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CHARACTERIZING DEGENERATE STURM-LIOUVILLE PROBLEMS

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Dedicated to the memory of F. V. Atkinson, - with profound respect and gratitude for the magic and beauty he brought to differential equations

Abstract. Consider the Dirichlet eigenvalue problem associated with the real two-term weighted Sturm-Liouville equation $-(p(x)y')' = \lambda r(x)y$ on the finite interval $[a, b]$. This eigenvalue problem will be called *degenerate* provided its spectrum fills the whole complex plane. Generally, in degenerate cases the coefficients $p(x), r(x)$ must each be sign indefinite on $[a, b]$. Indeed, except in some special cases, the quadratic forms induced by them on appropriate spaces must also be indefinite. In this note we present a necessary and sufficient condition for this boundary problem to be degenerate. Some extensions are noted.

1. Introduction

Ever since the pioneering work of Sturm, and Sturm and Liouville in the 1830's the study of eigenvalue problems for the Sturm-Liouville equation (1.1),

$$-(p(x)y')' + q(x)y = \lambda r(x)y, \tag{1.1}$$

$$y(a) = y(b) = 0, \tag{1.2}$$

and resultant spectral theory has acquired momentum and remains of great interest even today due, in part, to its original roots as a branch of applied mathematics (arising from the separation of variables method in the wave and heat equations etc.) In these cases of physical interest it is imperative that the spectrum of the problem (1.1)-(1.2), say, always be infinitely countable and having no finite point of accumulation. In such cases one can then ask questions about an eigenfunction expansion theorem, eigenvalue asymptotics, oscillation theorems for the eigenfunctions etc. as has been done in the past (cf., for example, [[2], Chapter 8], [9]). However, in the event where the spectrum fails to be infinitely countable few results are known. On the one hand, the spectrum may be finite perhaps even empty, since the problem may be restated so as to include three-term recurrence relations [[2],Theorem 8.4.5], and thus it reduces to a question about the eigenvalues of a

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generalized matrix eigenvalue problem (see also [10]). For a recent construction of such special examples see [8].

On the other hand, Sturm himself noted in [12] that if the coefficient $p(x)$ vanishes in the interior of $[a, b]$ then there may be a blow-up of the derivative at that point, a concern that was eventually handled using quasi-derivatives and Carathéodory conditions for existence and uniqueness of solutions in the case of Lebesgue measurable coefficients (e.g., [2]). For this reason among others (likely of a physical nature), Sturm restricted himself to cases where the leading term $p(x) > 0$ in the interval under consideration (and then also to cases where the weight $r(x) > 0$). See [9] for a review of cases where the weight $r(x)$ changes sign.

Allowing the quadratic form associated with the *principal part* (i.e., $-(p(x)y')'$) to be indefinite on a suitable space (which forces $p(x)$ to actually change its sign on sets of positive measure) led Atkinson and the author to construct the first example of a degenerate Sturm-Liouville operator in [3] (see also [1]). To set the scene we point out that whenever $1/p, q, r \in L^1[a, b]$ and all are real-valued the initial value problem associated with (1.1), for a fixed λ, and given initial conditions on $y(a, \lambda), (py')(a)$ has a unique solution with the property that $y(x)$ is absolutely continuous along with $(py')(x)$ and that this y satisfies the equation (1.1) almost everywhere (Lebesgue).

Example 1.1. Given an arbitrary p-term such that $1/p \in L^1[a, b]$, $1/p(x) \neq 0$, it is a simple matter for the reader to verify that the class of equations

$$-(p(x)y')' = \lambda\big(\frac{\alpha}{p(x)}\big)y,$$
$$y(a) = y(b) = 0,$$

where $\alpha \in \mathbf{R}$ is fixed, can lead to a class of degenerate Sturm-Liouville eigenvalue problems by noting that for given $\lambda \in \mathbb{C}$, $\lambda \neq 0$ an eigenfunction can be found by setting

$$y(x, \lambda) = \frac{1}{\sqrt{\lambda\alpha}} \sin\big(P(x)\sqrt{\lambda\alpha}\big),$$

where $P(x) \equiv \int_a^x p(s)^{-1} ds$ is chosen so that $P(b) = 0$. If $\lambda = 0$ an eigenfunction is obtained by setting

$$y(x, \lambda) = P(x).$$

Note that although the function y is absolutely continuous here, y' is not (recall Sturm's remark, above) but yet $(py')(x)$ is, and this y satisfies the equation a.e.

Next, whenever the zero-set of r is a set of measure zero, the operator T defined by setting

$$Tf(x) = -\frac{1}{r(x)}(p(x)f')',$$

on a suitable space (see [1]) is actually *self-adjoint* in the Kreĭn space $L^2_r[a, b]$, (see [5], [1]). Thus, in an interesting connection with operator theory, degenerate Sturm-Liouville operators give rise to self-adjoint operators on a Kreĭn space whose spectrum is all of $\mathbb{C}$ (see [1], Section 4] for more details about this connection. This particular class of degenerate problems may be expanded somewhat by relaxing the condition on $r(x)$ but this is not our concern at the moment.

The question at the core of this paper involves the determination of a set of explicit conditions that will ensure that the Sturm-Liouville problem

$$-(p(x)y')' = \lambda r(x)y, \tag{1.3}$$

$$y(a) = 0, \quad y(b) = 0, \tag{1.4}$$

is degenerate that is, its eigenvalue spectrum fills the whole complex plane, $\mathbb{C}$.

2. The Main Result

Theorem 2.1. *Consider the boundary value problem associated with the two-term Sturm-Liouville equation* (1.3)-(1.4) *on the finite interval* $[a, b]$. *The following statements are equivalent:*

1. *The collection of eigenvalues of (1.3)-(1.4) fills all of* $\mathbb{C}$,
2. *There exists an eigenvalue* $\lambda = \lambda^*$ *of* (1.3)-(1.4) *for which a sequence of eigenvalues* $\lambda_n \in \mathbb{C}$ *of* (1.3)-(1.4) *exists with the property that* $\lambda_n \to \lambda^*$ *as* $n \to \infty$.
3. *Define* $a_0(x) = P(x) \equiv \int_a^x p(s)^{-1} ds$ *and*

$$a_{n+1}(x) \equiv - \int_a^x P'(s) \int_a^s r(t) a_n(t)\, dt\, ds.$$

Then the coefficients p, r *in* (1.3) *satisfy*

$$a_n(b) = 0 \tag{2.1}$$

for each $n = 0, 1, 2, \dots$.

Notes: 1. Observe that the conditions in (2.1) are equivalent to solving the boundary problem for the differential equations

$$-(p(x){a'}_{n+1}(x))' = r(x)a_n(x), \quad a_{n+1}(a) = a_{n+1}(b) = 0,$$

for $n = 0, 1, 2, \dots$

2. In addition, more general separated homogeneous boundary conditions are easily handled using the same technique since the function $(py')(b, \lambda) + h \cdot y(b, \lambda)$ where h is a fixed constant, is an entire function of λ once again. In this case, there is a parallel theorem to Theorem 2.1 where the conditions (2.1) are more complicated looking but can nevertheless be written down. The technique can be modified easily to yield recursions of the form (2.1) in cases where the boundary conditions are even *non-linear* in the eigenvalue parameter (e.g., if the resulting boundary condition is an entire function of λ), subject to some simple yet lengthy calculations.

3. Finally, the addition of a potential term $q(x)$ as in (1.1) causes no serious difficulty to the technique presented here since an analogous set of conditions of the type described in (2.1) can be formulated. Once again the results are somewhat less elegant than those in the case considered here.

4. Note that one cannot find an example where we will "see" an accumulation of eigenvalues at a certain point in the complex plane without the spectrum being the whole complex plane, by item 2) in the Theorem. Furthermore, note that if either $P'(x)$ and/or $r(x)$ are a.e. zero on $[a, b]$, then (2.1) is automatically satisfied and so once again the spectrum fills the complex plane. Thus, in this sense, semidefinite problems may also lead to degeneracy. Generally, however, the coefficients p, r are sign-indefinite.

3. Proof of the Main Result

Lemma 3.1. *For a fixed initial condition $y(a,\lambda)$ and $(py')(a,\lambda)$, the function $y(b,\lambda)$, where y is a solution of (1.3), or more generally (1.1), is an entire function of $\lambda \in \mathbb{C}$ of order at most one-half, in some cases of order zero.*

Proof. Without loss of generality, we fix the initial condition for a given λ, at $x = a$ by means of $y(a,\lambda) = 0$, $(py')(a,\lambda) = 1$ and prove this for the case where $q(x) = 0$. This will then define a unique solution that is bounded on $[a,b]$ for each choice of $\lambda \in \mathbb{C}$. The eigenvalues of (1.3)-(1.4) are then given by the (possibly complex) zeros of the function $y(b,\lambda)$.

First, we note that even in cases where the principal part of (1.3) is indefinite the function $y(b,\lambda)$ is an entire function of λ. The argument is actually classical and we need only sketch the proof; one can either simulate the proof in Bôcher [4], or Ince [7] using care due to the indefiniteness of the sign of $p(x)$. For example, in [7, Section 10.72], one finds the derivation of the equation

$$\int_a^b r(x)y(x,\lambda_i)y(x,\lambda)\,dx = (py')(b,\lambda)\frac{y(b,\lambda)-y(b,\lambda_i)}{\lambda-\lambda_i},$$

without any reference to sign definiteness of any of the coefficients p, q, r in (1.1), for a given $\lambda \in \mathbb{C}$. It is easily seen that the limit as $\lambda \to \lambda_i$ exists and is finite (since $r \in L^1[a,b]$) and is given by the left side. Thus y is differentiable as a function of λ etc. The end result is that both $y(b,\lambda)$ and $(py')(b,\lambda)$ are entire functions of λ for $\lambda \in \mathbb{C}$. To obtain an estimate of the order of $y(b,\lambda)$ as an entire function in this sign-indefinite case, we adapt an argument in [11] (where there is also an extension to coefficients with a possibly nonlinear dependence on λ). To this end we write (1.3) as a first order system $u' = v/p$, $v' = -\lambda r u$. Then

$$\frac{d}{dx}\left\{|\lambda|u\bar{u} + v\bar{v}\right\} = \frac{|\lambda|}{p}(u\bar{v} + \bar{u}v) - r(\lambda u\bar{v} + \bar{\lambda}\bar{u}v),$$

from which we get

$$\Big|\frac{d}{dx}\left\{|\lambda||u|^2 + |v|^2\right\}\Big| \le 2|\lambda||uv|\{\frac{1}{|p|} + |r|\}.$$

But for $\lambda \neq 0$ we have $2|u||v| \le \left\{|\lambda||u|^2 + |v|^2\right\}/\sqrt{|\lambda|}$. It follows that

$$\Big|(\frac{d}{dx})\log\{|\lambda||u|^2 + |v|^2\}\Big| \le \sqrt{|\lambda|}\{\frac{1}{|p|} + |r|\}.$$

An integration with respect to x shows that the estimate

$$u(x,\lambda),\ v(x,\lambda)\ = O\{\exp(\mathrm{const}\cdot\sqrt{|\lambda|})\}$$

holds for every $x \in (a,b]$ therefore proving our estimate on the order. □

Remark 3.2. A *highly* degenerate case could be characterized by the (semidefinite) case where $1/p(x) = 0 = r(x)$ a.e. in $[a,b]$, or more generally, where $r(x) = 0$ a.e., and p has a primitive $P(x)$ with $P(a) = P(b) = 0$. In this case the bounds above would give that $y(b,\lambda)$ is an entire function of order zero. Thus, one can find examples of semidefinite problems that are degenerate. Whether there is an example where $y(b,\lambda)$ is not a polynomial in λ but still an entire function of order zero is an open problem.

Remark 3.3. It is of interest to determine conditions under which $u(b,\lambda)$, $v(b,\lambda)$, are either of order exactly $1/2$ or of order zero, and we leave this as an open problem. In connection with this we note the partial results [6], where it is proved that (1.1) is of order exactly $1/2$ when $p(x)=1$, and r,q are arbitrary, and [3] where it is shown that the order is again $1/2$ in the more general situation where $p(x)$ has finitely many sign changes and r,q are arbitrary. The question here deals with arbitrary p-terms: That is, there appears to be no example where the order is anything but 0 or $1/2$ for all cases (sign-definite or not) of p,q,r as above. The preceding indicates that if an example of such an entire function of order ν where $0<\nu<1/2$ does exist then, in indefinite cases, $p(x)$ changes sign an infinite number of times on intervals or more generally, sets of positive measure.

Proof of the Theorem 2.1. The equivalence of 1) and 2) is clear from the theory of analytic functions. The equivalence of 1) and 3) is proved as follows. Since $y(x,\lambda)$ is entire for each $x\in[a,b]$, there is a classical representation

$$y(x,\lambda)=\sum_{n=0}^{\infty}a_n(x)\,\lambda^n \tag{3.1}$$

where the series converges uniformly on every compact subset of the complex plane, and the $a_n(x)$ are to be determined. On the other hand, the solution $y(x,\lambda)$ satisfying $y(a,\lambda)=0$, $(py')(a,\lambda)=1$, is identical to the solution of the integral equation

$$y(x,\lambda)=P(x)-\int_a^x P'(s)\int_a^s \lambda\, r(t)y(t,\lambda)\,dt\,ds, \tag{3.2}$$

where $P(x)$ is defined above as a primitive of $1/p$. Inserting (3.1) into (3.2), expanding and collecting terms we obtain the (necessarily unique) representation for the $a_n(x)$, $n=0,1,2,\dots$, as

$$a_0(x)=P(x), \tag{3.3}$$

$$a_1(x)=(-1)\int_a^x P'(s)\int_a^s r(t)P(t)\,dt\,ds \tag{3.4}$$

$$a_2(x)=(-1)\int_a^x P'(s)\int_a^s r(t)a_1(t)\,dt\,ds \tag{3.5}$$

$$\vdots$$

$$a_{n+1}(x)=(-1)\int_a^x P'(s)\int_a^s r(t)a_n(t)\,dt\,ds, \tag{3.6}$$

$$\vdots$$

It follows that if condition 1) holds, then $y(b,\lambda)=0$ for every $\lambda\in\mathbb{C}$ and this implies that $a_n(b)=0$, for every $n=0,1,2,\dots$, which is 3). On the other hand, if 3) holds for every $n=0,1,\dots$, then the function $y(b,\lambda)=0$ for any value of $\lambda\in\mathbb{C}$. Hence every $\lambda\in\mathbb{C}$ is an eigenvalue and this completes the proof.

Note that the coefficients in Example 1.1 above satisfy the conditions of this theorem (so as to obtain an independent verification of the result). To see this fix an $\alpha\in\mathbf{R}$ and set $r(x)=\alpha/p(x)$, where $p(x)$ is chosen so that $P(b)=0$. Then

$a_0(b) = 0$, by our choice of P,

$$a_1(b) = -\alpha \int_a^b P'(s) \int_a^s P'(t)P(t)\,dt\,ds = 0,$$

since $P(a) = P(b) = 0$. Since $a_1(t) = -\alpha P^3(t)/3!$ we see that $a_2(t) = (-\alpha)^2 P^5(t)/5!$ so that once again, $a_2(b) = 0$. We can now proceed by induction in order to verify the remaining conditions, a calculation that we omit. This example includes the one presented in [3, p. 381], as a special case. □

Definition For given coefficients p, q, r as usual, we will call the problem (1.1)-(1.2) **totally non-definite** (as opposed to simply non-definite, see [9]) if the problem itself is non-definite and the quadratic form defined by the principal part of (1.1), that is,

$$\int_a^b p(x)|y'(x)|^2\,dx$$

is non-definite on the space of complex-valued functions y defined on the interval $[a, b]$ that are absolutely continuous along with the function py' and that satisfy the boundary conditions $y(a) = 0 = y(b)$. That is, the quadratic form defined above takes on both signs for suitable functions.

We produce a final example that indicates the kind of behaviour that can be expected when one is approximating the situation of degeneracy of the problem (1.1)-(1.2).

Example 3.4. To this end, let $\varepsilon > 0$, and consider the *totally* non-definite problem defined by setting $p(x) = r(x) = \operatorname{sgn} x$, the function that represents the sign of x, and $q(x) = \varepsilon$, is a function that is identically constant on $[-1, 1]$. Thus, we consider the eigenvalue problem

$$-(\operatorname{sgn} x\; y')' + \varepsilon\, y = \lambda \operatorname{sgn} x\, y, \tag{3.7}$$

$$y(-1) = y(1) = 0, \tag{3.8}$$

where $\lambda \in \mathbb{C}$ is a generally complex parameter. Recall that an eigenfunction of (3.7-3.8) is necessarily an absolutely continuous function y on $[-1, 1]$ along with py' such that $y(x)$ satisfies (3.7) almost everywhere along with the boundary conditions. This said, a straightforward calculation shows that the form of a typical eigenfunction of (3.7-3.8) normalized by setting $(py')(-1) = 1$ is given by

$$y(x,\lambda) = \begin{cases} -\frac{\sin\left((x+1)\sqrt{\lambda+\varepsilon}\right)}{\sqrt{\lambda+\varepsilon}} & \text{for } -1 \le x \le 0, \\ \frac{\sin(\sqrt{\lambda+\varepsilon})\,\sin\left((x-1)\sqrt{\lambda-\varepsilon}\right)}{\sqrt{\lambda+\varepsilon}\,\sin(\sqrt{\lambda-\varepsilon})} & \text{for } 0 \le x \le 1, \end{cases} \tag{3.9}$$

for suitable eigenvalues λ. These eigenvalues are obtained by solving the dispersion relation

$$\begin{aligned} f(\lambda,\varepsilon) \equiv {} & \sqrt{\lambda+\varepsilon}\cos\left(\sqrt{\lambda+\varepsilon}\right)\sin\left(\sqrt{\lambda-\varepsilon}\right) \\ & - \sqrt{\lambda-\varepsilon}\cos\left(\sqrt{\lambda-\varepsilon}\right)\sin\left(\sqrt{\lambda+\varepsilon}\right) = 0, \end{aligned} \tag{3.10}$$

obtained by setting $(py')(0-) = (py')(0+)$ in (3.9) as required for the existence of a solution. Observe that $f(\lambda, -\varepsilon) = -f(\lambda, \varepsilon)$ so that a typical eigenvalue λ (whenever it exists) is at the same time an eigenvalue of the problem (3.7-3.8) and its counterpart obtained by replacing ε by $-\varepsilon$ in (3.7). Also note that asymptotically the total number, $N(\lambda)$, of zeros of our solution (3.9) in $(-1,1)$ is independent of ε (even if the problem is degenerate) and in fact asymptotic to

$$N(\lambda) \sim 2\pi^{-1}\sqrt{\lambda},$$

as $\lambda \to \infty$ by the Atkinson-Mingarelli theorem (cf., [2]).

However, *there is no Sturm oscillation theorem* (or even its extension by Haupt and Richardson, cf., [9]) for the eigenfunctions as we show presently. Indeed, using elementary trigonometric identities we see that

$$\begin{aligned} f(\lambda,\varepsilon) &= \frac{\sqrt{\lambda+\varepsilon}-\sqrt{\lambda-\varepsilon}}{2}\sin\big(\sqrt{\lambda+\varepsilon}+\sqrt{\lambda-\varepsilon}\big) \\ &\quad - \frac{\sqrt{\lambda+\varepsilon}+\sqrt{\lambda-\varepsilon}}{2}\sin\big(\sqrt{\lambda+\varepsilon}-\sqrt{\lambda-\varepsilon}\big), \end{aligned}$$

an expression that can be rewritten as

$$f(\lambda,\varepsilon) = \varepsilon\,\frac{\sin\big(\sqrt{\lambda+\varepsilon}+\sqrt{\lambda-\varepsilon}\big)}{\sqrt{\lambda+\varepsilon}+\sqrt{\lambda-\varepsilon}} - \varepsilon\,\frac{\sin\big(\sqrt{\lambda+\varepsilon}-\sqrt{\lambda-\varepsilon}\big)}{\sqrt{\lambda+\varepsilon}-\sqrt{\lambda-\varepsilon}}. \tag{3.11}$$

However, the first term in (3.11) is $o(1)$ as $\lambda \to \infty$. On the other hand, since

$$\sqrt{\lambda+\varepsilon}-\sqrt{\lambda-\varepsilon} \to 0, \quad \lambda \to \infty,$$

the second term is easily seen to converge to the quantity $-\varepsilon$. It follows that

$$f(\lambda,\varepsilon) \to -\varepsilon, \quad \lambda \to \infty.$$

Thus, whenever $\varepsilon \neq 0$, we must have $f(\lambda,\varepsilon)$ of constant sign for all sufficiently large λ. In other words, given such an ε at the outset, there is a $\Lambda > 0$ such that there are no real eigenvalues in the region $|\lambda| > \Lambda$. On the other hand, since $f(\lambda,\varepsilon)$ is entire in λ, for given ε, there cannot be an accumulation of zeros either (i.e., eigenvalues) in the finite region $|\lambda| \leq \Lambda$. Thus, the collection of eigenvalues of the problem (3.7) is finite (and possibly empty). It follows that given $\varepsilon \neq 0$, there can be no oscillation theorem of traditional type for the eigenfunctions corresponding to these eigenvalues.

Incidentally, this construction provides an alternate example of a boundary problem of Sturm-Liouville type whose spectrum is finite and possibly empty (compare with [8]). In fact, numerical evidence here indicates that given a positive integer n there exists a value of ε (usually large) such that correpsonding eigenfunctions will oscillate $n-$times in $(-1,1)$. This is because one can show that the total number of eigenvalues of (3.7-3.8) tends to infinity as $|\varepsilon| \to \infty$.

As $|\varepsilon| \to 0$ we observe the following phenomena in the spectrum of (3.7-3.8). First, given $\varepsilon \neq 0$, $\lambda = \pm\varepsilon$ is an eigenvalue of (3.7-3.8) if and only if the corresponding *right-semidefinite* problem (see [9])

$$-(\operatorname{sgn}(x)\,y')' = \lambda\,(\operatorname{sgn}(x) \mp 1)\,y, \quad y(-1) = y(1) = 0,$$

admits $\lambda = \pm\varepsilon$ as an eigenvalue, respectively. The existence of a doubly infinite sequence of real eigenvalues for the right-semidefinite problem above (albeit left-indefinite) is known (see [1]), where this can be shown using the reciprocal transformation. Assimilating our results we get that given $\varepsilon \neq 0$, $\lambda = \pm\varepsilon$ is an eigenvalue for at most countably many such ε (where these are characterized above). Since these special ε-values form a countable set with no finite point of accumulation, there must be a smallest such set characterized by a particular $\varepsilon = \varepsilon_0$ such that whenever $|\varepsilon| < \varepsilon_0$, $\lambda = \pm\varepsilon$ cannot be an eigenvalue of (3.7-3.8). But as $|\varepsilon| \to 0$

the number of zeros of (3.9) decreases steadily. Thus, there is a critical value of $|\varepsilon|$ below which there is no real eigenvalue, that is the real spectrum is an empty set for all sufficiently small values of $|\varepsilon|$. The limiting case of $\varepsilon = 0$ gives a degenerate problem as is shown above (also because $f(\lambda, 0) \equiv 0$ for all λ, real or complex). Thus, the process leading from non-degeneracy to degeneracy may be a discontinuous one as exemplified here for three-term problems of the form (3.7-3.8).

Concluding Remarks. It follows that if for some given choice of coefficients p, r, at least one condition in part 3) of Theorem 2.1 above is violated, then the spectrum of (1.3)-(1.4) must be countable (finite or infinite or even empty) and may even consist of non-real eigenvalues (see e.g., [9]). Finally, we point out that any eigenvalue problem corresponding to a second order differential equation that is *transformable* into one of the type found in Example 1.1 will also have its spectrum equal to the whole complex plane. Thus, by transforming *back* starting from Example 1.1 such degenerate eigenvalue problems can be found at will.

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