$ is a multivalued operator from $D(G(t, \cdot)) \subset H$ into H which fulfills the following five conditions:

(G1) $D(\varphi^t) \subset D(G(t, \cdot)) \subset H$ for any $t \in R_+$. And for any interval $J \subset R_+$ and $v \in L^2_{\mathrm{loc}}(J; H)$ with $v(t) \in D(\varphi^t)$ for a.e. $t \in J$, there exists a strongly measurable function $g(\cdot)$ on J such that $g(t) \in G(t, v(t))$ for a.e. $t \in J$.
(G2) $G(t, z)$ is a convex subset of H for any $z \in D(\varphi^t)$ and $t \in R_+$.
(G3) There are positive constants C_2, C_3 such that

$$|g|^2_H \leq C_2\varphi^t(z) + C_3, \quad \forall t \in R_+,\ \forall z \in D(\varphi^t), \forall g \in G(t, z).$$

(G4) (demi-closedness) If $z_n \in D(\varphi^{t_n})$, $g_n \in G(t_n, z_n)$, $\{t_n\} \subset R_+$, $\{\varphi^{t_n}(z_n)\}$ is bounded, $z_n \to z$ in H, $t_n \to t$ and $g_n \to g$ weakly in H as $n \to +\infty$, then $g \in G(t, z)$.

(G5) For each bounded subset B of H, there exist positive constants $C_4(B)$ and $C_5(B)$ such that
$$\varphi^t(z) + (g, z-b)_H \ge C_4(B)|z|_H^2 - C_5(B),$$
for all $t \in R_+$, all $g \in G(t,z)$, all $z \in D(\varphi^t)$, and all $b \in B$.

For a given $\{\varphi^t\}$ in $\Phi(\{a_r\},\{b_r\})$, $\{G(t,\cdot)\}$ in $\mathcal{G}(\{\varphi^t\})$ and a forcing term f in $L^2_{\rm loc}(R_+;H)$, we consider the evolution equation
$$u'(t) + \partial\varphi^t(u(t)) + G(t,u(t)) \ni f(t) \quad \text{in } H, \quad t > s \tag{2.2}$$
for each $s \in R_+$.

Now we recall the known results on the existence and global estimates of solutions for the Cauchy problem of (2.2):

(A) [Existence of solution for (2.2)] (cf. [21, Theorem II, III]) The Cauchy problem for (2.2) has at least one solution u on $J = [s,+\infty)$ such that $(\cdot - s)^{\frac{1}{2}}u' \in L^2_{\rm loc}(J;H)$, $(\cdot - s)\varphi^{(\cdot)}(u(\cdot)) \in L^\infty_{\rm loc}(J)$ and $\varphi^{(\cdot)}(u(\cdot))$ is absolutely continuous on any compact subinterval of $(s,+\infty)$, provided that given initial value $u_0 \in \overline{D(\varphi^s)}$. In particular, if $u_0 \in D(\varphi^s)$, then the solution u satisfies that $u' \in L^2_{\rm loc}(J;H)$ and $\varphi^{(\cdot)}(u(\cdot))$ is absolutely continuous on any compact interval in J.

(B) [Global boundedness of solutions for (2.2)] (cf. [25, Theorem 2.2]) Suppose that
$$S_f := \sup_{t\in R_+} |f|_{L^2(t,t+1;H)} < +\infty.$$
Then, the solution u of the Cauchy problem for (2.2) on $[s,+\infty)$ satisfies the global estimate
$$\sup_{t\ge s}|u(t)|_H^2 + \sup_{t\ge s}\int_t^{t+1} \varphi^\tau(u(\tau))d\tau \le N_1(1 + S_f^2 + |u_0|_H^2),$$
where N_1 is a positive constant independent of f, $s \in R_+$ and a given initial value $u_0 \in \overline{D(\varphi^s)}$. Moreover, for each $\delta > 0$ and each bounded subset B of H, there is a constant $N_2(\delta,B) > 0$, depending only on $\delta > 0$ and B, such that
$$\sup_{t\ge s+\delta} |u'|^2_{L^2(t,t+1;H)} + \sup_{t\ge s+\delta}\varphi^t(u(t)) \le N_2(\delta,B)$$
for the solution u of the Cauchy problem for (2.2) on $[s,+\infty)$ with $s \in R_+$ and $u_0 \in \overline{D(\varphi^s)} \cap B$.

Next, we remember a notion of convergence for convex functions.

Definition 2.4 (cf. [20])**.** Let ψ, ψ_n $(n \in N)$ be proper l.s.c. and convex functions on H. Then we say that ψ_n converges to ψ on H as $n \to +\infty$ in the sense of Mosco [20], if the following two conditions are satisfied:

(i) For any subsequence $\{\psi_{n_k}\} \subset \{\psi_n\}$, if $z_k \to z$ weakly in H as $k \to +\infty$, then
$$\liminf_{k\to+\infty} \psi_{n_k}(z_k) \ge \psi(z).$$

(ii) For any $z \in D(\psi)$, there is a sequence $\{z_n\}$ in H such that
$$z_n \to z \text{ in } H \text{ as } n \to +\infty, \quad \lim_{n\to+\infty}\psi_n(z_n) = \psi(z).$$

Now, we recall a convergence result (cf. [25, Lemma 4.1]) as follows.

(C) Let $\{\varphi_n^t\} \in \Phi(\{a_r\},\{b_r\})$, $\{G_n(t,\cdot)\} \in \mathcal{G}(\{\varphi_n^t\})$ with common positive constants $C_1, C_2, C_3, C_4(B), C_5(B)$, $\{f_n\} \subset L^2(J;H)$, $J=[s,t_1] \subset R_+$, and $u_{0,n} \in \overline{D(\varphi_n^s)}$ for $n=1,2,\dots$. Assume that
  (i) φ_n^t converges to φ^t on H in the sense of Mosco [20] for each $t \in J$ (as $n \to +\infty$) and $\bigcup_{n=1}^{+\infty}\{z \in H; \varphi_n^t(z) \le k\}$ is relatively compact in H for every real $k>0$ and $t \in J$, where $\{\varphi^t\} \in \Phi(\{a_r\},\{b_r\})$ and $\varphi_n^t = \varphi^t$ if $n=+\infty$.
  (ii) If $z_n \in D(\varphi_n^{t_n})$, $g_n \in G_n(t_n, z_n)$, $\{t_n\} \subset R_+$, $\{\varphi_n^{t_n}(z_n)\}$ is bounded, $z_n \to z$ in H, $t_n \to t$ and $g_n \to g$ weakly in H as $n \to +\infty$, then $g \in G(t,z)$, where $\{G(t,\cdot)\} \in \mathcal{G}(\{\varphi^t\})$.
  (iii) $f_n \to f$ weakly in $L^2(J;H)$ for some $f \in L^2(J;H)$ and $u_{0,n} \to u_0$ in H for some $u_0 \in \overline{D(\varphi^s)}$.

  Denote by u the solution of the Cauchy problem for (2.2) on J with $u(s)=u_0$ and by u_n the solution of the Cauchy problem for (2.2) with φ^t, G, f replaced by φ_n^t, G_n, f_n, and with $u_n(s)=u_{0,n}$. Then u_n converges to u on J in the sense that
$$u_n \to u \text{ in } C(J;H), \quad (\cdot - s)^{\frac12} u_n' \to (\cdot - s)^{\frac12} u' \text{ weakly in } L^2(J;H),$$
$$\int_J \varphi_n^t(u_n(t))dt \to \int_J \varphi^t(u(t))dt \quad \text{as } n \to +\infty.$$

3. Attractor for periodic multivalued dynamical system

In this section we recall the known results obtained in [29] for a T_0-periodic system in H, of the form:
$$u'(t) + \partial\varphi_p^t(u(t)) + G_p(t,u(t)) \ni f_p(t) \quad \text{in } H, \quad t>s \tag{3.1}$$
for each $s \in R_+$, where φ_p^t, $G_p(t,\cdot)$ and $f_p(t)$ are T_0-periodic, namely periodic in time with the same period T_0, $0<T_0<+\infty$.

Definition 3.1. Let T_0 be a positive number. Then
(i) $\Phi_p(\{a_r\},\{b_r\};T_0)$ is the set of all $\{\varphi_p^t\} \in \Phi(\{a_r\},\{b_r\})$ satisfying the T_0-periodicity condition
$$\varphi_p^{t+T_0}(\cdot) = \varphi_p^t(\cdot) \quad \text{on } H, \quad \forall t \in R_+.$$
(ii) $\mathcal{G}_p(\{\varphi_p^t\};T_0)$ is the set of all $\{G_p(t,\cdot)\} \in \mathcal{G}(\{\varphi_p^t\})$ satisfying the T_0-periodicity condition
$$G_p(t+T_0,\cdot) = G_p(t,\cdot) \quad \text{in } H, \quad \forall t \in R_+.$$

For the rest of this section we assume that $\{\varphi_p^t\} \in \Phi_p(\{a_r\},\{b_r\};T_0)$, $\{G_p(t,\cdot)\} \in \mathcal{G}_p(\{\varphi_p^t\};T_0)$ and $f_p \in L^2_{\rm loc}(R_+;H)$ is T_0-periodic in time, namely
$$f_p(t+T_0) = f_p(t) \quad \text{in } H, \quad \forall t \in R_+. \tag{3.2}$$
Here we note that (3.1) can be considered as (2.2) in Section 2. So, by the result (A) in Section 2, the Cauchy problem for (3.1) has at least one solution u on $[s,+\infty)$. Hence we can define the multivalued dynamical process associated with (3.1) as follows:

Definition 3.2. For every $0 \le s \le t < +\infty$ we denote by $U(t,s)$ the mapping from $\overline{D(\varphi_p^s)}$ into $\overline{D(\varphi_p^t)}$ which assigns to each $u_0 \in \overline{D(\varphi_p^s)}$ the set

$$\begin{aligned} U(t,s)u_0 := \big\{&z \in H; \text{There is a solution } u \text{ of (3.1) on } [s,+\infty) \\ &\text{such that } u(s) = u_0 \text{ and } u(t) = z.\big\} \end{aligned} \tag{3.3}$$

Then we deduce easily the following properties of $\{U(t,s)\} := \{U(t,s); 0 \le s \le t < +\infty\}$:

- (U1) $U(s,s) = I$ on $\overline{D(\varphi_p^s)}$ for any $s \in R_+$.
- (U2) $U(t_2,s)z = U(t_2,t_1)U(t_1,s)z$ for any $0 \le s \le t_1 \le t_2 < +\infty$ and $z \in \overline{D(\varphi_p^s)}$.
- (U3) $U(t+T_0, s+T_0)z = U(t,s)z$ for any $0 \le s \le t < +\infty$ and $z \in \overline{D(\varphi_p^s)}$, that is, U is T_0-periodic.
- (U4) $\{U(t,s)\}$ has the following demi-closedness:
  If $0 \le s_n \le t_n < +\infty$, $s_n \to s$, $t_n \to t$, $z_n \in \overline{D(\varphi_p^{s_n})}$, $z \in \overline{D(\varphi_p^s)}$, $z_n \to z$ in H and an element $w_n \in U(t_n,s_n)z_n$ converges to some element $w \in H$ as $n \to +\infty$, then $w \in U(t,s)z$.

Next we define the discrete dynamical system in order to construct a global attractor for (3.1).

Definition 3.3. Let $U(\cdot,\cdot)$ be the solution operator for (3.1) defined in Definition 3.2. Then
(i) For each $\tau \in R_+$, we denote by U_τ the T_0-step mapping from $\overline{D(\varphi_p^\tau)}$ into $\overline{D(\varphi_p^{\tau+T_0})} = \overline{D(\varphi_p^\tau)}$, namely, $U_\tau := U(\tau+T_0,\tau)$.
(ii) For any $k \in Z_+ := N \cup \{0\}$, we define

$$U_\tau^k := \underbrace{U_\tau \circ U_\tau \circ \cdots \circ U_\tau}_{k \text{ iterations}}.$$

Clearly we have $U_\tau^k = U(\tau + kT_0, \tau)$ for any $\tau \in R_+$ and $k \in Z_+$.

Now, we recall the known result on the existence of global attractors for discrete multivalued dynamical systems U_τ associated with (3.1).

Theorem 3.4 ([29, Theorem 3.1])**.** *Assume that $\{\varphi_p^t\} \in \Phi_p(\{a_r\},\{b_r\};T_0)$, $\{G_p(t,\cdot)\} \in \mathcal{G}_p(\{\varphi_p^t\};T_0)$, and $f_p \in L^2_{\rm loc}(R_+;H)$ satisfies the T_0-periodicity condition* (3.2)*. Then, for each $\tau \in R_+$, there exists a subset $\mathcal{A}_\tau$ of $D(\varphi_p^\tau)$ such that*

- (i) *$\mathcal{A}_\tau$ is non-empty and compact in H;*
- (ii) *for each bounded set B in H and each number $\epsilon > 0$ there exists $N_{B,\epsilon} \in N$ such that*

  $$\operatorname{dist}_H(U_\tau^k z, \mathcal{A}_\tau) < \epsilon$$

  for all $z \in \overline{D(\varphi_p^\tau)} \cap B$ and all $k \ge N_{B,\epsilon}$;
- (iii) *$U_\tau^k \mathcal{A}_\tau = \mathcal{A}_\tau$ for any $k \in N$.*

Remark 3.5. By [29, Lemma 3.1] we can get the compact absorbing set $B_{0,\tau}$ of $\overline{D(\varphi_p^\tau)}$ for U_τ such that for each bounded subset B of H there is a positive integer n_B (independent of $\tau \in R_+$) satisfying

$$U_\tau^n \left(\overline{D(\varphi_p^\tau)} \cap B\right) \subset B_{0,\tau} \quad \text{for all } n \ge n_B.$$

Then we observe that the global attractor $\mathcal{A}_\tau$ is given by the ω-limit set of the absorbing set $B_{0,\tau}$ for U_τ, i.e.

$$\mathcal{A}_\tau = \bigcap_{n\in Z_+} \overline{\bigcup_{k\geq n} U_\tau^k B_{0,\tau}}.$$

The next theorem concerns a relationship between global attractors $\mathcal{A}_s$ and $\mathcal{A}_\tau$. For detail proof, see [29].

Theorem 3.6 ([29, Theorem 3.2]). *Suppose the same assumptions are made as in Theorem 3.4. Let $\mathcal{A}_s$ and $\mathcal{A}_\tau$ be global attractors for U_s and U_τ, with $0 \leq s \leq \tau \leq T_0$, respectively. Then, we have*

$$\mathcal{A}_\tau = U(\tau, s)\mathcal{A}_s,$$

where $U(\tau, s)$ is the T_0-periodic process given in Definition 3.2.

Remark 3.7. By Theorem 3.4 (iii) and Theorem 3.6, we see that the global attractor $\mathcal{A}_\tau$ for U_τ is T_0-periodic in τ. In fact, for each $\tau \in R_+$ choose $m_\tau \in Z_+$ and $\sigma_\tau \in [0, T_0)$ so that $\tau = \sigma_\tau + m_\tau T_0$. Then, we have $\mathcal{A}_\tau = \mathcal{A}_{\sigma_\tau}$.

The third known result is the existence of a global attractor for the T_0-periodic multivalued dynamical system (3.1).

Theorem 3.8 (cf. [29, Theorem 3.3]). *Under the assumptions of Theorem 3.4, put*

$$\mathcal{A} := \bigcup_{0\leq\tau\leq T_0} \mathcal{A}_\tau,$$

where $\mathcal{A}_\tau$ is as obtained in Theorem 3.4. Then, $\mathcal{A}$ has the following properties:

- (i) *$\mathcal{A}$ is non-empty and compact in H;*
- (ii) *for each bounded set B in H and each number $\epsilon > 0$ there exists a finite time $T_{B,\epsilon} > 0$ such that*

$$\operatorname{dist}_H(U(t+\tau, \tau)z, \mathcal{A}) < \epsilon$$

for all $\tau \in R_+$, all $z \in \overline{D(\varphi_p^\tau)} \cap B$ and all $t \geq T_{B,\epsilon}$.

Remark 3.9. In [29, Section 4] the characterization of the T_0-periodic global attractor was discussed. The author proved that for each time $\tau \in R_+$ the global attractor $\mathcal{A}_\tau$ for the discrete multivalued dynamical system U_τ coincides with the cross-section of the family of all global bounded complete trajectories for the T_0-periodic system (3.1).

4. Attractor for asymptotically periodic multivalued dynamical system

Throughout this section, let $M > 0$ be a fixed (sufficiently) large positive number. Now we put

$$\begin{aligned}\Psi_M := \big\{&\psi; \psi \text{ is proper, l.s.c. and convex on } H,\\ &\exists z \in D(\psi) \text{ s.t. } |z|_H \leq M,\ \psi(z) \leq M\big\}\end{aligned}$$

Then we state the notion of a metric topology on Ψ_M introduced in [16].

Given $\varphi, \psi \in \Psi_M$, we define $\rho(\varphi, \psi; \cdot) : D(\varphi) \to R$ by putting

$$\rho(\varphi, \psi; z) = \inf\{\max(|y - z|_H, \psi(y) - \varphi(z)); y \in D(\psi)\}$$

for each $z \in D(\varphi)$, and for each $r \geq M$

$$\rho_r(\varphi, \psi) := \sup_{z \in L_\varphi(r)} \rho(\varphi, \psi; z),$$

where $L_\varphi(r) := \{z \in D(\varphi); |z|_H \leq r, \varphi(z) \leq r\}$. Moreover, for each $r \geq M$, we define the functional $\pi_r(\cdot, \cdot)$ on $\Psi_M \times \Psi_M$ by

$$\pi_r(\varphi, \psi) := \rho_r(\varphi, \psi) + \rho_r(\psi, \varphi) \quad \text{for } \varphi, \psi \in \Psi_M.$$

Then, according to [16, Proposition 3.1], we can define a complete metric topology on Ψ_M so that the convergence $\psi_n \to \psi$ in Ψ_M (as $n \to +\infty$) if and only if

$$\pi_r(\psi_n, \psi) \to 0 \quad \text{for every } r \geq M.$$

Now by using the above topology on Ψ_M, we consider an asymptotically T_0-periodic system as follows.

Definition 4.1. Assume $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\}) \cap \Psi_M$, $\{G(t, \cdot)\} \in \mathcal{G}(\{\varphi^t\})$ and $f \in L^2_{\rm loc}(R_+; H)$. Then the system

$$v'(t) + \partial\varphi^t(v(t)) + G(t, v(t)) \ni f(t) \quad \text{in } H,\ t > s\ (\geq 0) \tag{4.1}$$

is asymptotically T_0-periodic, if there are $\{\varphi^t_p\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0) \cap \Psi_M$, $\{G_p(t, \cdot)\} \in \mathcal{G}_p(\{\varphi^t_p\}; T_0)$ and a T_0-periodic function $f_p \in L^2_{\rm loc}(R_+; H)$ such that

(A1) (Convergence of $\varphi^t - \varphi^t_p \to 0$ as $t \to +\infty$) For each $r \geq M$,

$$J^{(r)}_m := \sup_{\sigma \in [0, T_0]} \pi_r(\varphi^{mT_0+\sigma}, \varphi^\sigma_p) \to 0 \quad \text{as } m \to +\infty.$$

(A2) (Convergence of $G(t, \cdot) - G_p(t, \cdot) \to 0$ as $t \to +\infty$) If $\{\tau_n\} \subset [0, T_0]$, $\{m_n\} \subset Z_+$, $m_n \to +\infty$, $z_n \in D(\varphi^{m_nT_0+\tau_n})$, $g_n \in G(m_nT_0 + \tau_n, z_n)$, $\{\varphi^{m_nT_0+\tau_n}(z_n)\}$ is bounded, $z_n \to z$ in H, $\tau_n \to \tau$ and $g_n \to g$ weakly in H (as $n \to +\infty$), then $g \in G_p(\tau, z)$.

(A3) (Convergence of $f(t) - f_p(t) \to 0$ as $t \to +\infty$)

$$|f(mT_0 + \cdot) - f_p|_{L^2(0,T_0;H)} \to 0 \quad \text{as } m \to +\infty.$$

By Definition 4.1 we easily see that a limiting system for (4.1) is a T_0-periodic one (3.1) of the form:

$$u'(t) + \partial\varphi^t_p(u(t)) + G_p(t, u(t)) \ni f_p(t) \quad \text{in } H,\ t > s\ (\geq 0).$$

Here we note that (4.1) is also considered as (2.2). So, by the result (A) in Section 2, the Cauchy problem for (4.1) has at least one solution v on $[s, +\infty)$. Hence we can define the multivalued dynamical system associated with (4.1) as follows:

Definition 4.2. For every $0 \leq s \leq t < +\infty$ we denote by $E(t, s)$ the mapping from $\overline{D(\varphi^s)}$ into $\overline{D(\varphi^t)}$ which assigns to each $v_0 \in \overline{D(\varphi^s)}$ the set

$$\begin{aligned} E(t, s)v_0 := \big\{ & z \in H;\ \text{There is a solution } v \text{ of (4.1) on } [s, +\infty) \\ & \text{such that } v(s) = v_0 \text{ and } v(t) = z. \big\} \end{aligned}$$

Then we easily see that $\{E(t, s)\} := \{E(t, s); 0 \leq s \leq t < +\infty\}$ has the following evolution properties:

(E1) $E(s, s) = I$ on $\overline{D(\varphi^s)}$ for any $s \in R_+$.

(E2) $E(t_2, s)z = E(t_2, t_1)E(t_1, s)z$ for any $0 \leq s \leq t_1 \leq t_2 < +\infty$ and $z \in \overline{D(\varphi^s)}$.

(E3) $\{E(t,s)\}$ has the following demi-closedness:
If $0 \le s_n \le t_n < +\infty$, $s_n \to s$, $t_n \to t$, $z_n \in \overline{D(\varphi^{s_n})}$, $z \in \overline{D(\varphi^{s})}$, $z_n \to z$ in H and an element $w_n \in E(t_n, s_n)z_n$ converges to some element $w \in H$ as $n \to +\infty$, then $w \in E(t,s)z$.

Now we give the definition of a discrete ω-limit set for $E(\cdot,\cdot)$.

Definition 4.3 (Discrete ω-limit set for $E(\cdot,\cdot)$)**.** Let $\tau \in R_+$ be fixed. Let $\mathcal{B}(H)$ be a family of bounded subsets of H. Then for each $B \in \mathcal{B}(H)$, the set

$$\omega_\tau(B) := \bigcap_{n\in Z_+} \overline{\bigcup_{k\ge n, m\in Z_+} E(kT_0 + mT_0 + \tau, mT_0 + \tau)(\overline{D(\varphi^{mT_0+\tau})} \cap B)}$$

is called the discrete ω-limit set of B under $E(\cdot,\cdot)$.

Remark 4.4. By the definition of the discrete ω-limit set $\omega_\tau(B)$, it is easy to see that $x \in \omega_\tau(B)$ if and only if there exist sequences $\{k_n\} \subset Z_+$ with $k_n \uparrow +\infty$, $\{m_n\} \subset Z_+$, $\{z_n\} \subset B$ with $z_n \in \overline{D(\varphi^{m_nT_0+\tau})}$ and $\{x_n\} \subset H$ with $x_n \in E(k_nT_0 + m_nT_0 + \tau, m_nT_0 + \tau)z_n$ such that

$$x_n \to x \text{ in } H \text{ as } n \to +\infty.$$

Now we state the main theorems in this paper.

Theorem 4.5 (Discrete attractors of (4.1))**.** *For each $\tau \in R_+$, let $\mathcal{A}_\tau$ be the global attractor of T_0-periodic dynamical systems U_τ, which is obtained in Section 3. For $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\}) \cap \Psi_M$, $\{G(t,\cdot)\} \in \mathcal{G}(\{\varphi^t\})$ and $f \in L^2_{\rm loc}(R_+; H)$, we assume that the system* (4.1) *is asymptotically T_0-periodic. Here we put*

$$\mathcal{A}^*_\tau := \overline{\bigcup_{B\in\mathcal{B}(H)} \omega_\tau(B)}. \tag{4.2}$$

Then, we have

(i) *$\mathcal{A}^*_\tau (\subset D(\varphi^\tau_p))$ is non-empty and compact in H;*
(ii) *for each bounded set $B \in \mathcal{B}(H)$ and each number $\epsilon > 0$ there exists $N_{B,\epsilon} \in N$ such that*
$$\operatorname{dist}_H(E(kT_0 + \tau, \tau)z, \mathcal{A}^*_\tau) < \epsilon$$
for all $z \in \overline{D(\varphi^\tau)} \cap B$ and all $k \ge N_{B,\epsilon}$;
(iii) *$\mathcal{A}^*_\tau \subset U^l_\tau \mathcal{A}^*_\tau \subset \mathcal{A}_\tau$ for any $l \in N$, where U_τ is the discrete dynamical system for* (3.1) *given in Definition 3.3.*

Remark 4.6. By the definition of the discrete ω-limit set $\omega_\tau(B)$ and $\mathcal{A}^*_\tau$, we easily see that

$$\mathcal{A}^*_\tau = \mathcal{A}^*_{\tau+nT_0}, \quad \forall n \in N.$$

Hence $\mathcal{A}^*_\tau$ is T_0-periodic in time in the above sense.

The second main theorem concerns a relationship between attractors $\mathcal{A}^*_s$ and $\mathcal{A}^*_\tau$.

Theorem 4.7. *Suppose the same assumptions are made as in Theorem 4.5. Let $\mathcal{A}^*_s$ and $\mathcal{A}^*_\tau$ be discrete attractors for $E(\cdot,s)$ and $E(\cdot,\tau)$ with $0 \le s \le \tau < +\infty$, respectively. Then,*

$$\mathcal{A}^*_\tau \subset U(\tau,s)\mathcal{A}^*_s,$$

where $U(\tau,s)$ is the T_0-periodic process for (3.1) *which is given in Definition 3.2.*

By Theorems 4.5-4.7, we can get the attractor for the asymptotic T_0-periodic system (4.1).

Theorem 4.8 (Global attractor for (4.1))**.** *Suppose the same assumptions are made as in Theorem 4.5. For any $\tau \in R_+$, let $\mathcal{A}^*_\tau$ be the discrete attractor for $E(\cdot,\tau)$ obtained in Theorem 4.5. Here we put*

$$\mathcal{A}^* := \bigcup_{\tau\in[0,T_0]} \mathcal{A}^*_\tau. \tag{4.3}$$

Then, for any bounded set $B \in \mathcal{B}(H)$,

$$\bigcap_{s\geq 0} \overline{\bigcup_{t\geq s,\tau\in R_+} E(t+\tau,\tau)(\overline{D(\varphi^\tau)} \cap B)} \subset \mathcal{A}^*. \tag{4.4}$$

By Theorem 4.8, the set $\mathcal{A}^*$ can be called the global attractor of (4.1).

Here we give some key lemmas.

Lemma 4.9. *If $\{s_n\} \subset R_+$, $\{\tau_n\} \subset R_+$, $s \in R_+$, $\tau \in R_+$, $s_n \to s$, $\tau_n \to \tau$, $\{m_n\} \subset Z_+$ with $m_n \to +\infty$, $z_n \in \overline{D(\varphi^{m_nT_0+s_n})}$, $z \in \overline{D(\varphi^s_p)}$, $z_n \to z$ in H and an element $w_n \in E(m_nT_0+\tau_n+s_n, m_nT_0+s_n)z_n$ converges to some element $w \in H$ as $n \to +\infty$, then $w \in U(\tau+s,s)z$.*

Proof. Since $\tau_n \to \tau$, without loss of generality we may assume that there exists a finite time $T > 0$ such that $\{\tau_n\} \subset [0,T]$ and $\tau \in [0,T]$. By $w_n \in E(m_nT_0+\tau_n+s_n, m_nT_0+s_n)z_n$, there is a solution v_n of (4.1) on $[m_nT_0+s_n,+\infty)$ such that

$$v_n(m_nT_0+\tau_n+s_n) = w_n \text{ and } v_n(m_nT_0+s_n) = z_n.$$

Now we put $u_n(t) := v_n(t+m_nT_0+s_n)$, then we easily see that u_n is the solution for

$$\begin{gathered} u_n'(t) + \partial\varphi^{t+m_nT_0+s_n}(u_n(t)) + G(t+m_nT_0+s_n, u_n(t)) \\ \ni f(t+m_nT_0+s_n), \quad t > 0, \\ u_n(0) = z_n. \end{gathered}$$

Let $\delta \in (0,1)$ be fixed. Since $z_n \to z$ in H as $n \to +\infty$, $\{z_n\}$ is bounded in H. Hence, from global estimates of solutions (cf. (B) in Section 2) it follows that there is a positive constant $M_\delta > 0$ (independent of n) satisfying

$$\sup_{t\geq\delta} |u_n(t)|_H^2 + \sup_{t\geq\delta} |u_n'|^2_{L^2(t,t+1;H)} + \sup_{t\geq\delta} \varphi^{t+m_nT_0+s_n}(u_n(t)) \leq M_\delta. \tag{4.5}$$

By [16, Lemma 4.1] we note that the convergence assumption (A1) implies

$$\varphi^{t+m_nT_0+s_n} \to \varphi_p^{t+s} \tag{4.6}$$

in the sense of Mosco [20] for each $t \geq 0$ as $n \to +\infty$. Moreover by the same argument in [10, Lemma 3.1] we can prove that

$$\bigcup_{n=1}^{+\infty} \{z \in H; \varphi^{t+m_nT_0+s_n}(z) \leq k\} \tag{4.7}$$

is relatively compact in H for every real $k > 0$ and $t \geq 0$, where $\varphi^{t+m_nT_0+s_n} = \varphi_p^{t+s}$ if $n = +\infty$. Therefore, by (4.5)-(4.7), (A2), (A3) and the convergence result (C) in Section 2, (by taking a subsequence of $\{n\}$, if necessary) we see that there is a function u_δ such that

$$u'_\delta(t) + \partial\varphi_p^{t+s}(u_\delta(t)) + G_p(t+s, u_\delta(t)) \ni f_p(t+s), \quad t > \delta.$$

By the standard diagonal process and the same argument in [21, Lemma 3.10], we can construct the solution u on $[0, +\infty)$ satisfying

$$u'(t) + \partial\varphi_p^{t+s}(u(t)) + G_p(t+s, u(t)) \ni f_p(t+s), \quad t > 0,$$
$$u(0) = z$$

and

$$u_n \to u \text{ in } C([0,T]; H) \quad \text{as } n \to +\infty. \tag{4.8}$$

Then, by (4.8) and $u_n(\tau_n) = w_n$ we have $u(\tau) = w$, which implies $w \in U(\tau + s, s)z$. □

By (B) in Section 2, for each $B \in \mathcal{B}(H)$ we can choose constants $r_B > 0$ and $M_B > 0$ so that

$$|v|_H \le r_B \quad \text{and} \quad \varphi^{t+s}(v) \le M_B, \tag{4.9}$$

for any $s \in R_+$, $t \ge T_0$, $z \in \overline{D(\varphi^s)} \cap B$ and $v \in E(t+s, s)z$. Hence it follows from condition (A1) that for each $m \in Z_+$, $\tau \in [0, T_0]$, $n \in N$ and $z \in \overline{D(\varphi^{mT_0+\tau})} \cap B$ there is $\tilde{z} := \tilde{z}_{mT_0+\tau,z,nT_0} \in D(\varphi_p^\tau)$ such that

$$|\tilde{z} - v|_H \le J_{m+n}^{(r_B+M_B+M)},$$
$$\Big(\text{hence } |\tilde{z}|_H \le r_B + J_{m+n}^{(r_B+M_B+M)}\Big)$$

and

$$\varphi_p^\tau(\tilde{z}) - \varphi^{nT_0+mT_0+\tau}(v) \le J_{m+n}^{(r_B+M_B+M)},$$
$$\Big(\text{hence } \varphi_p^\tau(\tilde{z}) \le M_B + J_{m+n}^{(r_B+M_B+M)}\Big),$$

where $v \in E(nT_0 + mT_0 + \tau, mT_0 + \tau)z$.

Since $J_k^{(r_B+M_B+M)} \to 0$ as $k \to +\infty$, there is a number $N_0 \in N$ such that

$$J_k^{(r_B+M_B+M)} \le 1, \quad \forall k > N_0.$$

Now, put $J_0 := 1 + \sup_{1 \le k \le N_0} J_k^{(r_B+M_B+M)} < +\infty$. Then, we define the bounded set $\widetilde{B_\tau}$ by

$$\widetilde{B_\tau} := \{z \in H; |z|_H \le r_B + J_0\} \cap \overline{D(\varphi_p^\tau)}.$$

Let $B_{0,\tau}$ be the compact absorbing set for U_τ introduced by Remark 3.5. Then, we see that there exists a number $\widetilde{N} \in N$ so that

$$U_\tau^l \widetilde{B_\tau} \subset B_{0,\tau}, \quad \forall l \ge \widetilde{N}. \tag{4.10}$$

The next lemma is very important for proving Theorem 4.5 (iii).

Lemma 4.10. *Let $\tau \in R_+$ and $B_{0,\tau}$ be the compact absorbing set for U_τ. Then we have*

$$\omega_\tau(B) \subset B_{0,\tau}, \quad \forall B \in \mathcal{B}(H).$$

Proof. At first we assume $\tau \in [0, T_0]$. For each $B \in \mathcal{B}(H)$, let x be any element of $\omega_\tau(B)$. Then, it follows from Remark 4.4 that there exist sequences $\{k_n\} \subset Z_+$ with $k_n \to +\infty$, $\{m_n\} \subset Z_+$, $\{z_n\} \subset B$ with $z_n \in \overline{D(\varphi^{m_n T_0 + \tau})}$ and $\{x_n\} \subset H$ with $x_n \in E(k_n T_0 + m_n T_0 + \tau, m_n T_0 + \tau) z_n$ such that

$$x_n \to x \text{ in } H \quad \text{as } n \to +\infty. \tag{4.11}$$

Let $\widetilde{N}$ be the positive integer obtained in (4.10). Then by (E2) we have

$$\begin{aligned} x_n \in & E(k_n T_0 + m_n T_0 + \tau, k_n T_0 - \widetilde{N} T_0 + m_n T_0 + \tau) \\ & \circ E(k_n T_0 - \widetilde{N} T_0 + m_n T_0 + \tau, m_n T_0 + \tau) z_n \end{aligned} \tag{4.12}$$

for any n with $k_n \geq \widetilde{N} + 1$. Hence, there exists an element $y_n \in E(k_n T_0 - \widetilde{N} T_0 + m_n T_0 + \tau, m_n T_0 + \tau) z_n$ such that

$$x_n \in E(k_n T_0 + m_n T_0 + \tau, k_n T_0 - \widetilde{N} T_0 + m_n T_0 + \tau) y_n. \tag{4.13}$$

Since $\{z_n\} \subset B$, we see that $|y_n|_H \leq r_B$ and

$$\varphi^{k_n T_0 - \widetilde{N} T_0 + m_n T_0 + \tau}(y_n) \leq M_B$$

for any n with $k_n \geq \widetilde{N} + 1$, where r_B and M_B are same positive constants in (4.9).

From the convergence condition (A1) it follows that for $y_n \in E(k_n T_0 - \widetilde{N} T_0 + m_n T_0 + \tau, m_n T_0 + \tau) z_n$ there is $\widetilde{z}_n \in D(\varphi_p^\tau)$ such that

$$|\widetilde{z}_n - y_n|_H \leq J_{k_n - \widetilde{N} + m_n}^{(r_B + M_B + M)},$$

$$\Big(\text{hence } |\widetilde{z}_n|_H \leq r_B + J_{k_n - \widetilde{N} + m_n}^{(r_B + M_B + M)}\Big)$$

and

$$\varphi_p^\tau(\widetilde{z}_n) \leq M_B + J_{k_n - \widetilde{N} + m_n}^{(r_B + M_B + M)}.$$

Since $\{\widetilde{z}_n \in D(\varphi_p^\tau)\ ;\ n \in N \text{ with } k_n \geq \widetilde{N} + 1\} (\subset \widetilde{B_\tau})$ is relatively compact in H, we may assume that

$$\widetilde{z}_n \to \widetilde{z}_\infty \text{in } H \quad \text{ as } n \to +\infty$$

for some $\widetilde{z}_\infty \in H$. Then we easily see that $\widetilde{z}_\infty \in \widetilde{B_\tau}$ and

$$y_n \to \widetilde{z}_\infty \text{ in } H \quad \text{as } n \to +\infty. \tag{4.14}$$

By Lemma 4.9 and (4.11)-(4.14), we observe that $x \in U(\widetilde{N} T_0 + \tau, \tau) \widetilde{z}_\infty$, which implies that

$$x \in U(\widetilde{N} T_0 + \tau, \tau) \widetilde{B_\tau} = U_\tau^{\widetilde{N}} \widetilde{B_\tau} \subset B_{0,\tau}.$$

Hence we have $\omega_\tau(B) \subset B_{0,\tau}$.

For the general case of $\tau \in R_+$, choose positive numbers $i_\tau \in N$ and $\tau_0 \in [0, T_0]$ so that $\tau = \tau_0 + i_\tau T_0$. Then, we can show $\omega_\tau(B) \subset B_{0,\tau}$ by the same argument as above. $\square$

Proof of Theorem 4.5. On account of Lemma 4.10 we can get $\mathcal{A}_\tau^* \subset B_{0,\tau}$. Hence, Theorem 4.5 (i) holds. Also, by (4.2) and Remark 4.4 we observe that Theorem 4.5 (ii) holds.

Now, we prove Theorem 4.5 (iii). At first, let us prove that $\mathcal{A}_\tau^* \subset U_\tau^l \mathcal{A}_\tau^*$ for any $l \in N$. Let x be any element of $\mathcal{A}_\tau^*$. By the definition of $\mathcal{A}_\tau^*$, there are sequences $\{B_n\} \subset \mathcal{B}(H)$ and $\{x_n\} \subset H$ with $x_n \in \omega_\tau(B_n)$ such that

$$x_n \to x \quad \text{ in } H \quad \text{as } n \to +\infty. \tag{4.15}$$

Then, for each n it follows from Remark 4.4 that there exist sequences $\{k_{n,j}\} \subset Z_+$ with $k_{n,j} \to +\infty$, $\{m_{n,j}\} \subset Z_+$, $\{z_{n,j}\} \subset B_n$ with $z_{n,j} \in \overline{D(\varphi^{m_{n,j}T_0+\tau})}$ and $\{v_{n,j}\} \subset H$ with $v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau,\ m_{n,j}T_0 + \tau)z_{n,j}$ such that

$$v_{n,j} \to x_n \text{ in } H \quad \text{as } j \to +\infty. \tag{4.16}$$

Let l be any number in N, then we see that

$$\begin{aligned} v_{n,j} \in E(&k_{n,j}T_0 + m_{n,j}T_0 + \tau,\ k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau) \\ &\circ E(k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau,\ m_{n,j}T_0 + \tau)z_{n,j} \end{aligned}$$

for j with $k_{n,j} \ge l+1$. So, there exists an element $w_{n,j} \in E(k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j}$ such that

$$v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau)w_{n,j}. \tag{4.17}$$

By the global estimates (B) in Section 2, $\{w_{n,j} \in H\ ;\ j \in N \text{ with } k_{n,j} \ge l+1\}$ is relatively compact in H for each n. Therefore we may assume that the element $w_{n,j}$ converges to some element $\widetilde{w}_{n,\infty} \in H$ as $j \to +\infty$. Clearly, $\widetilde{w}_{n,\infty} \in \omega_\tau(B_n)$. Moreover, it follows from Lemma 4.9 and (4.16)-(4.17) that

$$x_n \in U(lT_0 + \tau, \tau)\widetilde{w}_{n,\infty} \subset U(lT_0 + \tau, \tau)\omega_\tau(B_n),$$

hence, we have

$$x_n \in \bigcup_{n\ge 1} U_\tau^l \omega_\tau(B_n), \quad \forall n \ge 1. \tag{4.18}$$

Here, by the closedness of $U(\cdot,\cdot)$ we note that for each subset X of $B_{0,\tau}$,

$$\overline{U_\tau^l X} \subset U_\tau^l \overline{X}, \quad \forall l \in N. \tag{4.19}$$

Taking into account Lemma 4.10, (4.15), (4.18) and (4.19), we observe that

$$x \in \overline{\bigcup_{n\ge 1} U_\tau^l \omega_\tau(B_n)} = \overline{U_\tau^l \bigcup_{n\ge 1} \omega_\tau(B_n)} \subset U_\tau^l \overline{\bigcup_{n\ge 1} \omega_\tau(B_n)} \subset U_\tau^l \mathcal{A}_\tau^*,$$

which implies that $\mathcal{A}_\tau^*$ is semi-invariant under the T_0-periodic dynamical systems U_τ, i.e.

$$\mathcal{A}_\tau^* \subset U_\tau^l \mathcal{A}_\tau^*, \quad \forall l \in N. \tag{4.20}$$

Next we shall prove that $U_\tau^l \mathcal{A}_\tau^* \subset \mathcal{A}_\tau$ for any $l \in N$. By (4.20), for each $l \in N$

$$U_\tau^l \mathcal{A}_\tau^* \subset U_\tau^l U_\tau^n \mathcal{A}_\tau^* = U_\tau^{l+n} \mathcal{A}_\tau^*, \quad \forall n \in N. \tag{4.21}$$

By $\mathcal{A}_\tau^* \subset B_{0,\tau}$, (4.21) and the attractive property of $\mathcal{A}_\tau$, we have

$$U_\tau^l \mathcal{A}_\tau^* \subset \mathcal{A}_\tau, \quad \forall l \in N.$$

Therefore, we conclude that $\mathcal{A}_\tau^* \subset U_\tau^l \mathcal{A}_\tau^* \subset \mathcal{A}_\tau$ for all $l \in N$. □

Proof of Theorem 4.7. Let x be any element of $\mathcal{A}_\tau^*$. Then by the definition of $\mathcal{A}_\tau^*$, there exist sequences $\{B_n\} \subset \mathcal{B}(H)$ and $\{x_n\} \subset H$ with $x_n \in \omega_\tau(B_n)$ such that

$$x_n \to x \text{ in } H \quad \text{as } n \to +\infty. \tag{4.22}$$

From Remark 4.4 it follows that for each n, there are sequences $\{k_{n,j}\} \subset Z_+$ with $k_{n,j} \to +\infty$, $\{m_{n,j}\} \subset Z_+$, $\{z_{n,j}\} \subset B_n$ with $z_{n,j} \in \overline{D(\varphi^{m_{n,j}T_0+\tau})}$ and $\{v_{n,j}\} \subset H$ with $v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j}$ such that

$$v_{n,j} \to x_n \text{ in } H \quad \text{as } j \to +\infty. \tag{4.23}$$

Note that for given $s,\tau \in R_+$ with $s \le \tau$ there is a positive number $l_s \in N$ satisfying

$$s \le \tau \le l_s T_0 + s.$$

Using the property (E2) we see that

$$\begin{aligned} v_{n,j} \in & E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 + m_{n,j}T_0 + s) \\ & \circ E(k_{n,j}T_0 + m_{n,j}T_0 + s, T_0 + m_{n,j}T_0 + l_sT_0 + s) \\ & \circ E(T_0 + m_{n,j}T_0 + l_sT_0 + s, m_{n,j}T_0 + \tau) z_{n,j} \end{aligned}$$

for any $j \in Z_+$ with $k_{n,j} \ge l_s + 2$. Here we can take elements $w_{n,j} \in H$ and $y_{n,j} \in H$ so that

$$v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 + m_{n,j}T_0 + s) w_{n,j}, \tag{4.24}$$

$$w_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + s, T_0 + m_{n,j}T_0 + l_sT_0 + s) y_{n,j}, \tag{4.25}$$

$$y_{n,j} \in E(T_0 + m_{n,j}T_0 + l_sT_0 + s, m_{n,j}T_0 + \tau) z_{n,j}. \tag{4.26}$$

By $\{z_{n,j}\} \subset B_n$ and the global boundedness result (B) in Section 2, we can get a positive constant $C_n := C_n(B_n) > 0$ satisfying

$$|y_{n,j}|_H \le C_n, \quad \forall y_{n,j} \in E(T_0 + m_{n,j}T_0 + l_sT_0 + s, m_{n,j}T_0 + \tau) z_{n,j}. \tag{4.27}$$

Here we define the bounded set B_{C_n} by

$$B_{C_n} := \{b \in H : |b|_H \le C_n\}.$$

From (4.27) and the result (B) in Section 2 it follows that the set

$$\begin{aligned} \big\{ & w_{n,j} \in H; w_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + s, T_0 + m_{n,j}T_0 + l_sT_0 + s) y_{n,j} \\ & \text{for any } j \in Z_+ \text{ with } k_{n,j} \ge l_s + 2 \big\} \end{aligned}$$

is relatively compact in H. Hence, we may assume that the element $w_{n,j}$ converges to some element $\widetilde{w}_{n,\infty} \in H$ as $j \to +\infty$. Clearly, $\widetilde{w}_{n,\infty} \in \omega_s(B_{C_n})$, and it follows from Lemma 4.10 that

$$\omega_s(B_{C_n}) \subset B_{0,s} \subset \overline{D(\varphi_p^s)}\,.$$

Moreover, by Lemma 4.9 and (4.23)-(4.24) we have

$$x_n \in U(\tau,s)\widetilde{w}_{n,\infty} \subset U(\tau,s)\omega_s(B_{C_n}), \quad \forall n \ge 1,$$

hence, we see that

$$x_n \in \bigcup_{n \ge 1} U(\tau,s)\omega_s(B_{C_n}), \quad \forall n \ge 1\,. \tag{4.28}$$

Here, by the closedness of $U(\cdot,\cdot)$, we note that for each subset X of $B_{0,s}$,

$$\overline{U(\tau,s)X} \subset U(\tau,s)\overline{X}. \tag{4.29}$$

On account of Lemma 4.10, (4.22), (4.28) and (4.29), we observe that

$$x \in \overline{\bigcup_{n \ge 1} U(\tau,s)\omega_s(B_{C_n})} = \overline{U(\tau,s)\bigcup_{n \ge 1} \omega_s(B_{C_n})} \subset U(\tau,s)\overline{\bigcup_{n \ge 1} \omega_s(B_{C_n})} \subset U(\tau,s)\mathcal{A}_s^*,$$

which implies that $\mathcal{A}_\tau^*$ is a subset of $U(\tau,s)\mathcal{A}_s^*$, namely $\mathcal{A}_\tau^* \subset U(\tau,s)\mathcal{A}_s^*$. $\square$

Proof of Theorem 4.8. For any $B \in \mathcal{B}(H)$, let z_0 be any element of the ω-limit set $\omega_E(B)$ which is define by

$$\omega_E(B) := \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s, \tau\in R_+} E(t+\tau,\tau)(\overline{D(\varphi^\tau)}\cap B)}.$$

Then we easily see that there exist sequences $\{t_n\} \subset R_+$ with $t_n \to +\infty$, $\{\tau_n\} \subset R_+$, $\{y_n\} \subset B$ with $y_n \in \overline{D(\varphi^{\tau_n})}$ and $\{z_n\} \subset H$ with $z_n \in E(t_n+\tau_n,\tau_n)y_n$ such that

$$\begin{gathered} t_n := k_nT_0 + t'_n, \quad k_n \in Z_+,\ k_n \to +\infty, \quad t'_n \in [T_0, 2T_0],\ t'_n \to t'_0, \\ \tau_n := l_nT_0 + \tau'_n, \quad l_n \in Z_+,\ \tau'_n \in [0,T_0],\ \tau'_n \to \tau'_0, \\ z_n \to z_0 \quad \text{in } H \end{gathered} \tag{4.30}$$

as $n \to +\infty$. Without loss of generality, we may assume that

(a) $t'_n + \tau'_n \nearrow t'_0 + \tau'_0$ or
(b) $t'_n + \tau'_n \searrow t'_0 + \tau'_0$.

Now, assume that (a) holds. Then let us consider the multivalued semiflow

$$v_n \in E(1 + k_nT_0 + l_nT_0 + t'_0 + \tau'_0,\ k_nT_0 + l_nT_0 + t'_n + \tau'_n)z_n. \tag{4.31}$$

Then, there is a solution u_n on $[k_nT_0 + l_nT_0 + t'_n + \tau'_n, +\infty)$ for

$$\begin{gathered} u'_n(t) + \partial\varphi^{t+k_nT_0+l_nT_0+t'_n+\tau'_n}(u_n(t)) + G(t + k_nT_0 + l_nT_0 + t'_n + \tau'_n, u_n(t)) \\ \ni f(t + k_nT_0 + l_nT_0 + t'_n + \tau'_n), \quad t > 0, \\ u_n(0) = z_n \quad \text{and} \quad u_n(1 + t'_0 + \tau'_0 - t'_n - \tau'_n) = v_n. \end{gathered}$$

Since $z_n \to z_0$ in H, $\{z_n\}$ is bounded in H. Therefore, by the global estimate (B) in Section 2, we see that the set

$$\begin{aligned} &\big\{v_n \in H; v_n \in E(1 + k_nT_0 + l_nT_0 + t'_0 + \tau'_0, k_nT_0 + l_nT_0 + t'_n + \tau'_n)z_n \\ &\text{for any } n \in N\big\} \end{aligned}$$

is relatively compact in H. Hence we may assume that

$$v_n \to v \text{ in } H \text{ for some } v \in H. \tag{4.32}$$

Now applying Lemma 4.9 with (4.30)-(4.32), we obtain

$$v \in U(1 + t'_0 + \tau'_0,\ t'_0 + \tau'_0)z_0,$$

more precisely, (taking the subsequence of $\{n\}$ if necessary) we observe that

$$u_n \to u \quad \text{in } C([0,1];H) \quad \text{as } n \to +\infty, \tag{4.33}$$

where u is the solution on $[t'_0 + \tau'_0, +\infty)$ satisfying

$$\begin{gathered} u'(t) + \partial\varphi_p^{t+t'_0+\tau'_0}(u(t)) + G_p(t + t'_0 + \tau'_0, u(t)) \ni f_p(t + t'_0 + \tau'_0), \quad t > 0, \\ u(0) = z_0 \quad \text{and} \quad u(1) = v. \end{gathered}$$

By (4.33) we easily see that

$$u_n(t'_0 + \tau'_0 - t'_n - \tau'_n) \to z_0 \quad \text{as } n \to +\infty. \tag{4.34}$$

Note that

$$\begin{aligned}
&u_n(t_0' + \tau_0' - t_n' - \tau_n') \\
&\in E(k_nT_0 + l_nT_0 + t_0' + \tau_0',\ k_nT_0 + l_nT_0 + t_n' + \tau_n')z_n \\
&= E(k_nT_0 + l_nT_0 + t_0' + \tau_0',\ l_nT_0 + \tau_n')y_n \\
&= E(k_nT_0 + l_nT_0 + t_0' + \tau_0',\ l_nT_0 + t_0' + \tau_0')E(l_nT_0 + t_0' + \tau_0',\ l_nT_0 + \tau_n')y_n.
\end{aligned}$$

So, we can take an element $x_n \in E(l_nT_0 + t_0' + \tau_0',\ l_nT_0 + \tau_n')y_n$ such that

$$u_n(t_0' + \tau_0' - t_n' - \tau_n') \in E(k_nT_0 + l_nT_0 + t_0' + \tau_0',\ l_nT_0 + t_0' + \tau_0')x_n. \tag{4.35}$$

By $\{y_n\} \subset B$ and the global estimate (B) in Section 2, we easily see that $\{x_n\}$ is bounded, i.e.

$$\{x_n\} \subset \widetilde{B} \text{ for some } \widetilde{B} \in \mathcal{B}(H). \tag{4.36}$$

Therefore, from Remarks 4.4-4.6 and (4.34)-(4.36) we observe that

$$z_0 \in \omega_{t_0'+\tau_0'}(\widetilde{B}) \subset \mathcal{A}^*_{t_0'+\tau_0'} \subset \mathcal{A}^*.$$

Thus (4.4) holds.

In the case (b) when $t_n' + \tau_n' \searrow t_0' + \tau_0'$, we can prove (4.4) by a slight modification of the proof as above. □

Note that Theorem 4.5 implies that the attracting set $\mathcal{A}^*_\tau$ for (4.1) is semi-invariant under U_τ associated with the limiting T_0-periodic system (3.1), in general. Moreover, from Theorem 4.7 we observe that

$$\mathcal{A}^*_\tau \subset U(\tau, s)\mathcal{A}^*_s \quad \text{for any } 0 \le s \le \tau < +\infty.$$

To get the invariance of $\mathcal{A}^*_\tau$ under U_τ and $\mathcal{A}^*_\tau = U(\tau, s)\mathcal{A}^*_s$, let us use a concept of a regular approximation, which was introduced in [17].

Definition 4.11 (Regular approximation)**.** Let $s \in R_+$ be fixed. Let $z \in D(\varphi^s_p)$. Then, we say that $U(t+s, s)z$ is regularly approximated by $E(t + kT_0 + s,\ kT_0 + s)$ as $k \to +\infty$, if for each finite $T > 0$ there are sequences $\{k_n\} \subset Z_+$ with $k_n \to +\infty$ and $\{z_n\} \subset H$ with $z_n \in D(\varphi^{k_nT_0+s})$ and $z_n \to z$ in H satisfying the following property: for any function $u \in W^{1,2}(0, T; H)$ satisfying $u(t) \in U(t + s, s)z$ for all $t \in [0, T]$ there is a sequence $\{u_n\} \subset W^{1,2}(0, T; H)$ such that $u_n(t) \in E(t + k_nT_0 + s,\ k_nT_0 + s)z_n$ for all $t \in [0, T]$ and $u_n \to u$ in $C([0, T]; H)$ as $n \to +\infty$.

Using the above concept, we can show the invariance of $\mathcal{A}^*_\tau$ under U_τ. Moreover we can get

$$\mathcal{A}^*_\tau = U(\tau, s)\mathcal{A}^*_s.$$

Theorem 4.12. *Suppose all assumptions in Theorem 4.5. Let $\mathcal{A}^*_s$ and $\mathcal{A}^*_\tau$ be discrete attractors for $E(\cdot, s)$ and $E(\cdot, \tau)$, with $0 \le s \le \tau < +\infty$, respectively. Assume that for any point z of $\mathcal{A}^*_s$, $U(t + s, s)z$ is regularly approximated by $E(t + kT_0 + s, kT_0 + s)$ as $k \to +\infty$. Then we have*

$$\mathcal{A}^*_\tau = U(\tau, s)\mathcal{A}^*_s.$$

Proof. By Theorem 4.7, we have only to show that

$$U(\tau, s)\mathcal{A}^*_s \subset \mathcal{A}^*_\tau.$$

To do so, let x be any element of $U(\tau, s)\mathcal{A}^*_s$.

At first, taking into account Definitions 3.2-3.3 and Theorem 4.5 (iii), we see that for each $n \in N$

$$\begin{aligned} &U_\tau^n U(\tau,s)\mathcal{A}_s^* \\ &= U(nT_0+\tau,\tau)U(\tau,s)\mathcal{A}_s^* = U(nT_0+\tau,nT_0+s)U(nT_0+s,s)\mathcal{A}_s^* \\ &= U(\tau,s)U_s^n\mathcal{A}_s^* \supset U(\tau,s)\mathcal{A}_s^*. \end{aligned} \tag{4.37}$$

Hence, there exists an element $y_n \in \mathcal{A}_s^*$ such that

$$x \in U_\tau^n U(\tau,s)y_n = U(nT_0+\tau-s+s,s)y_n.$$

Using our assumption as $t = nT_0+\tau-s$, we observe that for each n, there are sequences $\{k_{n,j}\} \subset Z_+$, $\{x_{n,j}\} \subset H$ and $\{y_{n,j}\} \subset H$ such that

$$k_{n,j} \to +\infty, \quad y_{n,j} \in D(\varphi^{k_{n,j}T_0+s}), \quad y_{n,j} \to y_n \quad \text{in } H$$

and

$$x_{n,j} \in E(nT_0+\tau-s+k_{n,j}T_0+s,\ k_{n,j}T_0+s)y_{n,j}, \quad x_{n,j} \to x \quad \text{in } H \tag{4.38}$$

as $j \to +\infty$. Therefore, by the usual diagonal argument, we can find a subsequence $\{j_n\}$ of $\{j\}$ such that $\widetilde{x}_n := x_{n,j_n}$, $\widetilde{y}_n := y_{n,j_n}$ and $\widetilde{k}_n := k_{n,j_n}$ satisfy

$$\begin{gathered} |\widetilde{x}_n - x|_H < \frac{1}{n}, \quad \widetilde{x}_n \in E(nT_0+\tau-s+\widetilde{k}_nT_0+s,\widetilde{k}_nT_0+s)\widetilde{y}_n, \\ |\widetilde{y}_n - y_n|_H < \frac{1}{n} \end{gathered} \tag{4.39}$$

for $n = 1,2,\dots$. Since $\{\widetilde{y}_n\}$ is bounded in H, there is a bounded set $B \in \mathcal{B}(H)$ so that $\{\widetilde{y}_n\} \subset B$. By (E2), we see that

$$\begin{aligned} \widetilde{x}_n &\in E(nT_0+\tau-s+\widetilde{k}_nT_0+s,\widetilde{k}_nT_0+s)\widetilde{y}_n \\ &= E(nT_0+\widetilde{k}_nT_0+\tau,T_0+\widetilde{k}_nT_0+\tau)E(T_0+\widetilde{k}_nT_0+\tau,\widetilde{k}_nT_0+s)\widetilde{y}_n, \end{aligned}$$

hence there is an element $\widetilde{z}_n \in E(T_0+\widetilde{k}_nT_0+\tau,\widetilde{k}_nT_0+s)\widetilde{y}_n$ such that

$$\widetilde{x}_n \in E(nT_0+\widetilde{k}_nT_0+\tau,T_0+\widetilde{k}_nT_0+\tau)\widetilde{z}_n. \tag{4.40}$$

Since $\{\widetilde{y}_n\} \subset B$ and the global estimate (B) in Section 2, we see that $\{\widetilde{z}_n\}$ is also bounded in H. Hence, there is a bounded set $\widetilde{B} \in \mathcal{B}(H)$ so that $\{\widetilde{z}_n\} \subset \widetilde{B}$. The above fact (4.38)-(4.40) implies (cf. Remark 4.4) that $x \in \omega_\tau(\widetilde{B}) \subset \mathcal{A}_\tau^*$. Thus we have $U(\tau,s)\mathcal{A}_s^* \subset \mathcal{A}_\tau^*$. □

By Remark 4.6 and the same argument in Theorem 4.12, we can get the following corollary.

Corollary 4.13. *(i) Suppose the same assumptions of Theorem 4.12. Then, $\mathcal{A}_s^*$ is invariant under the T_0-periodic dynamical system $U_s(:= U(T_0+s,s))$. Namely,*

$$\mathcal{A}_s^* = U_s^l\mathcal{A}_s^* \quad \text{for any } l \in N.$$

(ii) Assume that for any point z of $\mathcal{A}_\tau$, $U(t+\tau,\tau)z$ is regularly approximated by $E(t+kT_0+\tau,kT_0+\tau)$ as $k \to +\infty$. Then, $\mathcal{A}_\tau^* \supset \mathcal{A}_\tau$. Hence we have $\mathcal{A}_\tau^* = \mathcal{A}_\tau$ (cf. Theorem 4.5 (iii)).

Remark 4.14. If the solution operator $U(t,s)$ is single valued, namely the solution for the Cauchy problem of (3.1) is unique, the assumptions of Theorem 4.12 always hold. Thus, Theorem 4.12 implies the abstract results obtained in [11] which was concerned with the asymptotically T_0-periodic stability for the single valued dynamical system associated with time-dependent subdifferentials.

5. Applications to obstacle problems for PDE's

Let Ω be a bounded domain in R^N $(1 \le N < +\infty)$ with smooth boundary $\Gamma = \partial\Omega$, q be a fixed number with $2 \le q < +\infty$ and T_0 be a fixed positive number. We use the notation

$$a_q(v,z) := \int_\Omega |\nabla v|^{q-2}\nabla v \cdot \nabla z dx, \quad \forall v,\ z \in W^{1,q}(\Omega)$$

and denote by $(\cdot,\cdot)$ the usual inner product in $L^2(\Omega)$.

For prescribed obstacle functions $\sigma_0 \le \sigma_1$ and each $t \in R_+$ we define the set

$$K(t) := \big\{z \in W^{1,q}(\Omega); \sigma_0(t,\cdot) \le z \le \sigma_1(t,\cdot) \text{ a.e. on } \Omega\big\}.$$

Let f be a function in $L^2_{\rm loc}(R_+; L^2(\Omega))$ and h be a non-negative function on $R_+ \times R$.

Then for given $\mathbf{b} \in L^\infty(\Omega)^N$ we consider an interior asymptotically T_0-periodic double obstacle problem for each initial time $s \in R_+$:
Find functions $v \in C([s,+\infty); L^2(\Omega))$ and $\theta \in L^2_{\rm loc}((s,+\infty); L^2(\Omega))$ such that

$$\begin{gathered} v \in L^q_{\rm loc}((s,+\infty); W^{1,q}(\Omega)) \cap W^{1,2}_{\rm loc}((s,+\infty); L^2(\Omega)); \\ v(t) \in K(t) \quad \text{for a.e. } t \ge s; \\ 0 \le \theta(t,x) \le h(t,v(t,x)) \quad \text{a.e. on } (s,+\infty)\times\Omega; \\ (v'(t) + \theta(t) + \mathbf{b}\cdot\nabla v(t) - f(t), v(t) - z) + a_q(v(t), v(t)-z) \le 0 \\ \text{for } z \in K(t) \text{ and a.e. } t \ge s. \end{gathered} \tag{5.1}$$

The main object of this section is to consider the large-time behavior of solution for (5.1) under asymptotically T_0-periodicity assumptions

$$\sigma_i(t) - \sigma_{i,p}(t) \to 0\ (i=0,1), \quad h(t,\cdot) - h_p(t,\cdot) \to 0, \quad f(t) - f_p(t) \to 0$$

as $t \to \infty$ in the sense specified below, where $\sigma_{i,p}(t)$, $h_p(t,\cdot)$, $f_p(t)$ are periodic in time with the same period T_0. By the above assumptions, the limiting system of (5.1) is a T_0-periodic one as follows:
Find functions $u \in C([s,+\infty); L^2(\Omega))$ and $\theta \in L^2_{\rm loc}((s,+\infty); L^2(\Omega))$ such that

$$\begin{gathered} u \in L^q_{\rm loc}((s,+\infty); W^{1,q}(\Omega)) \cap W^{1,2}_{\rm loc}((s,+\infty); L^2(\Omega)); \\ u(t) \in K_p(t) \quad \text{for a.e. } t \ge s; \\ 0 \le \theta(t,x) \le h_p(t,u(t,x)) \quad \text{a.e. on } (s,+\infty)\times\Omega; \\ (u'(t) + \theta(t) + \mathbf{b}\cdot\nabla u(t) - f_p(t), u(t) - z) + a_q(u(t), u(t)-z) \le 0 \\ \text{for any } z \in K_p(t) \text{ and a.e. } t \ge s, \end{gathered} \tag{5.2}$$

where $K_p(t) := \big\{z \in W^{1,q}(\Omega) : \sigma_{0,p}(t,\cdot) \le z \le \sigma_{1,p}(t,\cdot) \text{ a.e. on } \Omega\big\}$.

Now we suppose the following conditions:

- σ_i and $\sigma_{i,p}$ are functions on $R_+ \times \Omega$ such that
$$\sup_{t\in R_+}\Big|\frac{d\sigma_i}{dt}\Big|_{L^2(t,t+1;W^{1,q}(\Omega))} + \sup_{t\in R_+}\Big|\frac{d\sigma_i}{dt}\Big|_{L^2(t,t+1;L^\infty(\Omega))} < +\infty,$$
$$\sup_{t\in R_+}\Big|\frac{d\sigma_{i,p}}{dt}\Big|_{L^2(t,t+1;W^{1,q}(\Omega))} + \sup_{t\in R_+}\Big|\frac{d\sigma_{i,p}}{dt}\Big|_{L^2(t,t+1;L^\infty(\Omega))} < +\infty$$
and $\sigma_{i,p}$ is a T_0-periodic obstacle function, i.e.
$$\sigma_{i,p}(t+T_0,x) = \sigma_{i,p}(t,x) \quad \text{for a.e. } x \in \Omega \text{ and any } t \in R_+$$
for $i = 0, 1$. Moreover, there are positive constants $k_1 > 0$ and $k_2 > 0$ such that
$$\sigma_1 - \sigma_0 \ge k_1 \quad \text{and} \quad \sigma_{1,p} - \sigma_{0,p} \ge k_1 \quad \text{a.e. on } R_+ \times \Omega$$
and
$$|\sigma_i|_{L^\infty(R_+;W^{1,q}(\Omega))} + |\sigma_i|_{L^\infty(R_+\times\Omega)} + |\sigma_{i,p}|_{L^\infty(R_+;W^{1,q}(\Omega))} + |\sigma_{i,p}|_{L^\infty(R_+\times\Omega)} \le k_2$$
for $i = 0, 1$.
- h and h_p are non-negative continuous functions on $R_+ \times R$. There is a positive constant L such that
$$\begin{gathered}|h(t,z_1) - h(t,z_2)| \le L|z_1 - z_2|\\ |h_p(t,z_1) - h_p(t,z_2)| \le L|z_1 - z_2|\end{gathered}$$
for all $t \in R_+$, $z_i \in R$ and $i = 1, 2$. Moreover, h_p is a T_0-periodic function, i.e. for any $z \in R$, $h_p(t+T_0,z) = h_p(t,z)$ for any $t \in R_+$.
- f, $f_p \in L^2_{\rm loc}(R_+;L^2(\Omega))$, and f_p is a T_0-periodic function, i.e.
$$f_p(t+T_0) = f_p(t) \quad \text{in } L^2(\Omega), \quad \forall t \in R_+.$$

Moreover, we suppose the following convergence conditions:

- (Convergence of $\sigma_i(t) - \sigma_{i,p}(t) \to 0$ as $t \to +\infty$) Put
$$\begin{aligned}I_m := &\sup_{t\in[0,T_0]}|\sigma_0(mT_0+t) - \sigma_{0,p}(t)|_{W^{1,q}(\Omega)}\\ &+ \sup_{t\in[0,T_0]}|\sigma_1(mT_0+t) - \sigma_{1,p}(t)|_{W^{1,q}(\Omega)}\\ &+ \sup_{t\in[0,T_0]}|\sigma_0(mT_0+t) - \sigma_{0,p}(t)|_{L^\infty(\Omega)}\\ &+ \sup_{t\in[0,T_0]}|\sigma_1(mT_0+t) - \sigma_{1,p}(t)|_{L^\infty(\Omega)}.\end{aligned}$$
Then, $I_m \to 0$ as $m \to +\infty$.
- (Convergence of $h(t,\cdot) - h_p(t,\cdot) \to 0$ as $t \to +\infty$) For any $z \in R$,
$$\sup_{t\in[0,T_0]}|h(mT_0+t,z) - h_p(t,z)| \to 0 \quad \text{as } m \to +\infty. \tag{5.3}$$
- (Convergence of $f(t) - f_p(t) \to 0$ as $t \to +\infty$)
$$|f(mT_0+\cdot) - f_p|_{L^2(0,T_0;L^2(\Omega))} \to 0 \quad \text{as } m \to +\infty. \tag{5.4}$$

Under the above assumptions, let us consider problems (5.1) and (5.2). To apply the abstract results in Sections 2-4, we choose $L^2(\Omega)$ as a real separable Hilbert space H. And we define a proper l.s.c. convex function φ^t on $L^2(\Omega)$ by

$$\varphi^t(z) = \begin{cases} \frac{1}{q}\int_\Omega |\nabla z|^q dx & \text{if } z \in K(t), \\ +\infty & \text{if } z \in L^2(\Omega) \setminus K(t), \end{cases} \tag{5.5}$$

and define φ_p^t by replacing $K(t)$ by $K_p(t)$ in (5.5).

Also, we define a multivalued operator $G(\cdot,\cdot)$ from $R_+ \times H^1(\Omega)$ into $L^2(\Omega)$ by

$$\begin{aligned} G(t,z) := \big\{ g \in L^2(\Omega); g = l + \mathbf{b}\cdot\nabla z \quad \text{in } L^2(\Omega) \\ 0 \le l(x) \le h(t, z(x)) \quad \text{a.e. on } \Omega \big\} \end{aligned} \tag{5.6}$$

for all $t \in R_+$ and $z \in H^1(\Omega)$. And we define $G_p(\cdot,\cdot)$ by replacing $h(t,\cdot)$ by $h_p(t,\cdot)$ in (5.6).

By the same argument as in [27, Lemma 5.1], we can obtain the following lemmas.

Lemma 5.1 (cf. [27, Lemma 5.1])**.** *For any $r > 0$ and $t \in R_+$, put*

$$\begin{aligned} a_r(t) &= b_r(t) \\ &:= k_3 \int_0^t \big\{ |\sigma'_{0,p}|_{L^\infty(\Omega)} + |\sigma'_{0,p}|_{W^{1,q}(\Omega)} + |\sigma'_{1,p}|_{L^\infty(\Omega)} + |\sigma'_{1,p}|_{W^{1,q}(\Omega)} \big\} d\tau \\ &\quad + k_3 \int_0^t \big\{ |\sigma'_0|_{L^\infty(\Omega)} + |\sigma'_0|_{W^{1,q}(\Omega)} + |\sigma'_1|_{L^\infty(\Omega)} + |\sigma'_1|_{W^{1,q}(\Omega)} \big\} d\tau, \end{aligned}$$

where k_3 is a (sufficiently large) positive constant. Then, $\{\varphi^t\} \in \Phi(\{a_r\},\{b_r\})$ and $\{\varphi_p^t\} \in \Phi_p(\{a_r\},\{b_r\};T_0)$. Moreover we have $\{G(t,\cdot)\} \in \mathcal{G}(\{\varphi^t\})$ and $\{G_p(t,\cdot)\} \in \mathcal{G}_p(\{\varphi_p^t\};T_0)$.

Lemma 5.2. *The convergence assumptions (A1)-(A3) hold.*

Proof. We see easily that (A2) and (A3) hold by assumptions (5.3) and (5.4). Now let us show (A1). For each $t \in R_+$ there are $m \in Z_+$ and $\tau \in [0, T_0]$ so that $t = mT_0 + \tau$. For each $z_p \in D(\varphi_p^t) = K_p(t)$, we put

$$z := (z_p - \sigma_{0,p}(t))\frac{\sigma_1(t) - \sigma_0(t)}{\sigma_{1,p}(t) - \sigma_{0,p}(t)} + \sigma_0(t).$$

Then we see that $z \in D(\varphi^t) = K(t)$. Moreover, by the same argument in [27, Lemma 5.1], we see that

$$|z - z_p|_{L^2(\Omega)} \le k_4 I_m \quad \text{and} \quad |\nabla z - \nabla z_p|_{L^q(\Omega)} \le k_4 I_m (1 + |\nabla z_p|_{L^q(\Omega)}) \tag{5.7}$$

for some constant $k_4 > 0$. Hence we have

$$\varphi^t(z) - \varphi_p^t(z_p) \le k_5 I_m (1 + \varphi_p^t(z_p)) \tag{5.8}$$

for a sufficiently large $k_5 > 0$.

Conversely, let $z \in D(\varphi^t) = K(t)$ and we put

$$z_p := (z - \sigma_0(t))\frac{\sigma_{1,p}(t) - \sigma_{0,p}(t)}{\sigma_1(t) - \sigma_0(t)} + \sigma_{0,p}(t).$$

Then, we observe that $z_p \in D(\varphi_p^t) = K_p(t)$ and

$$|z_p - z|_{L^2(\Omega)} \le k_4 I_m \quad \text{and} \quad \varphi_p^t(z_p) - \varphi^t(z) \le k_5 I_m (1 + \varphi^t(z)). \tag{5.9}$$

Therefore, by (5.7)-(5.9) we see that the convergence assumption (A1) holds. □

Clearly, the obstacle problem (5.1) can be reformulated as an evolution equation (4.1) involving the subdifferential of φ^t given by (5.5) and the multivalued operator $G(t, \cdot)$ defined by (5.6). Also, the limiting T_0-periodic problem (5.2) can be reformulated as an evolution equation (3.1). Therefore, by Lemmas 5.1-5.2 we can apply abstract results in Section 2-4. Namely, we can obtain an attractor $\mathcal{A}_s^*$ for (5.1), a T_0-periodic attractor $\mathcal{A}_s$ for (5.2) and the relationships between (5.1) and (5.2)

Additionally, we assume that $f(t) \equiv f_p(t)$ for any $t \in R_+$ and

$$\sigma_0(t, z) \equiv \sigma_{0,p}(t, z), \quad \sigma_{1,p}(t, z) \equiv \sigma_1(t, z), \quad h_p(t, z) \leq h(t, z)$$

for any $0 \leq t < +\infty$ and $z \in R$. Then we easily see that the assumptions of Theorem 4.12 and its Corollary hold. Hence we can get $\mathcal{A}_s^* = \mathcal{A}_s$ by the same argument in [30, Theorem 5.4].

Unfortunately, we do not give assumptions for $\sigma_i(t, \cdot)$, $h(t, \cdot)$ and $f(t)$ in order to get

$$U(\tau, s)\mathcal{A}_s^* = \mathcal{A}_\tau^* \subset \mathcal{A}_\tau \text{ for any } 0 \leq s \leq \tau < +\infty. \tag{5.10}$$

It seems difficult to show (5.10), so we leave it as an open problem.

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