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EXISTENCE AND MULTIPLICITY OF POSITIVE SOLUTIONS FOR A SINGULAR PROBLEM ASSOCIATED TO THE P-LAPLACIAN OPERATOR

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Abstract. Consider the problem

$$-\Delta_p u = g(u) + \lambda h(u) \quad \text{in } \Omega$$

with $u = 0$ on the boundary, where $\lambda \in (0, \infty)$, Ω is a strictly convex bounded and C^2 domain in $\mathbb{R}^N$ with $N \geq 2$, and $1 < p \leq 2$. Under suitable assumptions on g and h that allow a singularity of g at the origin, we show that for λ positive and small enough the above problem has at least two positive solutions in $C(\overline{\Omega}) \cap C^1(\Omega)$ and that $\lambda = 0$ is a bifurcation point from infinity. The existence of positive solutions for problems of the form $-\Delta_p u = K(x)g(u) + \lambda h(u) + f(x)$ in Ω, $u = 0$ on $\partial\Omega$ is also studied.

1. Introduction

This paper concerns problems of the form

$$\begin{aligned} -\Delta_p u &= Kg(u) + \lambda h(u) + f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \\ u &> 0 \quad \text{in } \Omega. \end{aligned} \tag{1.1}$$

Here λ is a nonnegative parameter, Δ_p is the p-laplacian operator defined by $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $1 < p < \infty$. We assume that

- (H1) Ω is a C^2 and bounded domain in $\mathbb{R}^N$ with $N \geq 2$
- (H2) $g : (0, \infty) \to (0, \infty)$ is a continuous and non increasing function (that may be singular at the origin)
- (H3) $h : [0, \infty) \to [0, \infty)$ is a continuous and non decreasing function
- (H4) K and f are nonnegative functions defined on Ω which satisfy that K is non identically zero, $K \in L^\infty(\Omega)$ and $f \in C(\overline{\Omega})$.

As usual, $g(u)$, $h(u)$ denote the Nemitskii operators associated with g and h respectively. The solutions of (1.1) will be understood in the following weak sense:

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$u \in W^{1,p}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$ satisfying $u = 0$ on $\partial\Omega$ and

$$\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla \varphi\rangle = \int_\Omega (Kg(u) + \lambda h(u) + f)\varphi$$

for all $\varphi \in C_c^\infty(\Omega)$.

Singular bifurcation problems of the form $-\Delta u = g(x,u) + h(x,\lambda u)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω have been considered in [4] for the case where, for some $\alpha > 0$ and $p > 0$ $g(x,u)$ and $h(x,\lambda u)$ behave like $u^{-\alpha}$ and $(\lambda u)^p$ respectively. There, existence of positive solutions for λ nonnegative and small enough is obtained via a sub and supersolutions method and non existence of such solutions is also shown for large values of λ. From these results it seems a natural question to ask for similar results when the laplacian is replaced by the degenerated operator Δ_p. Our aim in this paper is to study existence and (at least for the case $K = 1$, $f = 0$) multiplicity of positive solutions of (1.1). Our approach to this problem is somewhat different from the followed in [4] and it is more in the line of fixed point theorems for nonlinear eigenvalue problems. We first study in section 2, for a nonnegative $F \in C(\overline{\Omega})$, the problem $-\Delta_p u = Kg(u) + F$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω. Lemma 2.6 states that this problem has unique solution and Lemma 2.10 says that the corresponding solution operator S for this problem, defined by $S(F) := u$ is a compact, continuous and non decreasing map from $P \cup \{0\}$ into P, where P is the positive cone in $C(\overline{\Omega})$. These results (Lemmas 2.6 and 2.10) are suggested by the work of several authors in [2, 4, 5, 10, 11] where existence of positive solutions for this problem is obtained under different assumptions on K and f.

In section 3 we consider problem (1.1). We write it as $u = S(\lambda h(u) + f)$ with S as above. The above stated properties of S allow us to apply a classical fixed point theorem for nonlinear eigenvalue problems to obtain in Theorem 3.1 that for λ nonnegative and small enough there exists at least a (positive) solution of (1.1) and that the solution set for this problem (i.e., the set of the pairs (λ, u) that solve it) contains an unbounded subcontinuum (i.e., an unbounded connected subset) emanating from $(0, u^*)$, where u^* is the (unique) solution of the problem $-\Delta_p u = Kg(u) + f$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω.

Concerning multiplicity of positive solutions of (1.1), in section 4, Theorem 4.6, we prove that, if in addition,

- (H5) Ω is a strictly convex domain in R^N
- (H6) g and h are locally Lipchitz on $(0,\infty)$ and $[0,\infty)$ respectively
- (H7) $1 < p \le 2$, $\inf_{s>0} h(s)/s^{p-1} > 0$ and $\lim_{s\to\infty} h(s)/s^q < \infty$ for some $q \in (p-1, \frac{N(p-1)}{N-p}]$,

then the problem

$$\begin{gathered} -\Delta_p u = g(u) + \lambda h(u) \text{ in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \\ u > 0 \quad \text{in } \Omega \end{gathered} \tag{1.2}$$

has at least two positive solutions for λ positive and small enough and that $\lambda = 0$ is a bifurcation point from infinity for this problem.

To see this in section 4 we study, for $j \in \mathbb{N}$, the problem

$$\begin{gathered} -\Delta_p u = g(u + \frac{1}{j}) + \lambda h(u) \text{ in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \\ u > 0 \quad \text{in } \Omega. \end{gathered} \tag{1.3}$$

Lemma 4.1 provides, for a given $\lambda_0 > 0$, an a priori bound for the L^∞ norm of the solutions u of (1.3) corresponding to some $\lambda \geq \lambda_0$. On the other hand, from Theorem 3.1 we have an unbounded subcontinuum C_j of the solution set for (1.3) emanating now from $(0, u_j^*)$ where u_j^* is the (unique) solution of the problem $-\Delta_p v = g(v + \frac{1}{j}) + \lambda h(v)$ in Ω, $v = 0$ on $\partial\Omega$, $v > 0$ in Ω. Also (cf. Remark 3.2, part ii)) $C_j \subset [0, c) \times P$ for some positive constant c. Since C_j is connected and unbounded, from these facts we obtain, for λ positive and small enough, two positive solutions of (1.3) and then, going to the limit as j tends to infinity (perhaps after passing to a subsequence) we obtain two positive solutions for (1.3). Lemmas 4.2, 4.3, 4.5 and Remark 4.4 provide the necessary intermediate statements on order to do it.

2. Preliminaries

For this section, we assume that the conditions (H1), (H2), (H3) and (H4) stated at the introduction hold. Let us start with some preliminary remarks collecting some known facts about the p-Laplacian operator.

Remark 2.1. Let us recall [12, 6, 15] that for $v \in L^\infty(\Omega)$ and $1 < p < \infty$ the problem $-\Delta_p u = v$ in Ω, $u = 0$ on $\partial\Omega$ has a unique (weak) solution u which belongs to $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ and that the associated solution operator $(-\Delta_p)^{-1} : L^\infty(\Omega) \to C^1(\overline{\Omega})$ is a positive, continuous and compact map. Moreover, if $v \geq 0$ and $v \neq 0$ then u belongs to the interior of the positive cone in $C^1(\overline{\Omega})$ So $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$ and u is bounded from above and from below by positive multiples of the distance function

$$\delta(x) := \operatorname{dist}(x, \partial\Omega).$$

So $(-\Delta_p)^{-1}$ is a strongly positive operator on $C(\overline{\Omega})$, i.e., $v \in P$ implies $(-\Delta_p)^{-1} v \in Int(P)$ where P denotes the positive cone in $C(\overline{\Omega})$.

In addition, for the p-laplacian operator the following comparison principle holds: If U is a bounded domain (non necessarily regular) in $\mathbb{R}^N$ and if $u, v \in W^{1,p}_{\rm loc}(U) \cap C(\overline{U})$ with $1 < p < \infty$ satisfy (in weak sense) $-\Delta_p u \leq -\Delta_p v$ on U, $u \leq v$ on ∂U, then $u \leq v$.

Remark 2.2. If U is a bounded domain (i.e an open and connected set, non necessarily regular) in $\mathbb{R}^N$ and if $u, v \in W^{1,p}_{\rm loc}(U) \cap C(\overline{U})$ satisfy (in weak sense) $-\Delta_p u - Kg(u) \leq -\Delta_p v - g(v)$ in U with $u \leq v$ on ∂U, then $u \leq v$ on U. Indeed, suppose $u > v$ somewhere and consider the non empty open set $V = \{x \in U : u(x) > v(x)\}$. Since $-\Delta_p u + \Delta_p v \leq K(g(u) - g(v)) \leq 0$ in V and $u = v$ on ∂V the comparison principle gives a contradiction.

Lemma 2.3. *For a nonnegative $F \in L^\infty(\Omega)$ and for $j \in \mathbb{N}$, the problem*

$$\begin{gathered} -\Delta_p u_j = Kg(u+\frac{1}{j}) + F \text{ in } \Omega,\ u_j \in W^{1,p}_{\rm loc}(\Omega)\cap C(\overline{\Omega}), \\ u_j = 0 \quad \text{on } \partial\Omega, \\ u_j > 0 \quad \text{in } \Omega \end{gathered} \tag{2.1}$$

has a unique positive (weak) solution satisfying $u_j \in C^1(\overline{\Omega})$ and $j \to \frac{1}{j} + u_j$ is non increasing. Moreover, $u_j \geq c\,\delta$ for some positive constant c independent of j.

Proof. Let $g_j : \mathbb{R} \to \mathbb{R}$ be defined by $g_j(s) = g(s)$ for $s \geq \frac{1}{j}$ and $g_j(s) = g(\frac{1}{j})$ for $s < \frac{1}{j}$, let $T_j : C(\overline{\Omega}) \to C(\overline{\Omega})$ be given by $T_j(v) = (-\Delta_p)^{-1}(Kg(\frac{1}{j}+v)+F)$. Since for $v \in C(\overline{\Omega})$ we have

$$\|Kg(\frac{1}{j}+v)+F\|_{L^\infty(\Omega)}\|K\|_{L^\infty(\Omega)}g(\frac{1}{j}) + \|F\|_{L^\infty(\Omega)},$$

it follows that T_j is a compact operator. Moreover,

$$0 \leq T(v) \leq (-\Delta_p)^{-1}\big(g(\frac{1}{j})\|K\|_{L^\infty(\Omega)} + \|F\|_{L^\infty(\Omega)}\big) \quad \text{on } \Omega,$$

and so the existence assertion of the lemma follows from the Schauder fixed point theorem (as stated in [8, Corollary 11.2]) applied to T_j on a closed ball (in $C(\overline{\Omega})$) around 0 with radius large enough.

If v and w are two different solutions of (2.1) in $W^{1,p}_{\rm loc}(\Omega)\cap C(\overline{\Omega})$, consider the open set $\Omega' := \{x \in \Omega : v(x) > w(x)\}$. If $\Omega' \neq \emptyset$ then

$$-\Delta_p v + \Delta_p w = K\big(g_j(\frac{1}{j}+v) - g_j(\frac{1}{j}+w)\big) \quad \text{in } \Omega' \tag{2.2}$$

and also $v = w$ on $\partial\Omega'$, but, from our assumptions on K and g, the comparison principle gives $v \leq w$ on Ω' which is a contradiction. A similar contradiction is obtained if $v < w$ somewhere. thus the uniqueness assertion of the lemma holds. From the facts in Remark 2.1, the solution of (2.1) belongs to $C^1(\overline{\Omega})$ and it is positive because $(-\Delta_p)^{-1}$ is a positive operator Again by the comparison principle $\frac{1}{j+1} + u_{j+1} \leq \frac{1}{j} + u_j$. Indeed, consider the set $U = \{x \in \Omega : \frac{1}{j+1} + u_{j+1} > \frac{1}{j} + u_j\}$ and observe that $-\Delta_p(\frac{1}{j+1}+u_{j+1}) + \Delta_p(\frac{1}{j}+u_j) = Kg(\frac{1}{j+1}+u_{j+1}) - Kg(\frac{1}{j}+u_j) \leq 0$ in U and $\frac{1}{j+1} + u_{j+1} \leq \frac{1}{j} + u_j$ on ∂U, thus the comparison principle gives $U = \emptyset$.

Finally, $-\Delta_p(u_j) = Kg(\frac{1}{j}+u_j) + F \geq Kg(1+u_1)$, so the strong positivity of $(-\Delta_p)^{-1}$ gives the last assertion of the lemma. □

Remark 2.4 (Tolksdorf's estimates)**.** Let Ω' and Ω'' be open subsets of Ω such that $\Omega'' \subset\subset \Omega' \subset\subset \Omega$ and suppose that $u \in W^{1,p}_{\rm loc}(\Omega)\cap C(\overline{\Omega})$ satisfies $-\Delta_p u = v$ on Ω for some $v \in L^\infty(\Omega)$. Then there exist $\alpha \in (0,1)$ such that $u \in C^{1,\alpha}(\overline{\Omega''})$. Moreover, an upper bound of $\|u\|_{C^{1,\alpha}(\overline{\Omega''})}$ can be found depending only on p, Ω, Ω' Ω'', $\|u\|_{L^\infty(\Omega')}$ and $\|v\|_{L^\infty(\Omega')}$ (cf. [14, Theorem 1]).

The Tolksdorf's estimates imply the following result.

Remark 2.5. Assume that the sequences $\{F_j\}_{j\in\mathbb{N}}$ and $\{u_j\}_{j\in\mathbb{N}}$ are in $L^\infty_{\rm loc}(\Omega)$ and $W^{1,p}_{\rm loc}(\Omega)\cap C(\overline{\Omega})$ respectively with $1 < p < \infty$ and $u_j \geq 0$ such that $-\Delta_p u_j = F_j$ on Ω for all $j \in \mathbb{N}$. Assume also that for each open set $\Omega'' \subset\subset \Omega$ there exist positive constants $c_{\Omega''}$, $\widetilde{c}_{\Omega''}$ such that $\|F_j\|_{L^\infty(\Omega'')} \leq c_{\Omega''}$ and $\|u_j\|_{L^\infty(\Omega'')} \leq \widetilde{c}_{\Omega''}$ for all

$j \in \mathbb{N}$ and that $\lim_{j\to\infty} F_j = F$ a.e. in Ω for some $F : \Omega \to \mathbb{R}$. Then there exists a subsequence $\{u_{j_k}\}_{k\in\mathbb{N}}$ and a nonnegative function $v \in C^1(\Omega)$ satisfying $-\Delta_p v = F$ on Ω and such that $\{u_{j_k}\}_{k\in\mathbb{N}}$ converges, in the C^1 norm, to v on each compact subset of Ω.

Indeed, if $\Omega' \subset\subset \Omega$, let Ω'' be a domain such that $\Omega' \subset\subset \Omega'' \subset \Omega$. We have $\|F_j\|_{L^\infty(\Omega')} \le c_{\Omega''}$, $\|u_j\|_{L^\infty(\Omega'')} \le \widetilde{c}_{\Omega''}$. Taking into account the Tolksdorf's estimates in b), a Cantor diagonal process gives a subsequence $\{u_{j_k}\}_{k\in\mathbb{N}}$ that converges to some function $u \in C^1(\Omega)$ on each compact subset of Ω in the C^1 norm. So, we have, for all $\varphi \in C_c^\infty(\Omega)$

$$\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla\varphi\rangle = \lim_{j\to\infty}\int_\Omega |\nabla u_j|^{p-2}\langle \nabla u_j, \nabla\varphi\rangle = \lim_{j\to 0}\int_\Omega F_j\varphi = \int_\Omega F\varphi$$

and then u satisfies $-\Delta_p u = F$ on Ω.

Lemma 2.6. *For a nonnegative function $F \in L^\infty(\Omega)$ the problem*

$$\begin{gathered} -\Delta_p u = Kg(u) + F \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega,\\ u > 0 \quad \text{in } \Omega \end{gathered} \tag{2.3}$$

has a unique positive solution in $W^{1,p}_{\rm loc}(\Omega)\cap C(\overline{\Omega})$ and this solution belongs to $C^1(\Omega)\cap C(\overline{\Omega})$. Moreover, $u \ge c\delta$ where c is the positive constant given by Lemma 2.3 and $u = \lim_{j\to\infty} u_j$ (in the pointwise sense) with u_j as there.

Proof. Let u_j be as in Lemma 2.3 and let $u = \lim_{j\to\infty} u_j$. Since $\frac{1}{j} + u_j \ge c\delta$ (with c as there, and so independent of j) we have, for each subdomain $\Omega' \subset\subset \Omega$,

$$\|Kg(\frac{1}{j} + u_j) + F\|_{L^\infty(\Omega')} \le \|K\|_{L^\infty(\Omega)} g(c\delta) + \|F\|_{L^\infty(\Omega)}.$$

Also,

$$\|u_j\|_{L^\infty(\Omega')} \le \|\frac{1}{j} + u_j\|_{L^\infty(\Omega')} \le 1 + \|u_1\|_{L^\infty(\Omega)} < \infty.$$

After passing to a subsequence, from Remark 2.5 we can assume that $\{u_j\}_{j\in\mathbb{N}}$ converges, in the C^1 norm, on each compact subset of Ω, to a solution $u \in C^1(\Omega)$ of the problem $-\Delta_p u = Kg(u) + F$ in Ω.

Since (as shown in Lemma 2.3) $\frac{1}{j} + u_j$ is decreasing in j, we have $0 \le u \le \frac{1}{j} + u_j$ for all j. Also, $u_j \in C(\overline{\Omega})$, $u_j = 0$ on $\partial\Omega$ and so $u = 0$ on $\partial\Omega$ and u is continuous up to the boundary. Moreover, $\frac{1}{j} + u_j \ge c\delta$ gives, going to the limit, that $u \ge c\delta$.

If $z \in W^{1,p}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$ is another solution of (2.3), consider the open set $U := \{x \in \Omega : z(x) > u(x)\}$. From (2.3) we have $-\Delta_p z \le -\Delta_p(u)$ in U and $z = u$ in ∂U, the comparison principle leads to $U = \emptyset$. Then $z \le u$ in Ω. Similarly we see that $u \le z$. □

Remark 2.7. It is known [9, section 4] that if $m \in L^\infty(\Omega)$ and $|\{x \in \Omega : m(x) > 0\}| > 0$ then there exists a unique $\lambda = \lambda_1(-\Delta_p, m, \Omega) \in (0,\infty)$ such that the problem $-\Delta_p\Phi = \lambda m|\Phi|^{p-2}\Phi$ in Ω, $\Phi = 0$ in $\partial\Omega$, $\Phi > 0$ in Ω has a solution $\Phi \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$. This solution is unique up to a multiplicative constant, belongs to $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, satisfies that $\nabla\Phi(x) \ne 0$ for all $x \in \partial\Omega$ and there exists positive constants c_1 and c_2 such that $c_1\delta(x) \le \Phi(x) \le c_2\delta(x)$ for all $x \in \Omega$ (so $\Phi(x) > 0$ for all $x \in \Omega$). For the case $m = 1$ we will write $\lambda_1(-\Delta_p, \Omega)$ instead of $\lambda_1(-\Delta_p, m, \Omega)$.

We recall also that if $0 \le h \in L^\infty(\Omega)$, $\lambda > 0$ and if there exists a nonnegative and non identically zero solution $w \in W_0^{1,p}(\Omega)$ of the problem $-\Delta_p w = \lambda m w^{p-1} + h$ in Ω then $\lambda \ge \lambda_1(-\Delta_p, m)$ [9, Proposition 4.1]. This implies the following result.

Remark 2.8. Let $m \in L^\infty(\Omega)$ and let, as usual, $m^+ = \max(m, 0)$. Assume $m^+ \ne 0$ and let $\lambda \ge 0$ such that there exists a nonnegative and non identically zero function $w \in W^{1,p}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$ such that $-\Delta_p w \ge \lambda m w^{p-1}$ in Ω. Then $\lambda \le \lambda_1(-\Delta_p, m, \Omega)$. Indeed, Let $v \in W_0^{1,p}(\Omega)$ be the (positive) solution of the problem $-\Delta_p v = \lambda m w^{p-1}$ in Ω. Then $v \in C^1(\overline{\Omega})$ and $-\Delta_p w \ge -\Delta_p v$ in Ω, $w = v$ on $\partial\Omega$, Thus the comparison principle gives $w \ge v$ in Ω. Since $-\Delta_p v = \lambda m v^{p-1} + h$ with $h = \lambda m(w^{p-1} - v^{p-1})$, we have $0 \le h \in L^\infty(\Omega)$ and so Remark 2.7 applies to give that $\lambda \le \lambda_1(-\Delta_p, m, \Omega)$.

Remark 2.9. This remark concerns to the behavior near the boundary of the solution of problem (2.3). We will say that two functions $v_1, v_2 : \Omega \to (0, \infty)$ are comparable if there exist positive constants c_1, c_2 such that $c_1 v_1 \le v_2 \le c_2 v_1$. Consider in Lemma 2.6 the case $F = 0$ and assume that K is comparable with Φ^γ for some $\gamma \ge 0$ and that $0 < \liminf_{s\to 0^+} s^\alpha g(s) \le \limsup_{s\to 0^+} s^\alpha g(s) < \infty$ for some $\alpha > \gamma + 1$. Then the solution u given there is comparable with $\Phi^{\frac{\gamma+p}{\alpha+p-1}}$ (and so with $\delta^{\frac{\gamma+p}{\alpha+p-1}}$) where Φ is a positive principal eigenfunction for $-\Delta_p$ in Ω with homogeneous Dirichlet boundary condition associated to the weight $m \equiv 1$. Indeed, let $\beta = (\gamma + p)/(\alpha + p - 1)$ and let $v = \Phi^\beta$. Since $0 < \beta < 1$ it follows that $v \in C^1(\Omega) \cap C(\overline{\Omega})$. A computation shows that $-\Delta_p v = \widetilde{K} v^{-\alpha}$ on Ω, where

$$\widetilde{K} = \beta^{p-1}((1-\beta)(p-1)|\nabla\Phi|^p + \lambda_1 \Phi^p).$$

Taking into account that $0 < \beta < 1$, the properties of Φ stated in Remark 2.7 imply that $\widetilde{K}$ is comparable with 1 and so, from our assumptions on g, we can choose positive constants c and c' such that $-\Delta_p(cv) = c^{p-1}\widetilde{K} v^{-\alpha} \le g(v)$ and $-\Delta_p(c'v) = (c')^{p-1}\widetilde{K} v^{-\alpha} \ge g(v)$. Let $U = \{x \in \Omega : u(x) < cv(x)\}$. Thus U is open. Since g is non increasing we have $-\Delta_p u \ge -\Delta_p(cv)$ on U on Ω. Also $u = cv$ on ∂U and so the comparison principle implies $U = \emptyset$. Then $u \ge cv = c\Phi^\beta$ in Ω. Similarly, we obtain also that $u \le c'\Phi^\beta$ in Ω.

Let P be the positive cone in $C(\overline{\Omega})$. For $j \in \mathbb{N}$, let $S_j : P \cup \{0\} \to P$ be the solution operator for problem (2.1) gives by $S_j(f) = u$ and let $S : P \cup \{0\} \to P$ be the analogous solution map of (2.3).

Lemma 2.10. (i) *$S : P \cup \{0\} \to P$ is a continuous, non decreasing and compact map and the same is true for each S_j.*
(ii) *$0 < j \le k$ implies $S_k(u) \le S_j(u)$ for $u \in P \cup \{0\}$.*
(iii) *$S(u) \le S_j(u)$ for $u \in P \cup \{0\}$, $j \in \mathbb{N}$.*

Proof. To see that S is non decreasing, suppose $F_1, F_2 \in P$ with $F_1 \ge F_2 \ge 0$. Let $v_1 = S(F_1)$, $v_2 = S(F_2)$. If $v_1 < v_2$ somewhere in Ω, let $U := \{x \in \Omega : v_2(x) > v_1(x)\}$. Thus U is a non empty open set and, from our assumptions on g and K,

$$\begin{gathered} -\Delta_p v_1 = Kg(v_1) + F_1 \ge Kg(v_2) + F_2 = -\Delta_p v_2 \quad \text{in } U, \\ v_1 = v_2 \quad \text{on } \partial U. \end{gathered}$$

Then the comparison principle gives $v_1 \ge v_2$ on U which is a contradiction. Then S is non decreasing.

To see that S is continuous, consider $F \in P \cup \{0\}$ and a sequence $\{F_j\}_{j\in\mathbb{N}}$ in $P \cup \{0\}$ that converges to F in $C(\overline{\Omega})$. Let M be an upper bound of $\{F_j\}_{j\in\mathbb{N}}$. Then

$$0 < S(0) \leq S(F_j) \leq S(M). \tag{2.4}$$

Let $u_j = S(F_j)$, thus $-\Delta_p u_j = Kg(u_j) + F_j$ in Ω, $u_j = 0$ on $\partial\Omega$. Taking into account that by Lemma 2.6) $S(0) \geq c\delta$ and that S is non decreasing, we have

$$0 \leq g(u_j) = g(S(F_j)) \leq g(S(0)) \leq g(c\delta) \in L^\infty_{\rm loc}(\Omega).$$

Also $0 \leq u_j \leq S(M) \in C(\overline{\Omega})$. Then Remark 2.5 gives a subsequence $\{F_{j_k}\}_{k\in\mathbb{N}}$ such that $S(F_{j_k})$ converges, in the C^1 norm, on each compact subset of Ω to a positive solution $z \in C^1(\Omega)$ of the problem

$$-\Delta_p z = Kg(z) + F \quad \text{in } \Omega.$$

Since $u_{j_k} = S(F_{j_k}) \geq S(0) \geq S(c\delta)$, we have $z \geq c\delta$. Also, $z \leq S(M) \in C(\overline{\Omega})$. Since $S(M) = 0$ on $\partial\Omega$ it follows that z is continuous up to the boundary and $z = 0$ on $\partial\Omega$. Thus $z = S(F)$.

Let $\varepsilon > 0$ and let $\eta = \eta(\varepsilon) > 0$ such that $S(M) \leq \varepsilon$ on $\Omega - \Omega_\eta$ where

$$\Omega_\eta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \eta\}. \tag{2.5}$$

We have $S(F_{j_k}) \leq S(M) \leq \varepsilon$ on $\Omega - \Omega_\eta$ for all k. Also $S(F) \leq \varepsilon$ on $\Omega - \Omega_\eta$, thus $\|S(F_{j_k}) - S(F)\|_{L^\infty(\Omega-\Omega_\eta)} \leq 2\varepsilon$ for all k. On the other hand, since $\{S(F_{j_k})\}$ converges in $C^1(\overline{\Omega}_\eta)$ to $S(F)$, for k large enough we have $\|S(F_{j_k}) - S(F)\|_{L^\infty(\Omega_\eta)} \leq \varepsilon$. Then $\{S(F_{j_k})\}$ converges in $C(\overline{\Omega})$ to $S(F)$. Then S is continuous.

To prove that S is a compact map, consider a bounded sequence $\{F_j\}$ in $P \cup \{0\}$ and let $M \in (0, \infty)$ be an upper bound of $\{F_j\}$. For $\varepsilon > 0$ let $\eta = \eta(\varepsilon)$ be chosen as above. As before, Remark 2.5 gives a subsequence $\{F_{j_k}\}$ that converges, in the C^1 norm, on each compact subset of Ω. Thus, for k and s large enough,

$$\|S(F_{j_k}) - S(F_{j_s})\|_{C(\overline{\Omega}_\eta)} \leq \varepsilon$$

and

$$\begin{aligned}\|S(F_{j_k}) - S(F_{j_s})\|_{C(\Omega-\Omega_\eta)} &\leq \|S(F_{j_k})\|_{C(\Omega-\Omega_\eta)} + \|S(F_{j_s})\|_{C(\Omega-\Omega_\eta)} \\ &\leq 2\|S(F_{j_s})\|_{C(\Omega-\Omega_\eta)} \leq 2\varepsilon\end{aligned}$$

Then $\{S(F_{j_k})\}_{k\in\mathbb{N}}$ is a Cauchy's sequence in $C(\overline{\Omega})$ and the compactness of S follows. Since for each j, $g(. + \frac{1}{j})$ satisfies the assumptions made for on g, (i) holds for each S_j. Finally, (ii) is a direct consequence of the comparison principle and, since $S(u) = \lim_{j\to\in} S_j(u)$ (by Lemma 2.6), (iii) follows from (ii). $\square$

3. An existence result

Our assumptions for this section are those stated at the beginning of the Section 2. Let us introduce some additional notations. Consider, for $j \in \mathbb{N}$ and $\lambda \geq 0$ the problem

$$\begin{gathered}-\Delta_p u = Kg(u + \frac{1}{j}) + \lambda h(u) + f \quad \text{in } \Omega,\ u \in W^{1,p}_{\rm loc}(\Omega) \cap C(\overline{\Omega}), \\ u = 0 \quad \text{on } \partial\Omega, \\ u > 0 \quad \text{in } \Omega\end{gathered} \tag{3.1}$$

Let $\pi : [0,\infty) \times P \to [0,\infty)$ be defined by $\pi(\lambda,u) = \lambda$ and for j as above, let

$$\Sigma_j = \{(\lambda,u) \in [0,\infty) \times P : u \in W^{1,p}_{\rm loc}(\Omega) \cap C(\overline{\Omega}) \text{ and } u \text{ solves } (3.1)\},$$
$$\Lambda_j = \pi(\Sigma_j), \quad u_j^* = S_j(f)$$

and let Σ_∞, Λ_∞ and u^*_∞ be the sets and the function analogously defined replacing (3.1) by (1.1). Finally, let C_j (respectively C_∞) be the connected component of Σ_j containing u_j^* (respectively of Σ_∞ containing u^*_∞).

With this notation, we have the following theorem.

Theorem 3.1. (i) $(\lambda,u) \in \Sigma_\infty$ *implies that* $u \in C^1(\Omega)$.

(ii) *For* $f \in P \cup \{0\}$ *it holds that* C_∞ *is unbounded in* $[0,\infty) \times P$.

(iii) Λ_∞ *is an interval containing* 0.

(iv) *For* $j \in N$*, (i), (ii) and (iii) hold with* Σ_∞*,* Λ_∞ *and* u^*_∞ *replaced by* Σ_j*,* Λ_j *and* u_j^* *respectively and with* (3.1) *replaced by* (1.1).

(v) *There exists* $\widetilde{\lambda} > 0$ *such that* $[0,\widetilde{\lambda}] \subset \Lambda_\infty$ *and* $[0,\widetilde{\lambda}] \subset \Lambda_j$ *for each* j.

Proof. (i) is given by Lemma 2.3. To see (ii) and (iii), observe that (1.1) is equivalent to $S(\lambda h(u) + f) = u$. Let $T : [0,\infty) \times (P \cup \{0\}) \to C(\overline{\Omega})$ be defined by $T(\lambda,v) = S(\lambda h(v + u^*_\infty) + f) - u^*_\infty$ (since S is non decreasing we have $T(\lambda,v) \geq 0$ for $v \geq 0$). Lemma 2.10 implies that T is a continuous, non decreasing and compact map. Moreover, $T(0,0) = 0$ and, since $T(0,v) = 0$ for all $v \in P \cup \{0\}$, $v = 0$ is the unique fixed point of $T(0,.)$. For each $\sigma \geq 1$ and $\rho > 0$, we have also that $T(0,u) \neq \sigma u$ for $u \in P \cap \rho\partial B$, where B denotes the open unit ball centered at 0 in $C(\overline{\Omega})$. Since u solves (1.1) if and only if $u = v + u^*_\infty$ with v a fixed point for T, in [1, Theorem 17.1], applied to T, gives that C_∞ is unbounded in $[0,\infty) \times P$ and that Λ_∞ is an interval. Thus (i), (ii) and (iii) hold for S and, replacing in the above argument g by $g(. + \frac{1}{j})$, we see that the same is true for each S_j.

To prove (v) one observes that, by Lemma 2.3 the problem

$$-\Delta_p u = Kg(1+u) + f \quad \text{in } \Omega,\ u \in W^{1,p}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$$
$$u = 0 \quad \text{on } \partial\Omega$$

has a unique solution $u = u_1$ that belongs to $C^1(\overline{\Omega})$. Thus $0 \in \Lambda_1$. Since, by ii), C_1 is unbounded, it follows that there exists $\widetilde{\lambda} > 0$ such that $\widetilde{\lambda} \in \Lambda_1 - \{0\}$. Thus, by (iii), for $0 < \lambda < \widetilde{\lambda}$ there exists a positive solution $u_{\lambda,1}$ of

$$-\Delta_p u_{\lambda,1} = Kg(1+u_{\lambda,1}) + \lambda h(u_{\lambda,1}) + f \quad \text{in } \Omega,\ u_{\lambda,1} \in W^{1,p}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$$
$$u_{\lambda,1} = 0 \quad \text{on } \partial\Omega$$

Now, by Lemma 2.10, $S(\lambda h(u_{\lambda,1}) + f) \leq S_1(\lambda h(1+u_{\lambda,1}) + f)$ and since the operator $u \to U(u) := S(\lambda h(u) + f)$ is a positive, non decreasing, continuous and compact map. [1, Theorem 17.1] implies that the sequence $\{U^j(0)\}_{j\in\mathbb{N}}$ converges to a fixed point of U. Then $\lambda \in \Sigma_\infty$. Similarly, by considering S_j instead of S we get that $\lambda \in \Lambda_j$ for all j. □

Remark 3.2. (i) If for some $\lambda_0 > 0$ and $j \in \mathbb{N}$ we know that an a priori estimate $\|u\|_{L^\infty(\Omega)} \leq c$ holds for each positive solution of (3.1) associated to each $\lambda \geq \lambda_0$ then an upper bound for Λ_j can be given. Indeed, in this case we have

$$-\Delta_p u = g(\frac{1}{j} + u) + \lambda h(u) + f \geq \lambda c^{1-p} h(u) u^{p-1} \quad \text{in } \Omega.$$

Also, $u = S_j(\lambda h(u)+f) \geq S(0) \geq c\delta$ for some positive constant c (cf. Lemmas 2.10 and 2.6), then $h(u) \geq h(c\delta)$ and so, by Remark 2.8, λ does not exceed the principal eigenvalue for $-\Delta_p$ associated to the weight function $c^{1-p}h(c\delta)$.

(ii) On the other hand if $\inf_{s>0} \frac{h(s)}{s^{p-1}} > 0$ a similar result holds. Indeed, in this case from (3.1) we have $-\Delta_p u \geq \lambda \inf_{s>0} \frac{h(s)}{s^{p-1}} u^{p-1}$ in Ω, $u = 0$ on $\partial\Omega$ and so, again by Remark 2.8, $\lambda \inf_{s>0} \frac{h(s)}{s^{p-1}} \leq \lambda_1(-\Delta_p, \Omega)$.

The following proposition gives some additional information about the regularity of the solutions of (1.1).

Proposition 3.3. *Assume that* $\sup_{s>0} s^\alpha g(s) < \infty$ *for some* $\alpha \in [0, \frac{2p-1}{p-1})$. *Then* $u \in W_0^{1,p}(\Omega)$ *for all positive weak solution* $u \in W_{\rm loc}^{1,p}(\Omega) \cap C(\overline{\Omega})$ *of* (1.1) *with* $\lambda > 0$.

Proof. We have

$$\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla\varphi\rangle = \int_\Omega (Kg(u) + \lambda h(u) + f)\varphi \tag{3.2}$$

for all $\varphi \in C_c^\infty(\Omega)$ and so, since $u \in W_{\rm loc}^{1,p}(\Omega)$ this equality holds also for all $\varphi \in W_0^{1,p}(\Omega)$ such that $\operatorname{supp}\varphi \subset \Omega$. For $\varepsilon > 0$, let $u_\varepsilon := \max(u, \varepsilon) - \varepsilon$. Since $u \in C(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$, we have $\operatorname{supp} u_\varepsilon \subset \Omega$. So we can take $\varphi = u_\varepsilon$ as test function in (1.1) to obtain

$$\begin{aligned} \int_\Omega \chi_{\{u>\varepsilon\}} |\nabla u|^p &= \int_\Omega (\lambda h(u) + Kg(u) + f) u_\varepsilon \\ &\leq \int_{u>\varepsilon} (\lambda h(u) + Kg(u) + f) u \\ &\leq \int_\Omega (\lambda h(u) + Kg(u) + f) u \end{aligned} \tag{3.3}$$

We claim that the last integral is finite. Indeed, it is enough to show that $ug(u) \in L^1(\Omega)$ and clearly this holds if $\alpha \leq 1$. Suppose now $\alpha > 1$. We have

$$\begin{aligned} -\Delta_p u &= \lambda h(u) + Kg(u) + f \\ &\leq \Big(\big(\lambda h(\|u\|_{L^\infty(\Omega)}) + \|f\|_{L^\infty(\Omega)}\big) \|u\|^\alpha_{L^\infty(\Omega)} + c_1 \|K\|_{L^\infty(\Omega)} \Big) u^{-\alpha} \\ &= cu^{-\alpha} \end{aligned} \tag{3.4}$$

where $c = c_{\lambda,u} = (\lambda h(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}))\|u\|_{L^\infty(\Omega)} + c_1\|K\|_{L^\infty(\Omega)}$.

Let $w \in W_{\rm loc}^{1,p}(\Omega) \cap C(\overline{\Omega})$ be the solution (provided by Lemma 2.6) of the problem $-\Delta_p w = cw^{-\alpha}$ in Ω, $w = 0$ on $\partial\Omega$. Then, from (3.4), $-\Delta_p u - cu^{-\alpha} \leq -\Delta_p w - cw^{-\alpha}$ in Ω, also $u = w = 0$ on $\partial\Omega$ and so Remark 2.1 gives $u \leq w$ in Ω. On the other hand, Remark 2.8 gives $w \leq c'\Phi^{\frac{p}{\alpha+p-1}}$ for some constant c' where Φ is a positive principal eigenfuntion for $-\Delta_p$ on Ω. Then

$$0 \leq ug(u) \leq c''\Phi^{\frac{p}{\alpha+p-1}}\Phi^{-\frac{\alpha p}{\alpha+p-1}} = c''\Phi^{-\frac{p(\alpha-1)}{\alpha+p-1}}.$$

Since $0 \leq \alpha < \frac{2p-1}{p-1}$ implies $\frac{p(\alpha-1)}{\alpha+p-1} < 1$ our claim holds. By Lemma 2.6, $u(x) > 0$ for all $x \in \Omega$ and so, from (3.4) and from the monotone convergence Theorem, we get $|\nabla u|^p \in L^1(\Omega)$. $\square$

4. A multiplicity result

In this section we assume that in addition to the conditions (H1) (H2) and (H3) stated at the introduction, the conditions (H5), (H6) and (H7) also hold.

In [3, Proposition 4.1] it is proved that if Ω is a strictly convex and bounded domain with C^2 boundary and if $G : \mathbb{R} \to \mathbb{R}$ is a locally Lipchitz function, then there exists $\rho > 0$, with ρ depending only on Ω and N, such that if $1 < p \leq 2$ and $u \in C^1(\overline{\Omega})$ is a positive weak solution of the problem $-\Delta_p u = G(u)$ in Ω, $u = 0$ on $\partial\Omega$ then the global maximum of u in $\overline{\Omega}$ is achieved at least at some point $y \in \Omega$ satisfying $\operatorname{dist}(y, \partial\Omega) \geq \rho$. From this fact and using the Gidas Spruck blow up technique [7], in [3, Lemmas 3.1 and 3.2] is obtained an a priori estimate for the solutions of the above problem. Following a similar approach, Lemma 4.1 below adapts to our actual setting, with a similar purpose, the arguments in [3].

Lemma 4.1. *Let Ω be a strictly convex, C^2 and bounded domain in $\mathbb{R}^N, N \geq 2$. Assume that $1 < p \leq 2$ and that, in addition to the hypothesis stated at the introduction, g and h are locally Lipchitz on their domains and that $\inf_{s>0} s^{-p+1}h(s) > 0$ and $0 < \lim_{s\to\infty} s^{-q}h(s) < \infty$ for some $q \in (p-1, \frac{N(p-1)}{N-p}]$. Then for each $\lambda_0 > 0$ there exists a positive constant c_{λ_0} such that for all j and for all positive solution u of the problem*

$$
\begin{gathered}
-\Delta_p u = g(u + \frac{1}{j}) + \lambda h(u) \quad \text{in } \Omega,\ u \in W^{1,p}_{\rm loc}(\Omega) \cap C(\overline{\Omega}), \\
u = 0 \quad \text{on } \partial\Omega, \\
u > 0 \quad \text{in } \Omega,
\end{gathered} \tag{4.1}
$$

with $\lambda \geq \lambda_0$ it holds that $\|u\|_{L^\infty(\Omega)} < c_{\lambda_0}$

Proof. Let $c = \inf_{s>0}(h(s)/s^{p-1})$ and let u be a positive solution of (4.1) corresponding to some $\lambda > 0$. We have $-\Delta_p u = c\lambda u^{p-1} + H$ in Ω with $H := g(u + \varepsilon) + \lambda(h(u) - cu^{p-1})$. Since $H \in L^\infty(\Omega)$ and $H > 0$ in Ω, Remark 2.7 gives that $\lambda \leq c^{-1}\lambda_1(-\Delta_p, \Omega)$.

To prove the Lemma we proceed by contradiction. Assume that there exists a sequence $\{u_n, \lambda_n, \varepsilon_n\}_{n\in\mathbb{N}}$ such that $j_n \in \mathbb{N}$, $\lambda_n \geq \lambda_n$, with u_n satisfying (4.1) for $\lambda = \lambda_n$ and such that $\|u_n\|_{L^\infty(\Omega)} \geq n$ and let $G_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$
G_n(s) = \begin{cases} g(s + \frac{1}{j_n}) + \lambda_n h(s) & \text{for } s > 0, \\ g(\frac{1}{j_n}) + \lambda_n h(0) & \text{for } s \leq 0. \end{cases}
$$

Thus each G_n is locally Lipchitz and so, by [3, Proposition 4.1], there exists $x_n \in \Omega$ such that $\|u_n\|_{L^\infty(\Omega)} = u_n(x_n)$ and $\operatorname{dist}(x_n, \partial\Omega) \geq \rho$ with ρ as described at the beginning of the section.

Let $\alpha_n = \|u_n\|_{L^\infty(\Omega)}$ and let $\Omega_n = \alpha_n^k(\Omega - x_n)$ where $\Omega_n := \{x - x_n : x \in \Omega\}$ and $k = \frac{q-p+1}{p}$. Observe that, since $q > p - 1$, we have $k > 0$.

Let $w_n : \Omega_n \to \mathbb{R}$ be defined by

$$
w_n(y) = \frac{1}{\alpha_n} u_n\big(\frac{1}{\alpha_n^k} y + x_n\big).
$$

Lemma 2.3 applied to $F := \lambda_n h(u_n) \in C(\overline{\Omega})$ gives $u_n \in C^1(\overline{\Omega})$ and so $w_n C^1(\overline{\Omega}_n)$. Let $v \in C^1(\overline{\Omega})$ be the solution of the problem

$$\begin{gathered} -\Delta_p v = \lambda_0 h(v) \quad \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

We have, for some positive constant c_1 that $v \geq c_1\delta$ in Ω. Since $-\Delta_p u_n \geq \lambda_0 h(u_n) = -\Delta_p v$ in Ω and $u_n = v$ on $\partial\Omega$ we get that $u_n \geq c_1\delta$ and so $w_n(y) \neq 0$ for $y \in \Omega_n$. A computation gives that

$$\begin{gathered} -\Delta_p w_n(y) = \frac{1}{\alpha_n^q} g(\alpha_n w_n + \frac{1}{j_n}) + \lambda_n w_n^q \frac{h(\alpha_n w_n)}{(\alpha_n w_n)^q} \quad \text{in } \Omega_n \\ w_n = 0 \quad \text{on } \partial\Omega_n. \end{gathered} \tag{4.2}$$

For $r > 0$, let $\overline{B}_r(0)$ be the closed ball in $\mathbb{R}^n$ centered at 0 with radius r. Since (by our contradiction hypothesis) $\lim_{n\to\infty} \alpha_n^k = \infty$, by our choice of x_n there exists $n_0 = n_0(r)$ such that $\overline{B}_r(0) \subset \Omega_n$ for all $n \geq n_0$.

For c_1 as above and for n large enough we have

$$u_n(\frac{1}{\alpha_n^k} y + x_n) \geq c_1\delta(\frac{1}{\alpha_n^k} y + x_n) \geq c_1\delta(\frac{\rho}{2})$$

for all $y \in \Omega_n$. Then (recalling that, by Remark 3.2, $\lambda_n \leq c_2^{-1}\lambda_1(-\Delta_p, \Omega)$ with $c_2 = \inf_{s>0}(h(s)/s^{p-1})$) we get that, for $y \in \overline{B}_r(0)$,

$$\begin{aligned} 0 &\leq \alpha_n^{-q} 1 g(\alpha_n w_n(y) + \frac{1}{j_n}) + \lambda_n h(\alpha_n w_n(y)) \\ &\leq \alpha_n^{-q} g(c_1 \frac{\rho}{2}) + \frac{1}{c_2}\lambda_1(-\Delta_p, \Omega)\alpha_n^q u_n^q(\alpha_n^{-k} 1 y + x_n) \frac{h(u_n(\alpha_n^{-k} y + x_n))}{u_n^q(\alpha_n^{-k} y + x_n)} \\ &\leq \alpha_n^{-q} g(c_1 \frac{\rho}{2}) + \frac{1}{c_2}\lambda_1(-\Delta_p, \Omega) \sup_{s > c_1 \frac{\rho}{2}} \frac{h(s)}{s^q} \end{aligned}$$

Thus, from (4.2) and Remark 2.4, there exist positive constants c_2 and $\alpha \in (0,1)$ such that $\|w_n\|_{C^{1,\alpha}(\overline{B}_{r/2}(0))} \leq c_2$ for all n large enough. Then we can find a subsequence $\{w_{n_j}\}_{j\in\mathbb{N}}$ that converges in $C^1(\overline{B}_{r/2}(0))$ to some nonnegative $w \in C^1(\overline{B}_r(0))$ with $\|w\|_{L^\infty(\overline{B}_r(0))} = 1$. After passing to a furthermore subsequence we can assume that λ_{n_j} converges to some $\lambda^* \in [\lambda_0, \lambda_1(-\Delta_p, \Omega)]$. We take test functions in $C_c^\infty(\overline{B}_{r/2}(0))$ in (4.2) and taking the limit as n tends to ∞ we get $-\Delta_p w \geq \lambda^* B w^q$ on $\overline{B}_{r/2}(0)$, where $B = \lim_{s\to\infty}(s^{-q}h(s))$. Since $w \not\equiv 0$, Remark 2.4 gives $w(x) > 0$ for all $x \in \overline{B}_{r/2}(0)$ and so, again taking test functions in $C_c^\infty(\overline{B}_{r/2}(0))$ in (4.2) and going to the limit as n tends to ∞, we obtain now

$$-\Delta_p w = B\lambda^* w^q \text{ on } \overline{B}_{r/2}(0). \tag{4.3}$$

Taking a sequence of balls $\overline{B}_{r_i}(0)$ with radius increasing to ∞ and repeating the above argument on the subsequence w_{n_j} obtained in the previous step, we can obtain a Cantor diagonal subsequence, still denoted by w_{n_j}, which converges in the C^1 norm on each compact set in $\mathbb{R}^N$ to a function $\widetilde{w} \in C^1(\mathbb{R}^N)$ satisfying $-\Delta_p \widetilde{w} = B\lambda^* \widetilde{w}^q$ on $\mathbb{R}^N$. Since, under our assumptions on p and q, this problem has no solution [13] we obtain a contradiction. $\square$

Lemma 4.2. *For $\sigma > \|S(0)\|_{L^\infty(\Omega)}$ there exists λ_σ and $j_\sigma \in \mathbb{N}$ such that for $j > j_\sigma$ and $0 \leq \lambda < \lambda_\sigma$, problem (4.1) has no positive solution u satisfying $\|u\|_{L^\infty(\Omega)} = \sigma$.*

Proof. We proceed by contradiction. Suppose that there exists a sequence $\{j_n, u_n, \lambda_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} j_n = \infty$, $\lambda_n > 0$, $\lim_{n\to\infty} \lambda_n = 0$, and where u_n is a positive solution of (4.1) for $\lambda = \lambda_n$ and $j = j_n$ satisfying $\|u_n\|_{L^\infty(\Omega)} = \sigma$. Let $M > 0$ be an upper bound of $\{\lambda_n\}$. By Lemma 2.10 we have

$$0 < S(0) \le S(\lambda_n h(u_n)) = u_n \le S_1(\lambda_n h(u_n)) \le S_1(Mh(\sigma)).$$

Then $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $C(\overline{\Omega})$. Also, for $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ we have

$$\|g(u_n + \frac{1}{j_n}) + \lambda_n h(u_n)\|_{L^\infty(\Omega'')} \le \|g(S(0))\|_{L^\infty(\Omega'')} + Mh(\sigma)$$

thus Remark 2.5 gives a subsequence $\{u_{j_k}\}$ that converges, in the C^1 norm, to some function $u \in C^1(\Omega)$ on each compact subset of Ω.

Since $0 < S(0) < u_n < S_1(Mh(\sigma))$ and also $S_1(Mh(\sigma)) \in C(\overline{\Omega})$, $S_1(Mh(\sigma)) = 0$ on $\partial\Omega$, we get that $u \in C(\overline{\Omega}) \cap C^1(\Omega)$ and $u = 0$ on $\partial\Omega$ Going to the limit in the weak form of

$$-\Delta_p u_{n_k} = g(u_{n_k} + \frac{1}{j_{n_k}}) + \lambda h(u_{n_k})$$

we find that $-\Delta_p u = g(u) + \lambda h(u)$ in Ω. So $u = S(0)$.

Observe that $\{u_{n_k}\}$ converges to u in $C(\overline{\Omega})$. Indeed, given $\varepsilon > 0$, let $\eta = \eta(\varepsilon) > 0$ such that $S_1(Mh(\sigma)) < \varepsilon$ on $\Omega - \Omega_\eta$ (with Ω_η defined by (2.5)). Proceeding as in the proof of the continuity of S in Lemma 2.10, we get that $\|u_{n_k} - u\|_{L^\infty(\Omega_\eta)} < \varepsilon$ for k large enough and that $\|u_{n_k} - u\|_{L^\infty(\Omega-\Omega_\eta)} < 2\varepsilon$ for all k. Then $\{u_{n_k}\}_{k\in\mathbb{N}}$ converges to u in $C(\overline{\Omega})$.

Since $\|S(0)\|_{L^\infty(\Omega)} < \sigma = \|u_n\|_{L^\infty(\Omega)}$ for all n, we get a contradiction. $\square$

Lemma 4.3. *Let $\widetilde{\lambda}$ be as in Theorem 3.1 and let $\widetilde{u}$ be a positive solution of* (3.1) *corresponding to $j = 1$ and $\lambda = \widetilde{\lambda}$ (taking there $K = 1$ and $f = 0$). Then for $\sigma > \|\widetilde{u}\|_{L^\infty(\Omega)}$, $0 \le \lambda \le \widetilde{\lambda}$ and $j \in \mathbb{N}$ there exists a positive solution u of (4.1) satisfying $u \in C^1(\Omega) \cap C(\overline{\Omega})$, $u \ge S(0)$ and $\|u\|_{L^\infty(\Omega)} \le \sigma$.*

Proof. For $0 < \lambda \le \widetilde{\lambda}$, $j \in \mathbb{N}$, Lemma 2.10 gives

$$0 < S(0) < S_j(\lambda h(\widetilde{u})) \le S_1(\widetilde{\lambda} h(\widetilde{u})) = \widetilde{u} \le \sigma. \tag{4.4}$$

Let $T : P \cup \{0\} \to P$ be defined by $T(v) = S_j(\lambda h(v))$. Then T is a non decreasing continuous and compact map and (4.4) says that $T(\widetilde{u}) \le \widetilde{u}$ and [1, Theorem 17.1] applies to see that $\{T^k(0)\}_{k\in\mathbb{N}}$ is a non decreasing sequence that converges in $C(\overline{\Omega})$ to a fixed point $u \in P$ for T, which solves (4.1) and since $T^k(0) \le T^k(\widetilde{u}) \le \widetilde{u} \le \sigma$ we get $\|u\|_{L^\infty(\Omega)} \le \sigma$. Also $u \ge T^k(0) = S_j(\lambda T^{k-1}(0)) \ge S_j(0) \ge S(0)$ (the last inequality by Lemma 2.10 applied with $F := \lambda h(u)$) and since $\lambda h(u) \in C(\overline{\Omega})$, from (4.1) Lemma 2.3 gives $u \in C^1(\overline{\Omega})$ $\square$

Remark 4.4. The following analogous of the Lemmas 4.2 and 4.3 hold:

(i) For $\sigma > \|\widetilde{u}\|_{L^\infty(\Omega)}$ there exists $\lambda_\sigma > 0$ such that for $0 \le \lambda \le \widetilde{\lambda}$ (1.2) has a positive solution u satisfying $u \in C^1(\Omega) \cap C(\overline{\Omega})$ and $\|u\|_{L^\infty(\Omega)} = \sigma$.

(ii) For $\sigma > \|\widetilde{u}\|_{L^\infty(\Omega)}$ and for $0 \le \lambda \le \widetilde{\lambda}$ there exists a positive solution u of (1.2) satisfying $u \in C^1(\Omega) \cap C(\overline{\Omega})$, $u \ge S(0)$ and $\|u\|_{L^\infty(\Omega)} \le \sigma$.

Indeed, the proofs are the same, replacing there S_j by S and $g(\frac{1}{j} + .)$ by g whenever they appear and using Lemma 2.6 instead of Lemma 2.3.

Lemma 4.5. *For $\sigma \geq \|\widetilde{u}\|_{L^\infty(\Omega)} + \|S(0)\|_{L^\infty(\Omega)}$ we have*

(i) *There exist $\eta > 0$ and $j_\sigma \in N$ such that for $0 < \lambda < \eta$ and $j \geq j_\sigma$, problem (4.1) has a positive solution u_j satisfying $\|u_j\|_{L^\infty(\Omega)} \geq \sigma$.*

(ii) *There exist $\eta > 0$ such that for $0 < \lambda < \eta$ problem (1.2) has a positive solution u satisfying $\|u\|_{L^\infty(\Omega)} \geq \sigma$.*

Proof. Let Σ_j, C_j, u_j^*, and $\widetilde{\lambda}$ be as in Theorem 3.1 and let $\widetilde{u}$ be as in Lemma 4.3. Let σ, j_σ, λ_σ be as in Lemma 4.2 and let $\eta = \min(\lambda_\sigma, \widetilde{\lambda})$. For $0 < \lambda < \eta$ and $j \geq j_\sigma$ let $\lambda_0 \in (0, \lambda)$ and let c_{λ_0} be the constant provided by Lemma 4.1. Clearly we can assume that $c_{\lambda_0} \geq \sigma$. Let $O_1 = O_{11} \cup O_{12} \cup O_{13}$ with

$$
\begin{gathered}
O_{11} = \{(\overline{\lambda}, \overline{u}) \in \Sigma_j : 0 \leq \overline{\lambda} < \lambda \text{ and } \|\overline{u}\|_{L^\infty(\Omega)} < \sigma\}, \\
O_{12} = \{(\overline{\lambda}, \overline{u}) \in \Sigma_j : \lambda < \overline{\lambda} < \lambda_1(-\Delta_p, \Omega) \text{ and } \|\overline{u}\|_{L^\infty(\Omega)} < c_{\lambda_0}\}, \\
O_{13} = \{(\overline{\lambda}, \overline{u}) \in \Sigma_j : \overline{\lambda} = \lambda \text{ and } \|\overline{u}\|_{L^\infty(\Omega)} < c_{\lambda_0}\},
\end{gathered}
$$

and let

$$
O_2 = \{(\overline{\lambda}, \overline{u}) \in \Sigma_j : 0 \leq \overline{\lambda} < \lambda \text{ and } \|\overline{u}\|_{L^\infty(\Omega)} > \sigma\}\,.
$$

Suppose, by contradiction, that there not exists a positive solution u_j of problem (4.1) such that $\|u_j\|_{L^\infty(\Omega)} \geq \sigma$. Clearly, this assumption implies that O_1 and O_2 are disjoint relative open sets in Σ_j. Moreover, $\Sigma_j \subset O_1 \cup O_2$. Indeed, suppose that $(\overline{\lambda}, \overline{u}) \in \Sigma_j$ and consider the case $\overline{\lambda} < \lambda$. Then $\overline{\lambda} < \lambda_\sigma$ and so, by Lemma 4.2, $\|\overline{u}\|_{L^\infty(\Omega)} \neq \sigma$. Thus $(\overline{\lambda}, \overline{u}) \in O_{11} \cup O_2$ In the case $\overline{\lambda} = \lambda$, taking into account that $\lambda > \lambda_0$ and Lemma 4.1 we get that $(\overline{\lambda}, \overline{u}) \in O_{13}$ and in the case $\overline{\lambda} > \lambda$, again by Lemma 4.1 we get that $(\overline{\lambda}, \overline{u}) \in O_{12}$. Then $\Sigma_j \subset O_1 \cup O_2$. Let C_j be the unbounded connected component of Σ_j containing $(0, u_j^*)$. Thus $C_j \subset O_1 \cup O_2$. Since, by Theorem 3.1, C_j is unbounded and since O_1 is bounded, we get that $C_j \cap O_2 \neq \emptyset$. Since C_j is connected this implies that $C_j \cap O_1 = \emptyset$. But, since $(0, u_j^*) \in C_j$ and

$$
\|u_j^*\|_{L^\infty(\Omega)} = \|S_j(0)\|_{L^\infty(\Omega)} \leq \|S_1(0)\|_{L^\infty(\Omega)} \leq \|S_1(\widetilde{\lambda} h(\widetilde{u}))\|_{L^\infty(\Omega)} = \|\widetilde{u}\|_{L^\infty(\Omega)} < \sigma
$$

and so we get that $u_j^* \in O_1$. Then $C_j \cap O_1 \neq \emptyset$ which is a contradiction. Thus i) holds.

To prove (ii), consider for $j \geq j_\sigma$ the solution u_j given by the part (i) and observe that

$$
u_j = S_j(\lambda h(u_j)) \geq S_j(0) \geq S(0) \geq \widetilde{c}\delta
$$

where the constant $\widetilde{c}$ is independent of j (these inequalities follow from Lemma 2.10 part (i) and from Lemma 2.6 applied with $K = 1$ and $f = 0$). Also, by Lemma 4.1, $u_j \leq c_{\frac{\lambda}{2}}$ and so

$$
\begin{gathered}
-\Delta_p u_j = g(\frac{1}{j} + u_j) + \lambda h(u_j) \leq g(\widetilde{c}\delta) + \lambda_1(-\Delta_p, m, \Omega) h(c_{\frac{\lambda}{2}}) \quad \text{in } \Omega \\
u_j = 0 \quad \text{on } \partial\Omega
\end{gathered}
\tag{4.5}
$$

Since $0 \leq u_j \leq c_{\frac{\lambda}{2}}$, from Remark 2.5, after passing to some subsequence, we can assume that $\{u_j\}_{j \in \mathbb{N}}$ converges, in the C^1 norm, on each compact subset of Ω, to some function $u \in C^1(\Omega)$ satisfying $u \geq \widetilde{c}\delta$ (and so $u(x) > 0$ for $x \in \Omega$) which is a solution of the problem $-\Delta_p u = g(u) + \lambda h(u)$ in Ω. Let $w = (-\Delta_p)^{-1}(h(c_{\frac{\lambda}{2}}))$. From(4.5) we have $0 \leq u_j \leq w$. Since $w \in C(\overline{\Omega})$ and $w = 0$ on $\partial\Omega$ we obtain that u is continuous up to the boundary and that $u = 0$ on $\partial\Omega$.

Finally, let $\rho = \rho(\Omega, N)$ be as at the beginning of this section. Thus $\|u_j\|_{L^\infty(\Omega)} = \|u_j\|_{L^\infty(\overline{\Omega_\rho})}$ for all j and since $\{u_j\}_{j\in\mathbb{N}}$ converges in $C^1(\overline{\Omega_\rho})$ to u we get that $\|u\|_{L^\infty(\Omega)} \geq \|u\|_{L^\infty(\overline{\Omega_\rho})} = \lim_{j\to\infty} \|u_j\|_{L^\infty(\overline{\Omega_\rho})} = \lim_{j\to\infty} \|u_j\|_{L^\infty(\Omega)} \geq \sigma$ and the proof of the lemma is complete. $\square$

Theorem 4.6. *Assume the conditions (H1), (H2), (H3), (H5), (H6) and (H7) are satisfied. Then*

(i) *For λ positive and small enough there exist at least two positive solutions of the problem* (1.2).
(ii) $\lambda = 0$ *is a bifurcation point from infinity.*

Proof. To prove (i) observe that for λ positive and small enough, taking into account Lemma 4.5 (ii) we have a solution $u \in C(\overline{\Omega}) \cap C^1(\Omega)$ of (1.2) which satisfies $\|u\|_{L^\infty(\Omega)} \geq \sigma + 1$ and, by Remark 4.4 (ii), a solution $v \in C(\overline{\Omega}) \cap C^1(\Omega)$ such that $\|v\|_{L^\infty(\Omega)} \leq \sigma$. To prove (ii) note that, proceeding as in Remark 3.2, we have $\Lambda_\infty \subset [0, c^{-1}\lambda_1(-\Delta_p, \Omega)]$ with $c = 1/\inf_{s>0}(h(s)/s^{p-1})$. Since by Theorem 3.1 C_∞ is unbounded, (ii) follows). $\square$

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