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EXISTENCE OF POSITIVE SOLUTIONS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH RIEMANN-LIOUVILLE LEFT-HAND AND RIGHT-HAND FRACTIONAL DERIVATIVES

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Abstract. Combining properties of Riemann-Liouville fractional calculus and fixed point theorems, we obtain three existence results of one positive solution and of multiple positive solutions for initial value problems with fractional differential equations.

1. Introduction

Let s be a real number and $n = [s] + 1$ where $[s]$ the integer part of s. For a function $f : [a, b] \to \mathbb{R}$ the expressions

$$D_{a+}^s f(x) = \frac{1}{\Gamma(n-s)} (\frac{d}{dx})^n \int_a^x \frac{f(t)}{(x-t)^{s-n+1}} dt$$

$$D_{b-}^s f(x) = \frac{(-1)^n}{\Gamma(n-s)} (\frac{d}{dx})^n \int_x^b \frac{f(t)}{(t-x)^{s-n+1}} dt$$

are called, respectively, the Riemann-Liouville left-hand and right-hand fractional derivative of order s. If s is an integer, the derivative of order s is understood in the sense of usual differentiation:

$$D_{a+}^s = (\frac{d}{dx})^s, \quad D_{b-}^s = (-1)^s (\frac{d}{dx})^s \quad s = 1, 2, 3, \dots$$

Here we consider the initial-value problem

$$\begin{gathered} D_{1-}^{\alpha} D_{0+}^{\delta} u(t) = g(t, u(t)) \quad 0 < t < 1, 0 < \alpha, \delta < 1 \\ t^{1-\delta} u(t)\big|_{t=0} = a \geq 0, \quad (1-t)^{1-\alpha} D_{0+}^{\delta} u(t)|_{t=0} = b \geq 0, \end{gathered} \tag{1.1}$$

where $D_{1-}^{\alpha}, D_{0+}^{\delta}$ are the Riemann-Liouville right-hand and left-hand fractional derivatives.

For $x > 0$, the expressions

$$I_{a+}^s f(x) = \frac{1}{\Gamma(s)} \int_a^x \frac{f(t)}{(x-t)^{1-s}} dt, \quad x > a,$$

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$$I_{b-}^s f(x) = \frac{1}{\Gamma(s)} \int_x^b \frac{f(t)}{(t-x)^{1-s}} dt, \quad x < b$$

are called, respectively, the Riemann-Liouville left-hand and right-hand fractional integral of order s; see [6]

Proposition 1.1 ([6, theorem 2.4]). *If $s > 0$ then $D_{a+}^s I_{a+}^s f(x) = f(x)$ for any $f \in L_1(a,b)$, while*

$$I_{a+}^s D_{a+}^s f(x) = f(x) \tag{1.2}$$

is satisfied for $f \in I_{a+}^s(L_1(a,b))$ with

$$I_{a+}^s(L_1(a,b)) = \{g(x) : g(x) = I_{a+}^s \varphi(x), \quad \varphi \in L_1(a,b)\}$$

If $f, D_{a+}^s f \in L_1(a,b)$, then (1.2) is not true in general. However

$$I_{a+}^s D_{a+}^s f(x) = f(x) - \Sigma_{k=0}^{n-1} \frac{(x-a)^{s-k-1}}{\Gamma(s-k)} f_{n-s}^{(s-k-1)}(a)$$

where $n = [s] + 1, f_{n-s}(x) = I_{a+}^{n-s} f(x)$. In particular for $0 < Res < 1$, we have

$$I_{a+}^s D_{a+}^s f(x) = f(x) - \frac{f_{1-s}(a)}{\Gamma(s)} (x-a)^{s-1}.$$

Remark 1.2. Similar results hold for right-hand fractional derivatives.

The following theorems play major role in this article.

Theorem 1.3 ([7]). *Let X be a Banach space, and let $P \subset X$ be a cone in X. If Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $S : P \to P$ be a completely continuous operator such that either*

(1) $\|Sw\| \le \|w\|$, $w \in P \cap \partial\Omega_1$, $\|Sw\| \ge \|w\|$, $w \in P \cap \partial\Omega_2$, *or*
(2) $\|Sw\| \ge \|w\|$, $w \in P \cap \partial\Omega_1$, $\|Sw\| \le \|w\|$ $w \in P \cap \partial\Omega_2$

then S has a fixed point in $P \cap \overline{\Omega}_2 \backslash \Omega_1$.

Theorem 1.4 ([4]). *Let K be a cone and $K_c = \{y \in K | \|y\| \le c\}$, and $A : \overline{K_c} \to \overline{K_c}$ be completely continuous and α be a nonnegative continuous concave function on K such that $\alpha(y) \le \|y\|$, for all $y \in \overline{K_c}$. If there exist $0 < a < b < d \le c$ such that*

(C1) *$\{y \in K(\alpha, b, d) | \alpha(y) > b\} \ne \emptyset$ and $\alpha(Ay) > b$, for all $y \in K\{\alpha, b, d\}$,*
(C2) *$\|Ay\| < a$, for $\|y\| \le a$*
(C3) *$\alpha(Ay) > b$, for $y \in K\{\alpha, b, c\}$ with $\|Ay\| > d$,*

then A has at least three fixed points y_1, y_2, y_3 satisfying

$$\|y_1\| < a, \quad b < \alpha(y_2), \quad \|y_3\| > a \quad \text{with} \quad \alpha(y_3) < b.$$

2. Main Results

Let $X = \{u \in C(0,1) : t^{1-\delta}(1-t)^{1-\alpha} u \in C[0,1]\}$ be the Banach space endowed with the norm

$$\|u\| = \max_{0 \le t \le 1} t^{1-\delta}(1-t)^{1-\alpha} |u(t)|.$$

Let K be the cone $K = \{u \in X; u(t) \ge 0, \ 0 \le t \le 1\}$. Applying I_{1-}^α to the first equation in (1.1) it follows that $(1-t)^{\alpha-1} D_{0+}^\delta u(t)|_{t=0} = b$ that

$$D_{0+}^\delta u(t) = b(1-t)^{\alpha-1} + I_{1-}^\alpha g(t, u(t)) \tag{2.1}$$

From this equation, Proposition 1.1 and the condition $t^{1-\delta}u(t)|_{t=0} = a$, we have

$$
\begin{aligned}
u(t) &= at^{\delta-1} + I^{\delta}_{0+}(b(1-t)^{\alpha-1} + I^{\alpha}_{1-}g(t,u(t))) \\
&= at^{\delta-1} + \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1} b(1-s)^{\alpha-1} ds \\
&\quad + \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1} I^{\alpha}_{1-}g(s,u(s)) ds \\
&= at^{\delta-1} + \frac{b}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}(1-s)^{\alpha-1} ds \\
&+ \quad \frac{1}{\Gamma(\delta)\Gamma(\alpha)}\int_0^t (t-s)^{\delta-1}\int_s^1 (\tau-s)^{\alpha-1} g(\tau,u(\tau)) d\tau\, ds
\end{aligned}
$$

Defining $T : X \to X$ by

$$
\begin{aligned}
Tu(t) = at^{\delta-1} &+ \frac{b}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}(1-s)^{\alpha-1} ds \\
&+ \frac{1}{\Gamma(\delta)\Gamma(\alpha)}\int_0^t (t-s)^{\delta-1}\int_s^1 (\tau-s)^{\alpha-1} g(\tau,u(\tau)) d\tau\, ds,
\end{aligned}
$$

we see that u is a solution to (1.1) if and only if u is a fixed point of T.

Lemma 2.1. *If g is continuous, then T is a completely continuous operator.*

Proof. Since g is continuous, T transforms X into X. Let $M = \{u \in X; \|u\| \le l, l > 0\}$. For $u \in M$,

$$
\begin{aligned}
& t^{1-\delta}(1-t)^{1-\alpha}|Tu(t)| \\
&= |a(1-t)^{1-\alpha} + \frac{bt^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}(1-s)^{\alpha-1} ds \\
&+ \quad \frac{t^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)\Gamma(\alpha)}\int_0^t (t-s)^{\delta-1}\int_s^1 (\tau-s)^{\alpha-1} g(\tau,u(\tau)) d\tau\, ds| \\
&\le a + \frac{bt^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}(1-t)^{\alpha-1} ds \\
&+ \quad \frac{t^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)\Gamma(\alpha)}\int_0^t (t-s)^{\delta-1}\int_s^1 (\tau-s)^{\alpha-1} |g(\tau,u(\tau))| d\tau\, ds \\
&\le a + \frac{b}{\Gamma(1+\delta)} + \frac{L}{\Gamma(\delta)\Gamma(\alpha)}\int_0^t (t-s)^{\delta-1}\int_s^1 (\tau-s)^{\alpha-1} d\tau\, ds \\
&= a + \frac{b}{\Gamma(1+\delta)} + \frac{L}{\Gamma(\delta)\Gamma(1+\alpha)}\int_0^t (t-s)^{\delta-1}(1-s)^{\alpha} ds \\
&\le a + \frac{b}{\Gamma(1+\delta)} + \frac{L}{\Gamma(\delta)\Gamma(1+\alpha)}\int_0^t (t-s)^{\delta-1} ds \\
&= a + \frac{b}{\Gamma(1+\delta)} + \frac{L}{\Gamma(1+\delta)\Gamma(1+\alpha)} t^{\delta} \\
&\le a + \frac{b}{\Gamma(1+\delta)} + \frac{L}{\Gamma(1+\delta)\Gamma(1+\alpha)}
\end{aligned}
$$

where

$$L = \max_{0 \le t \le 1, \|u\| \le l} |g(t,u(t))| + 1\,.$$

So $T(M)$ is bounded.

Let us see that $\overline{T(M)}$ is equicontinuous. For $u \in M, \varepsilon > 0, t_1, t_2 \in [0,1], t_1 < t_2$, let

$$\eta < \{\frac{\varepsilon}{4(a+1)}, \big(\frac{\varepsilon\Gamma(1+\delta)}{4(2b+1)}\big)^{1/\delta}, \big(\frac{\varepsilon\Gamma(1+\alpha)\Gamma(1+\delta)}{8L}\big)^{1/\delta}, \frac{\varepsilon\Gamma(1+\delta)\Gamma(1+\alpha)}{4(b\Gamma(1+\alpha)+L)}\}$$

For $t_2 - t_1 \le \max\{t_2 - t_1, t_2^{1-\delta} - t_1^{1-\delta}, (1-t_1)^{1-\alpha} - (1-t_2)^{1-\alpha}\} < \eta$, we have

$$\begin{aligned}
&|t_2^{1-\delta}(1-t_2)^{1-\alpha}|Tu(t_2)| - t_1^{1-\delta}(1-t_1)^{1-\alpha}|Tu(t_1)||\\
&= |a(1-t_2)^{1-\alpha} - a(1-t_1)^{1-\alpha} + \frac{bt_2^{1-\delta}(1-t_2)^{1-\alpha}}{\Gamma(\delta)}\int_0^{t_2}(t_2-s)^{\delta-1}(1-s)^{\alpha-1}ds\\
&\quad - \frac{bt_1^{1-\delta}(1-t_1)^{1-\alpha}}{\Gamma(\delta)}\int_0^{t_1}(t_1-s)^{\delta-1}(1-s)^{\alpha-1}ds\\
&\quad + \frac{t_2^{1-\delta}(1-t_2)^{1-\alpha}}{\Gamma(\delta)\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\delta-1}\int_s^1(\tau-s)^{\alpha-1}g(\tau,u(\tau))d\tau\, ds\\
&\quad - \frac{t_1^{1-\delta}(1-t_1)^{1-\alpha}}{\Gamma(\delta)\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\delta-1}\int_s^1(\tau-s)^{\alpha-1}g(\tau,u(\tau))d\tau\, ds|\\
&\le \frac{bt_1^{1-\delta}(1-t_1)^{1-\alpha}}{\Gamma(\delta)}\int_0^{t_1}((t_1-s)^{\delta-1} - (t_2-s)^{\delta-1})(1-t_1)^{\alpha-1}ds\\
&\quad + \frac{bt_2^{1-\delta}(1-t_2)^{1-\alpha}}{\Gamma(\delta)}\int_{t_1}^{t_2}(t_2-s)^{\delta-1}(1-t_2)^{\alpha-1}ds + a((1-t_1)^{1-\alpha} - (1-t_2)^{1-\alpha})\\
&\quad + \frac{Lt_1^{1-\delta}(1-t_1)^{1-\alpha}}{\Gamma(\delta)\Gamma(\alpha)}\int_0^{t_1}((t_1-s)^{\delta-1} - (t_2-s)^{\delta-1})\int_s^1(\tau-s)^{\alpha-1}d\tau\, ds\\
&\quad + \frac{Lt_2^{1-\delta}(1-t_2)^{1-\alpha}}{\Gamma(\delta)\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\delta-1}\int_s^1(\tau-s)^{\alpha-1}d\tau\, ds\\
&\quad + \frac{b(t_2^{1-\delta}(1-t_2)^{1-\alpha} - t_1^{1-\delta}(1-t_1)^{1-\alpha})}{\Gamma(\delta)}\int_0^{t_1}(t_2-s)^{\delta-1}(1-t_2)^{\alpha-1}ds\\
&\quad + \frac{L(t_2^{1-\delta}(1-t_2)^{1-\alpha} - t_1^{1-\delta}(1-t_1)^{1-\alpha})}{\Gamma(\delta)\Gamma(\alpha)}\int_0^{t_1}(t_2-s)^{\delta-1}\int_s^1(\tau-s)^{\alpha-1}d\tau\, ds\\
&\le \frac{bt_1^{1-\delta}}{\Gamma(1+\delta)}(t_1^\delta + (t_2-t_1)^\delta - t_2^\delta)\\
&\quad + \frac{bt_2^{1-\delta}}{\Gamma(1+\delta)}(t_2-t_1)^\delta + a((1-t_1)^{1-\alpha} - (1-t_2)^{1-\alpha})\\
&\quad + \frac{Lt_1^{1-\delta}(1-t_1)^{1-\alpha}}{\Gamma(\delta)\Gamma(1+\alpha)}\int_0^{t_1}((t_1-s)^{\delta-1} - (t_2-s)^{\delta-1})ds\\
&\quad + \frac{Lt_2^{1-\delta}(1-t_2)^{1-\alpha}}{\Gamma(\delta)\Gamma(1+\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\delta-1}ds\\
&\quad + \frac{b(t_2^{1-\delta}(1-t_2)^{1-\alpha} - t_1^{1-\delta}(1-t_1)^{1-\alpha})}{\Gamma(1+\delta)}t_2^\delta(1-t_2)^{\alpha-1}\\
&\quad + \frac{L(t_2^{1-\delta}(1-t_2)^{1-\alpha} - t_1^{1-\delta}(1-t_1)^{1-\alpha})}{\Gamma(\delta)\Gamma(1+\alpha)}\int_0^{t_1}(t_2-s)^{\delta-1}ds
\end{aligned}$$

$$\begin{aligned}
&\le \frac{bt_1^{1-\delta}}{\Gamma(1+\delta)}(t_2-t_1)^\delta + \frac{bt_2^{1-\delta}}{\Gamma(1+\delta)}(t_2-t_1)^\delta \\
&+ \quad a((1-t_1)^{1-\alpha} - (1-t_2)^{1-\alpha}) + \frac{Lt_1^{1-\delta}(1-t_1)^{1-\alpha}}{\Gamma(1+\delta)\Gamma(1+\alpha)}(t_1^\delta + (t_2-t_1)^\delta - t_2^\delta) \\
&+ \frac{Lt_2^{1-\delta}(1-t_2)^{1-\alpha}}{\Gamma(1+\delta)\Gamma(1+\alpha)}(t_2-t_1)^\delta + \frac{b(t_2^{1-\delta}-t_1^{1-\delta})}{\Gamma(1+\delta)} + \frac{L(t_2^{1-\delta}-t_1^{1-\delta})}{\Gamma(1+\delta)\Gamma(1+\alpha)} \\
&\le \frac{b}{\Gamma(1+\delta)}(t_2-t_1)^\delta + \frac{b}{\Gamma(1+\delta)}(t_2-t_1)^\delta + a((1-t_1)^{1-\alpha} - (1-t_2)^{1-\alpha}) \\
&+ \frac{L}{\Gamma(1+\delta)\Gamma(1+\alpha)}(t_2-t_1)^\delta + \frac{L}{\Gamma(1+\delta)\Gamma(1+\alpha)}(t_2-t_1)^\delta \\
&+ \frac{b(t_2^{1-\delta}-t_1^{1-\delta})}{\Gamma(1+\delta)} + \frac{L(t_2^{1-\delta}-t_1^{1-\delta})}{\Gamma(1+\delta)\Gamma(1+\alpha)} \\
&< \frac{2b+1}{\Gamma(1+\delta)}\eta^\delta + (a+1)\eta + \frac{2L}{\Gamma(1+\delta)\Gamma(1+\alpha)}\eta^\delta \\
&+ \Big(\frac{b}{\Gamma(1+\delta)} + \frac{L}{\Gamma(1+\delta)\Gamma(1+\alpha)}\Big)\eta \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\
&= \varepsilon
\end{aligned}$$

By Arzela-Azcoli's theorem $\overline{TM}$ is equicontinuous, so the operator T is completely continuous. □

Theorem 2.2. *If g is continuous, $a+b \neq 0$, and there exists $0 < \mu \le 1$ such that*
$$\lim_{|u|\to\infty} \frac{g(t,u(t))}{|u|^\mu} = 0, \tag{2.2}$$
then problem (1.1) *has one positive solution.*

Proof. As pointed out above, we only need to prove the existence of fixed point of operator T in K. It follows from the Lemma 2.1 that $T : K \to K$ is a completely continuous operator. From (2.2), there exists $N > 0$, such that for $0 < \varepsilon < \frac{\Gamma(1+\mu(\alpha-1)+\alpha)\Gamma(1+\mu(\delta-1)+\delta)}{4\Gamma(1+\mu(\alpha-1))\Gamma(1+\mu(\delta-1))}$,
$$g(t,u(t)) \le \varepsilon|u|^\mu, \quad \text{for } t \in [0,1], |u| \ge N$$
So we have
$$g(t,u(t)) \le \varepsilon|u|^\mu + c, \quad \text{for } t \in [0,1], u \in [0,+\infty)$$
where
$$c = \max_{0\le t\le 1, |u|\le N} |g(t,u(t))| + 1.$$
Let $\Omega_1 = \{u \in K; \|u\| < R_1\}$, where $R_1 > \{1, 4a, \frac{4b}{\Gamma(1+\delta)}, \frac{4c}{\Gamma(1+\delta)\Gamma(1+\alpha)}\}$, for $u \in \partial\Omega_1$, we have

$$\begin{aligned}
&t^{1-\delta}(1-t)^{1-\alpha}|Tu(t)| \\
&= |a(1-t)^{1-\alpha} + \frac{bt^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}(1-s)^{\alpha-1}ds \\
&+ \quad \frac{t^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)\Gamma(\alpha)}\int_0^t (t-s)^{\delta-1}\int_s^1 (\tau-s)^{\alpha-1} g(\tau,u(\tau))d\tau\, ds|
\end{aligned}$$

$$\begin{aligned}
&\le a+\frac{bt^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}(1-t)^{\alpha-1}ds\\
&\quad+\frac{t^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)\Gamma(\alpha)}\int_0^t (t-s)^{\delta-1}\int_s^1(\tau-s)^{\alpha-1}|g(\tau,u(\tau))|d\tau\, ds\\
&\le a+\frac{b}{\Gamma(1+\delta)}+\frac{t^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)\Gamma(\alpha)}\int_0^t (t-s)^{\delta-1}\int_s^1(\tau-s)^{\alpha-1}(\varepsilon|u(\tau)|^\mu+c)d\tau\, ds\\
&\le \frac{t^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)\Gamma(\alpha)}\int_0^t (t-s)^{\delta-1}\int_s^1(\tau-s)^{\alpha-1}(\varepsilon\tau^{\mu(\delta-1)}(1-\tau)^{\mu(\alpha-1)}\|u\|^\mu+c)d\tau\, ds\\
&+a+\frac{b}{\Gamma(1+\delta)}\\
&\le \frac{t^{1-\delta}(1-t)^{1-\alpha}\varepsilon\|u\|^\mu}{\Gamma(\delta)\Gamma(\alpha)}\int_0^t (t-s)^{\delta-1}\int_s^1(\tau-s)^{\alpha-1}s^{\mu(\delta-1)}(1-\tau)^{\mu(\alpha-1)}d\tau\, ds\\
&+\quad \frac{c}{\Gamma(\delta)\Gamma(\alpha)}\int_0^t (t-s)^{\delta-1}\int_s^1(\tau-s)^{\alpha-1}d\tau\, ds+a+\frac{b}{\Gamma(1+\delta)}\\
&=\frac{t^{1-\delta}(1-t)^{1-\alpha}\varepsilon\|u\|^\mu\Gamma(1+\mu(\alpha-1))}{\Gamma(\delta)\Gamma(1+\alpha+\mu(\alpha-1))}\int_0^t (t-s)^{\delta-1}s^{\mu(\delta-1)}(1-s)^{\mu(\alpha-1)+\alpha}ds\\
&\quad+\frac{c}{\Gamma(\delta)\Gamma(1+\alpha)}\int_0^t (t-s)^{\delta-1}(1-s)^{\alpha}ds+a+\frac{b}{\Gamma(1+\delta)}
\end{aligned}$$

If $\mu(\alpha-1)+\alpha<0$, then the first equality of above becomes

$$\begin{aligned}
&\frac{t^{1-\delta}(1-t)^{1-\alpha}\varepsilon\|u\|^\mu\Gamma(1+\mu(\alpha-1))}{\Gamma(\delta)\Gamma(1+\alpha+\mu(\alpha-1))}\int_0^t (t-s)^{\delta-1}s^{\mu(\delta-1)}(1-s)^{\mu(\alpha-1)+\alpha}ds\\
&\le \frac{t^{1-\delta}(1-t)^{1-\alpha}\varepsilon\|u\|^\mu\Gamma(1+\mu(\alpha-1))}{\Gamma(\delta)\Gamma(1+\mu(\alpha-1)+\alpha)}\int_0^t (t-s)^{\delta-1}s^{\mu(\delta-1)}(1-t)^{\mu(\alpha-1)+\alpha}ds\\
&=\frac{t^{1-\delta}(1-t)^{1+\mu(\alpha-1)}\varepsilon\|u\|^\mu\Gamma(1+\mu(\alpha-1))}{\Gamma(\delta)\Gamma(1+\mu(\alpha-1)+\alpha)}\int_0^t (t-s)^{\delta-1}s^{\mu(\delta-1)}ds\\
&=\frac{t^{1+\mu(\delta-1)}(1-t)^{1+\mu(\alpha-1)}\varepsilon\|u\|^\mu\Gamma(1+\mu(\alpha-1))\Gamma(1+\mu(\delta-1))}{\Gamma(1+\mu(\alpha-1)+\alpha)\Gamma(1+\mu(\delta-1)+\delta)}\\
&\le \frac{\varepsilon R_1^\mu\Gamma(1+\mu(\alpha-1))\Gamma(1+\mu(\delta-1))}{\Gamma(1+\mu(\alpha-1)+\alpha)\Gamma(1+\mu(\delta-1)+\delta)}\,.
\end{aligned}$$

If $\mu(\alpha-1)+\alpha\ge 0$, then the first equality of above becomes

$$\begin{aligned}
&\frac{t^{1-\delta}(1-t)^{1-\alpha}\varepsilon\|u\|^\mu\Gamma(1+\mu(\alpha-1))}{\Gamma(\delta)\Gamma(1+\alpha+\mu(\alpha-1))}\int_0^t (t-s)^{\delta-1}s^{\mu(\delta-1)}(1-s)^{\mu(\alpha-1)+\alpha}ds\\
&\le \frac{t^{1-\delta}(1-t)^{1-\alpha}\varepsilon\|u\|^\mu\Gamma(1+\mu(\alpha-1))}{\Gamma(\delta)\Gamma(1+\mu(\alpha-1)+\alpha)}\int_0^t (t-s)^{\delta-1}s^{\mu(\delta-1)}ds\\
&=\frac{t^{1+\mu(\delta-1)}(1-t)^{1-\alpha}\varepsilon\|u\|^\mu\Gamma(1+\mu(\alpha-1))\Gamma(1+\mu(\delta-1))}{\Gamma(1+\mu(\alpha-1)+\alpha)\Gamma(1+\mu(\delta-1)+\delta)}\\
&\le \frac{\varepsilon R_1^\mu\Gamma(1+\mu(\alpha-1))\Gamma(1+\mu(\delta-1))}{\Gamma(1+\mu(\alpha-1)+\alpha)\Gamma(1+\mu(\delta-1)+\delta)}\,.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
t^{1-\delta}(1-t)^{1-\alpha}|Tu(t)| &\leq \frac{\varepsilon\|u\|^{\mu}\Gamma(1+\mu(\alpha-1))\Gamma(1+\mu(\delta-1))}{\Gamma(1+\mu(\alpha-1)+\alpha)\Gamma(1+\mu(\delta-1)+\delta)} \\
&\quad + \frac{c}{\Gamma(\delta)\Gamma(1+\alpha)}\int_0^t (t-s)^{\delta-1}ds + a + \frac{b}{\Gamma(1+\delta)} \\
&\leq \frac{\varepsilon R_1^{\mu}\Gamma(1+\mu(\alpha-1))\Gamma(1+\mu(\delta-1))}{\Gamma(1+\mu(\alpha-1)+\alpha)\Gamma(1+\mu(\delta-1)+\delta)} \\
&\quad + \frac{c}{\Gamma(1+\delta)\Gamma(1+\alpha)}t^{\delta} + a + \frac{b}{\Gamma(1+\delta)} \\
&\leq \frac{\varepsilon R_1\Gamma(1+\mu(\alpha-1))\Gamma(1+\mu(\delta-1))}{\Gamma(1+\mu(\alpha-1)+\alpha)\Gamma(1+\mu(\delta-1)+\delta)} \\
&\quad + \frac{c}{\Gamma(1+\delta)\Gamma(1+\alpha)} + a + \frac{b}{\Gamma(1+\delta)} \\
&\leq \frac{R_1}{4} + \frac{R_1}{4} + \frac{R_1}{4} + \frac{R_1}{4} = R_1
\end{aligned}$$

and $\|Tu\| \leq R_1 = \|u\|$. Taking

$$\Omega_2 = \{u \in K; \|u\| < R_2\}$$

where $R_2 < \{a(\frac{1}{2})^{\delta-1} + \frac{b}{\Gamma(1+\delta)}(\frac{1}{2})^{\delta}, R_1\}$, then for $u \in \partial\Omega_2$, we obtain

$$\begin{aligned}
Tu(\frac{1}{2}) =& a(\frac{1}{2})^{\delta-1} + \frac{b}{\Gamma(\delta)}\int_0^{\frac{1}{2}}(\frac{1}{2}-s)^{\delta-1}(1-s)^{\alpha-1}ds \\
&+ \frac{1}{\Gamma(\delta)\Gamma(\alpha)}\int_0^{\frac{1}{2}}(\frac{1}{2}-s)^{\delta-1}\int_s^1(\tau-s)^{\alpha-1}g(\tau,u(\tau))d\tau\, ds \\
&\geq a(\frac{1}{2})^{\delta-1} + \frac{b}{\Gamma(\delta)}\int_0^{\frac{1}{2}}(\frac{1}{2}-s)^{\delta-1}ds \\
&= a(\frac{1}{2})^{\delta-1} + \frac{b}{\Gamma(1+\delta)}(\frac{1}{2})^{\delta} \\
&\geq R_2\,.
\end{aligned}$$

Therefore, $\|Tu\| \geq R_2 = \|u\|$. Theorem 1.3 implies that operator T has one fixed point $u^*(t) \in \overline{\Omega_1}\backslash\Omega_2$, then $u^*(t)$ is one positive solution of problem (1.1). $\square$

Theorem 2.3. *If g is continuous, and there exists constant $c_1, c_2 > 0$ and $1 \leq \lambda < \min\{\frac{\alpha}{1-\alpha}, \frac{1}{1-\delta}\}$ with $\alpha \geq \frac{1}{2}$ or $1 \leq \lambda < \min\{\frac{1}{1-\alpha}, \frac{1}{1-\delta}\}$ with $0 < \alpha \leq \frac{1}{2}$ such that*

$$g(t,u(t)) \leq c_1 + c_2|u(t)|^{\lambda}, \quad \text{for all } t \in [0,1], u \in [0,+\infty) \tag{2.3}$$

then problem (1.1) has at least one solution.

Proof. As in Theorem 2.2, we only need to consider existence of fixed point of operator T. By Lemma 2.1, T is a completely continuous operator. We will make use of the Schauder Fixed Point Theorem to prove this theorem. Let $0 < R < 1$, and

$$B_R = \{u \in C([0,\gamma],[0,+\infty)); \|u - (at^{\delta-1} + \frac{b}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}(1-s)^{\alpha-1}ds\| \leq R\}$$

be a convex bounded and closed subset of the Banach space $C[0,\gamma]$, where

$$
\begin{aligned}
\gamma < \min\big\{1, \big(&\frac{R\Gamma(1+\delta)\Gamma(1+\alpha)}{2c_1}\big)^{1/\delta},\\
&\big(\frac{\Gamma(1+\delta+\lambda(\delta-1))\Gamma(1+\alpha+\lambda(\alpha-1))}{2c_2\Gamma(1+\lambda(\delta-1))\Gamma(1+\lambda(\alpha-1))}\big)^{\frac{1}{1+\lambda(\delta-1)}}\big\}
\end{aligned}
$$

Note that for all $u\in B_R$,

$$
\begin{aligned}
&t^{1-\delta}(1-t)^{1-\alpha}|Tu(t)-(at^{\delta-1}+\frac{b}{\Gamma(\delta)}\int_0^t(t-s)^{\delta-1}(1-s)^{\alpha-1}ds)|\\
&=\frac{t^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)\Gamma(\alpha)}\int_0^t(t-s)^{\delta-1}\int_s^1(\tau-s)^{\alpha-1}g(\tau,u(\tau))d\tau\,ds\\
&\le\frac{t^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)\Gamma(\alpha)}\int_0^t(t-s)^{\delta-1}\int_s^1(\tau-s)^{\alpha-1}(c_1+c_2|u(\tau)|^\lambda)d\tau\,ds\\
&\le\frac{t^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)\Gamma(\alpha)}\int_0^t(t-s)^{\delta-1}\int_s^1(\tau-s)^{\alpha-1}\\
&\quad\times(c_1+c_2\tau^{\lambda(\delta-1)}(1-\tau)^{\lambda(\alpha-1)}\|u\|^\lambda)d\tau\,ds\\
&\le\frac{c_1t^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)\Gamma(1+\alpha)}\int_0^t(t-s)^{\delta-1}(1-s)^{\alpha}ds\\
&\quad+\frac{c_2\|u\|^\lambda t^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)\Gamma(\alpha)}\int_0^t(t-s)^{\delta-1}\int_s^1(\tau-s)^{\alpha-1}s^{\lambda(\delta-1)}(1-\tau)^{\lambda(\alpha-1)}d\tau\,ds\\
&=\frac{c_1t^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)\Gamma(1+\alpha)}\int_0^t(t-s)^{\delta-1}(1-s)^{\alpha}ds\\
&\quad+\frac{c_2\|u\|^\lambda t^{1-\delta}(1-t)^{1-\alpha}\Gamma(1+\lambda(\alpha-1))}{\Gamma(\delta)\Gamma(1+\alpha+\lambda(\alpha-1))}\int_0^t(t-s)^{\delta-1}s^{\lambda(\delta-1)}(1-s)^{\lambda(\alpha-1)+\alpha}ds
\end{aligned}
$$

If $1\le\lambda<\min\{\frac{\alpha}{1-\alpha},\frac{1}{1-\delta}\}$, then for the second formula of above becomes

$$
\begin{aligned}
&\frac{c_2\|u\|^\lambda t^{1-\delta}(1-t)^{1-\alpha}\Gamma(1+\lambda(\alpha-1))}{\Gamma(\delta)\Gamma(1+\alpha+\lambda(\alpha-1))}\int_0^t(t-s)^{\delta-1}s^{\lambda(\delta-1)}(1-s)^{\lambda(\alpha-1)+\alpha}ds\\
&\le\frac{c_2\|u\|^\lambda t^{1-\delta}(1-t)^{1-\alpha}\Gamma(1+\lambda(\alpha-1))}{\Gamma(\delta)\Gamma(1+\alpha+\lambda(\alpha-1))}\int_0^t(t-s)^{\delta-1}s^{\lambda(\delta-1)}ds\\
&=\frac{c_2\|u\|^\lambda(1-t)^{1-\alpha}\Gamma(1+\lambda(\alpha-1))\Gamma(1+\lambda(\delta-1))}{\Gamma(1+\delta+\lambda(\delta-1))\Gamma(1+\alpha+\lambda(\alpha-1))}t^{\lambda(\delta-1)+1}\\
&\le\frac{c_2\|u\|^\lambda\Gamma(1+\lambda(\alpha-1))\Gamma(1+\lambda(\delta-1))}{\Gamma(1+\delta+\lambda(\delta-1))\Gamma(1+\alpha+\lambda(\alpha-1))}\gamma^{\lambda(\delta-1)+1}
\end{aligned}
$$

Similarly, if $1\le\lambda<\min\{\frac{1}{1-\alpha},\frac{1}{1-\delta}\}$, then for the second formula of above becomes

$$
\begin{aligned}
&\frac{c_2\|u\|^\lambda t^{1-\delta}(1-t)^{1-\alpha}\Gamma(1+\lambda(\alpha-1))}{\Gamma(\delta)\Gamma(1+\alpha+\lambda(\alpha-1))}\int_0^t(t-s)^{\delta-1}s^{\lambda(\delta-1)}(1-s)^{\lambda(\alpha-1)+\alpha}ds\\
&\le\frac{c_2\|u\|^\lambda t^{1-\delta}(1-t)^{1-\alpha}\Gamma(1+\lambda(\alpha-1))}{\Gamma(\delta)\Gamma(1+\alpha+\lambda(\alpha-1))}\int_0^t(t-s)^{\delta-1}s^{\lambda(\delta-1)}(1-t)^{\lambda(\alpha-1)+\alpha}ds\\
&=\frac{c_2\|u\|^\lambda(1-t)^{1+\lambda(\alpha-1)}\Gamma(1+\lambda(\alpha-1))\Gamma(1+\lambda(\delta-1))}{\Gamma(1+\delta+\lambda(\delta-1))\Gamma(1+\alpha+\lambda(\alpha-1))}t^{\lambda(\delta-1)+1}
\end{aligned}
$$

$$\le \frac{c_2\|u\|^\lambda\Gamma(1+\lambda(\alpha-1))\Gamma(1+\lambda(\delta-1))}{\Gamma(1+\delta+\lambda(\delta-1))\Gamma(1+\alpha+\lambda(\alpha-1))}\gamma^{\lambda(\delta-1)+1}$$

Therefore, we have

$$\begin{aligned}
&t^{1-\delta}(1-t)^{1-\alpha}|Tu(t)-(at^{\delta-1}+\frac{b}{\Gamma(\delta)}\int_0^t(t-s)^{\delta-1}(1-s)^{\alpha-1}ds)| \\
&= \frac{t^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)\Gamma(\alpha)}\int_0^t(t-s)^{\delta-1}\int_s^1(\tau-s)^{\alpha-1}g(\tau,u(\tau))d\tau\, ds \\
&\le \frac{c_1}{\Gamma(\delta)\Gamma(1+\alpha)}\int_0^t(t-s)^{\delta-1}ds+\frac{c_2\|u\|^\lambda\Gamma(1+\lambda(\alpha-1))\Gamma(1+\lambda(\delta-1))}{\Gamma(1+\delta+\lambda(\delta-1))\Gamma(1+\alpha+\lambda(\alpha-1))}\gamma^{\lambda(\delta-1)+1} \\
&= \frac{c_1}{\Gamma(1+\delta)\Gamma(1+\alpha)}t^\delta+\frac{c_2\|u\|^\lambda\Gamma(1+\lambda(\alpha-1))\Gamma(1+\lambda(\delta-1))}{\Gamma(1+\delta+\lambda(\delta-1))\Gamma(1+\alpha+\lambda(\alpha-1))}\gamma^{\lambda(\delta-1)+1} \\
&\le \frac{c_1}{\Gamma(1+\delta)\Gamma(1+\alpha)}\gamma^\delta+\frac{c_2R\Gamma(1+\lambda(\alpha-1))\Gamma(1+\lambda(\delta-1))}{\Gamma(1+\delta+\lambda(\delta-1))\Gamma(1+\alpha+\lambda(\alpha-1))}\gamma^{\lambda(\delta-1)+1} \\
&< \frac{R}{2}+\frac{R}{2}=R
\end{aligned}$$

Therefore, $T(B_R)\subset B_R$.Now by the Schauder Fixed Point Theorem there exists $u^*(t)\in B_R$ such that $Tu^*(t)=u^*(t)$, and this completes the proof. □

The following result establishes the existence of multiple positive solutions for the initial value problem (1.1). Let

$$f(u)=u(\frac{1}{2}),\quad u\in K$$

Obviously f is a nonnegative concave function that satisfies $f(u)\le\|u\|$, for $u\in K$ and

$$\begin{gathered}
K_c=\{u\in K;\|u\|\le c\},\\
K(f,d,e)=\{u\in K; d\le f(u),\|u\|\le e\}\,.
\end{gathered}$$

Theorem 2.4. *Let g be continuous. If $a(\frac{1}{2})^{\delta-1}+\frac{b}{\Gamma(1+\delta)}<\bar a<\bar b<d=2\bar b=c$, if g satisfies:*

- (H4) $g(t,u)<(\bar a-a-\frac{b}{\Gamma(1+\delta)})\Gamma(1+\delta)\Gamma(1+\alpha)$ *for* $0\le t\le 1$, $0\le u\le\bar a$
- (H5) $g(t,u)<(c-a-\frac{b}{\Gamma(1+\delta)})\Gamma(1+\delta)\Gamma(1+\alpha)$ *for* $0\le t\le 1$, $0\le u\le c$
- (H6) $g(t,u)>(\frac{1}{2})^{-1-\delta}(\bar b-a(\frac{1}{2})^{\delta-1}-\frac{b(\frac{1}{2})^\delta}{\Gamma(1+\delta)})\frac{(1+\delta)\Gamma(1+\delta)\Gamma(1+\alpha)}{\delta}$ *for* $0\le t\le 1$, $\bar b\le u\le 2\bar b$,

then the initial-value problem (1.1) *has three positive solutions u_1,u_2,u_3 satisfying*

$$\|u_1\|<\bar a,\quad \bar b<f(u_2),\quad \|u_3\|>\bar a\quad with\quad f(u_3)<\bar b \tag{2.4}$$

Proof. We apply Theorem 1.4. Since g is continuous, by Lemma 2.1, operator T is completely continuous. Now we choose $u\in\overline{K_c}$, then $\|u\|\le c$, and $g(t,u(t))<(c-a-\frac{b}{\Gamma(1+\delta)})\Gamma(1+\delta)\Gamma(1+\alpha)$ for $t\in[0,1]$ by (H5), so we have

$$\begin{aligned}
&t^{1-\delta}(1-t)^{1-\alpha}|Tu(t)| \\
&= |a(1-t)^{1-\alpha}+\frac{bt^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)}\int_0^t(t-s)^{\delta-1}(1-s)^{\alpha-1}ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{t^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)\Gamma(\alpha)}\int_0^t (t-s)^{\delta-1}\int_s^1 (\tau-s)^{\alpha-1}g(\tau,u(\tau))d\tau\,ds| \\
&\le a(1-t)^{1-\alpha}+\frac{bt^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}(1-s)^{\alpha-1}ds \\
&\quad+\frac{t^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)\Gamma(\alpha)}\int_0^t (t-s)^{\delta-1}\int_s^1 (\tau-s)^{\alpha-1}|g(\tau,u(\tau))|d\tau\,ds \\
&< a(1-t)^{1-\alpha}+\frac{bt^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}(1-s)^{\alpha-1}ds \\
&\quad+\frac{t^{1-\delta}(1-t)^{1-\alpha}(c-a-\frac{b}{\Gamma(1+\delta)})\Gamma(1+\delta)\Gamma(1+\alpha)}{\Gamma(\delta)\Gamma(\alpha)}\int_0^t (t-s)^{\delta-1}\int_s^1 (\tau-s)^{\alpha-1}d\tau\,ds \\
&= a(1-t)^{1-\alpha}+\frac{bt^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}(1-s)^{\alpha-1}ds \\
&\quad+\frac{t^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)\Gamma(1+\alpha)}\int_0^t (t-s)^{\delta-1}(c-a-\frac{b}{\Gamma(1+\delta)})\Gamma(1+\delta)\Gamma(1+\alpha)(1-s)^{\alpha}ds \\
&\le a+\frac{bt^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}(1-t)^{\alpha-1}ds \\
&\quad+\frac{t^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)\Gamma(1+\alpha)}\int_0^t (t-s)^{\delta-1}(c-a-\frac{b}{\Gamma(1+\delta)})\Gamma(1+\delta)\Gamma(1+\alpha)ds \\
&\le a+\frac{bt}{\Gamma(1+\delta)}+t(1-t)^{1-\alpha}(c-a-\frac{b}{\Gamma(1+\delta)}) \\
&\le a+\frac{b}{\Gamma(1+\delta)}+c-a-\frac{b}{\Gamma(1+\delta)}=c
\end{aligned}
$$

That is $\|Tu\|\le c$. On the other hand, $f(u)\le\|u\|$ for $u\in K_c$, and $T:K_c\to K_c$ is completely continuous by the above deduction and Lemma 2.1. Similarly, if $u\in\overline{K_{\overline{a}}}$, then $\|u\|\le\overline{a}$, and $g(t,u(t))<(\overline{a}-a-\frac{b}{\Gamma(1+\delta)})\Gamma(1+\delta)\Gamma(1+\alpha)$ for each $t\in[0,1]$ by (H4), we obtain

$$
\begin{aligned}
&t^{1-\delta}(1-t)^{1-\alpha}|Tu(t)| \\
&= |a(1-t)^{1-\alpha}+\frac{bt^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}(1-s)^{\alpha-1}ds \\
&\quad+\frac{t^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)\Gamma(\alpha)}\int_0^t (t-s)^{\delta-1}\int_s^1 (\tau-s)^{\alpha-1}g(\tau,u(\tau))d\tau\,ds| \\
&\le a(1-t)^{1-\alpha}+\frac{bt^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}(1-s)^{\alpha-1}ds \\
&\quad+\frac{t^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)\Gamma(\alpha)}\int_0^t (t-s)^{\delta-1}\int_s^1 (\tau-s)^{\alpha-1}|g(\tau,u(\tau))|d\tau\,ds \\
&< a(1-t)^{1-\alpha}+\frac{bt^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}(1-s)^{\alpha-1}ds \\
&\quad+\frac{t^{1-\delta}(1-t)^{1-\alpha}(\overline{a}-a-\frac{b}{\Gamma(1+\delta)})\Gamma(1+\delta)\Gamma(1+\alpha)}{\Gamma(\delta)\Gamma(\alpha)}\int_0^t (t-s)^{\delta-1}\int_s^1 (\tau-s)^{\alpha-1}d\tau\,ds
\end{aligned}
$$

$$
\begin{aligned}
&= a(1-t)^{1-\alpha} + \frac{bt^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1}(1-s)^{\alpha-1} ds \\
&\quad + \frac{t^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)\Gamma(1+\alpha)} \int_0^t (t-s)^{\delta-1}(\overline{a} - a - \frac{b}{\Gamma(1+\delta)})\Gamma(1+\delta)\Gamma(1+\alpha)(1-s)^{\alpha} ds \\
&\le a + \frac{bt^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1}(1-t)^{\alpha-1} ds \\
&\quad + \frac{t^{1-\delta}(1-t)^{1-\alpha}}{\Gamma(\delta)\Gamma(1+\alpha)} \int_0^t (t-s)^{\delta-1}(\overline{a} - a - \frac{b}{\Gamma(1+\delta)})\Gamma(1+\delta)\Gamma(1+\alpha) ds \\
&= a + \frac{bt}{\Gamma(1+\delta)} + t(1-t)^{1-\alpha}(\overline{a} - a - \frac{b}{\Gamma(1+\delta)}) \\
&\le a + \frac{b}{\Gamma(1+\delta)} + \overline{a} - a - \frac{b}{\Gamma(1+\delta)} = \overline{a}\,.
\end{aligned}
$$

So $\|Tu\| < \overline{a}$ for $\|u\| \le \overline{a}$ which proves the condition (C2) of Theorem 1.4. We note that $u(t) = 2\overline{b}$, $0 \le t \le 1$ belong to $K(f, \overline{b}, 2\overline{b})$. In fact $f(u) = f(2\overline{b}) = 2\overline{b} > \overline{b}$, so $\{u \in K(f, \overline{b}, 2\overline{b}) | f(u) > \overline{b}\} \ne \emptyset$. In addition, if we choose $u \in K(f, \overline{b}, 2\overline{b})$, then we have $\overline{b} < f(u) = u(\frac{1}{2}) \le \|u\| \le 2\overline{b}$, and by (H6)

$$
\begin{aligned}
f(Tu) &= Tu(\frac{1}{2}) \\
&= a(\frac{1}{2})^{\delta-1} + \frac{b}{\Gamma(\delta)} \int_0^{\frac{1}{2}} (\frac{1}{2} - s)^{\delta-1}(1-s)^{\alpha-1} ds \\
&\quad + \frac{1}{\Gamma(\delta)\Gamma(\alpha)} \int_0^{\frac{1}{2}} (\frac{1}{2} - s)^{\delta-1} \int_s^1 (\tau - s)^{\alpha-1} g(\tau, u(\tau)) d\tau\, ds \\
&> \frac{(\frac{1}{2})^{-1-\delta}(\overline{b} - a(\frac{1}{2})^{\delta-1} - \frac{b(\frac{1}{2})^{\delta}}{\Gamma(1+\delta)})\frac{(1+\delta)\Gamma(1+\delta)\Gamma(1+\alpha)}{\delta}}{\Gamma(\delta)\Gamma(\alpha)} \\
&\quad \times \int_0^{\frac{1}{2}} (\frac{1}{2} - s)^{\delta-1} \int_s^1 (\tau - s)^{\alpha-1} d\tau\, ds + a(\frac{1}{2})^{\delta-1} + \frac{b}{\Gamma(\delta)} \int_0^{\frac{1}{2}} (\frac{1}{2} - s)^{\delta-1} ds \\
&= \frac{(\frac{1}{2})^{-1-\delta}(\overline{b} - a(\frac{1}{2})^{\delta-1} - \frac{b(\frac{1}{2})^{\delta}}{\Gamma(1+\delta)})\frac{(1+\delta)\Gamma(1+\delta)}{\delta}}{\Gamma(\delta)} \int_0^{\frac{1}{2}} (\frac{1}{2} - s)^{\delta-1}(1-s)^{\alpha} ds \\
&\quad + a(\frac{1}{2})^{\delta-1} + \frac{b(\frac{1}{2})^{\delta}}{\Gamma(1+\delta)} \\
&\ge \frac{(\frac{1}{2})^{-1-\delta}(\overline{b} - a(\frac{1}{2})^{\delta-1} - \frac{b(\frac{1}{2})^{\delta}}{\Gamma(1+\delta)})\frac{(1+\delta)\Gamma(1+\delta)}{\delta}}{\Gamma(\delta)} \int_0^{\frac{1}{2}} (\frac{1}{2} - s)^{\delta-1}(1-s) ds \\
&\quad + a(\frac{1}{2})^{\delta-1} + \frac{b(\frac{1}{2})^{\delta}}{\Gamma(1+\delta)} \\
&> \frac{(\overline{b} - a(\frac{1}{2})^{\delta-1} - \frac{b(\frac{1}{2})^{\delta}}{\Gamma(1+\delta)})(1+\delta)}{\delta} - \frac{\overline{b} - a(\frac{1}{2})^{\delta-1} - \frac{b(\frac{1}{2})^{\delta}}{\Gamma(1+\delta)}}{\delta} \\
&\quad + a(\frac{1}{2})^{\delta-1} + \frac{b(\frac{1}{2})^{\delta}}{\Gamma(1+\delta)} = \overline{b}\,.
\end{aligned}
$$

So $f(Tu) > \bar{b}$ for $u \in K(f, \bar{b}, 2\bar{b})$ which proves the condition (C1) of Theorem 1.4. Choosing $u \in K(f, \bar{b}, c)$ such that $\|Tu\| \geq d = 2\bar{b}$, then $\bar{b} < f(u) = u(\frac{1}{2}) \leq \|u\| \leq c = d = 2\bar{b}$, therefore by the above deduction that $f(Tu) = Tu(\frac{1}{2}) > \bar{b}$, which proves the condition (C3) of Theorem 1.4. Thus by Theorem 2.3, T has three fixed points in K, which proves the theorem. $\square$

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