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NEW CHARACTERIZATIONS OF ASYMPTOTIC STABILITY FOR EVOLUTION FAMILIES ON BANACH SPACES

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Abstract. We generalize the Datko - Rolewicz theorem on exponential stability in the non-autonomous case. Also, we extend the results obtained by Jan van Neerven [18].

1. Introduction

Let $\mathbb{R}_+$ be the set of non-negative real numbers, $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X and $\omega_0(\mathbf{T}) := \inf_{t>0} \frac{\ln \|T(t)\|}{t}$ its uniform exponential growth. It is well known the autonomous version of Datko theorem ([12]) which says that

> If for each $x \in X$ the map $t \mapsto \|T(t)x\|$ belongs to the space $L^2(\mathbb{R}_+)$ then the semigroup $\mathbf{T}$ is exponentially stable, that is $\omega_0(\mathbf{T})$ is strictly negative.

This result was generalized by Pazy ([19]) who proved that the exponent $p = 2$ from the autonomous version of Datko theorem may be replaced by every $1 \leq p < \infty$. Moreover, from the Pazy proof follows an interesting individual stability result. Namely *if a trajectory of the semigroup* $\mathbf{T}$, *(i.e. a map* $t \mapsto T(t)x$ *with* $x \in X$*), belongs to the space* $L^p(\mathbb{R}_+)$, *then it decay to* 0 *at* ∞. On the other hand a classical result says that if a real valued function f on $\mathbb{R}_+$ is uniformly continuous and $\int_0^\infty |f(t)|dt < \infty$ then it decay to 0 at ∞, see for example [1]. Then we can say that each trajectory of a strongly continuous semigroup which belongs to the space $L^p(\mathbb{R}_+)$is uniformly continuous on $\mathbb{R}_+$ if and only if it decay to 0 at ∞. In order to introduce the nonautonomous results of this type we recall the notion of solid space over $\mathbb{R}_+$.

The set of all $\mathbb{R}$-valued functions f defined on $\mathbb{R}_+$ will be denoted by $\mathcal{F}(\mathbb{R}_+, \mathbb{R})$. Let $\rho : \mathcal{F}(\mathbb{R}_+, \mathbb{R}) \to [0, \infty]$ be a map with the following properties:

- (N1) $\rho(f) = 0$ if and only if $f = 0$.
- (N2) $\rho(af) = |a|\rho(f)$ for every real scalar a and every $f \in \mathcal{F}(\mathbb{R}_+, \mathbb{R})$ with $\rho(f) < \infty$.
- (N3) $\rho(f + g) \leq \rho(f) + \rho(g)$ for all $f, g \in \mathcal{F}(\mathbb{R}_+, \mathbb{R})$.

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We will denote by $F = F_\rho$ the set $\{f \in \mathcal{F}(\mathbb{R}_+, \mathbb{R}) : |f|_F := \rho(f) < \infty\}$. It is clear that the pair $(F, |\cdot|_F)$ is a linear normed space. Every normed subspace E of F will be called *normed function space*. A normed function space is called *solid* if for each $f \in \mathcal{F}(\mathbb{R}_+, \mathbb{R})$ and each $g \in E$ for which $|f| \le |g|$ we have that $f \in E$ and $|f|_E \le |g|_E$. For more details about Banach functions spaces we refer to the books [14, 22, 2, 24].

Let X be a Banach space and $\mathcal{L}(X)$ the Banach algebra of all linear and bounded operators acting on X. The norm on X and on $\mathcal{L}(X)$ will be denoted by $\|\cdot\|$. Recall that a family $\mathcal{U} = \{U(t,s) : t \ge s \ge 0\}$ in $\mathcal{L}(X)$ is called *evolution family with exponential growth* if $U(t,t) = Id$, (Id is the identity operator in $\mathcal{L}(X)$), $U(t,s)U(s,r) = U(t,r)$ for all $t \ge s \ge r \ge 0$ and there exist the real constants ω and M such that

$$\|U(t,s)\| \le M e^{\omega(t-s)} \quad \text{for all } t \ge s \ge 0. \tag{1.1}$$

We may suppose that $\omega > 0$ and $M \ge 1$. The evolution family $\mathcal{U}$ is called *uniformly bounded* if we can choose $\omega = 0$ in (1.1) and *uniformly exponentially stable* if there exist a negative ω such that (1.1) holds. Let E be a solid space. For the moment we suppose that for every positive T the space E contains the characteristic function of the interval $[0,T]$. We will see that this is not a restriction.

We will suppose that the space E satisfies one or more of the following hypotheses:

(H1) $\lim_{T\to\infty} |\chi_{[0,T]}|_E = \infty$.
(H2) For every positive t, the function $h \mapsto |\chi_{[h,t+h]}|_E$ is nondecreasing on $\mathbb{R}_+$.
(H3) There exists a positive number δ such that

$$K_\delta := \inf_{t\ge 0} |\chi_{[t,t+\delta]}|_E > 0\,. \tag{1.2}$$

(H4) There exists a positive function h, with $h(\infty) = \infty$, such that

$$1 + |\chi_{[s,t]}|_E \ge h(t-s) \text{ for all } t \ge s \ge 0. \tag{1.3}$$

It is easily to see that (H1) does not imply (H2), but (H2) implies (H3), and (H4) implies (H1). Moreover (H3) and (H4) do not imply (H2). To see this, let a be a strictly decreasing function on $\mathbb{R}_+$ with $a(\infty) = 1$, and E be the solid space consisting by all real-valued and locally measurable functions f (we identify every two functions which are equal almost everywhere) for which

$$|f|_E := \int_0^\infty a(r)|f(r)|dr < \infty.$$

Then the space E is solid, satisfies (H4) and (H3) (because the infimum from (1.2) is equal to $\delta > 0$), but it does not satisfy (H2).

Let $\mathcal{U}$ be an evolution family with exponential growth and let $s \ge 0$ and $x \in X$, be fixed. When $0 \le t < s$ we put $U(t,s)x = 0$. By U_s^x we will denote the real-valued map

$$r \mapsto U_s^x(r) := \chi_{[s,\infty)}(r)\|U(r,s)x\|, \quad r \in \mathbb{R}_+. \tag{1.4}$$

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing and continuous function such that $\varphi(t) > 0$ for all $t > 0$. The non-autonomous version of Datko theorem ([13, 11, 21]) says that the evolution family $\mathcal{U}$ is exponentially stable if and only if there exists a real number $1 \le p < \infty$ such that for each $s \ge 0$ and each $x \in X$, the map U_s^x belongs to $L^p(\mathbb{R}_+)$ and $\sup_{s\ge 0} |U_s^x|_p < \infty$. The non-autonomous version of Rolewicz theorem

([20, 21]) says that if for each $s \geq 0$ and each $x \in X$, $\phi \circ U_s^x$ belongs to $L^1(\mathbb{R}_+)$ and for each $x \in X$ we have that

$$\sup_{s \geq 0} |\varphi \circ U_s^x|_1 < \infty$$

then the evolution family $\mathcal{U}$ is exponentially stable. The reverse statement of the Rolewicz theorem is not true. We mention that Datko and Rolewicz used in their proofs the continuity of the map $t \mapsto U(t,s)x : [s, \infty) \to X$ for every $x \in X$. In the papers [5, 8] it is shown that the spaces L^p and L^1 in the above theorems can be replaced by a solid space satisfying (H1) and (H2). Moreover by an example in [5] it is shown that (H1) and (H2) cannot be removed. However, in this paper we will prove that it is possible to put (H3) and (H4) instead of (H2).

If E is *rearrangement invariant* solid function space over $\mathbb{R}_+$ (see e. g. [14] or [17, page 222] for this class of spaces) then the hypotheses (H2) and (H3) are equivalent and these hypotheses are satisfied automatically. Moreover (H1) and (H4) are equivalent in this case.

2. The Datko theorem for weighted spaces

To prove the main results we need the following Lemma whose proof can be found in [4, Lemma 4].

Lemma 2.1. *Let $\mathcal{U}$ be an evolution family which has exponential growth. If there exist a function $g : \mathbb{R}_+ \to (0, \infty)$ and a $t_0 > 0$ such that $g(t_0) < 1$ and if in addition*

$$\|U(t,s)\| \leq g(t-s) \quad \text{for all } t \geq s \geq 0$$

then $\mathcal{U}$ is uniformly exponentially stable.

Theorem 2.2. *Let $\mathcal{U}$ be an evolution family with exponentially growth on a Banach space X. If for each $s \geq 0$ and each $x \in X$ the map $t \mapsto (U_s^x)(t)$ belongs to a solid space E which verifies the hypotheses (H3) and if*

$$\sup_{s \geq 0} |U_s^x|_E := M(x) < \infty$$

then the evolution family $\mathcal{U}$ is uniformly bounded. If, in addition, the space E satisfies (H4) then the evolution family $\mathcal{U}$ is uniformly exponentially stable.

Proof. Let $s \geq 0, t \geq s + \delta, x \in X$ and $t - \delta \leq \tau < t$. Using inequality (1.1) we obtain

$$\begin{aligned} e^{-\omega} \chi_{[t-\delta,t]}(\tau) \|U(t,s)x\| &= e^{-\omega} \|U(t,\tau)U(\tau,s)x\| \\ &\leq e^{-\omega(t-\tau)} \|U(t,\tau)\| \cdot \|U(\tau,s)x\| \\ &\leq M \|U(\tau,s)x\|. \end{aligned}$$

Then

$$e^{-\omega} \chi_{[t-\delta,t]}(\cdot) \|U(t,s)x\| \leq M U_s^x(\cdot),$$

and because E is a solid space, it follows that the function $\chi_{[t-\delta,t]}(\cdot)\|U(t,s)x\|$ belongs to E. Moreover in view of (1.2) and of (H3) we have

$$K_\delta e^{-\omega} \|U(t,s)x\| \leq M |U_s^x|_E \leq M \cdot M(x).$$

Now using the Principle of Uniform Boundedness it is easily to see that the family $\mathcal{U}$ is uniformly bounded, that is, there exists a positive constant L such that

$$\sup_{v \geq u \geq 0} \|U(v,u)\| \leq L. \tag{2.1}$$

Let $t \geq s \geq 0$ and $x \in X$, be fixed. Using (2.1) we obtain

$$\chi_{[s,t]}(\tau)\|U(t,s)x\| \leq \|U(t,\tau)\|\|U(\tau,s)x\| \leq L \cdot \|U(\tau,s)x\|$$

for every $t \geq \tau \geq s$. On the other hand $\|U(t,s)x\| \leq L \cdot \|x\|$ for every $t \geq s$. Now it is easy to derive the inequality

$$\|U(t,s)x\| \leq L \cdot (1 + |\chi_{[s,t]}|_E)^{-1}[M(x) + \|x\|].$$

Using again the Uniform Boundedness Principle it follows that there exist a positive constant K, such that

$$\|U(t,s)\| \leq K(1 + |\chi_{[s,t]}|_E)^{-1}.$$

In view of the hypothesis (H4) there exists a function $g : \mathbb{R}_+ \to (0,\infty)$ such that

$$\inf_{r \in [0,\infty)} g(r) < 1 \quad \text{and} \quad \|U(t,s)\| \leq g(t-s).$$

Then from the previous Lemma, it follows that $\mathcal{U}$ is uniformly exponentially stable. $\square$

The following Corollary extends a similar result in [5].

Corollary 2.3. *Let $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\}$ be an evolution family with exponential growth such that for each $s \geq 0$ and each $x \in X$, the map U_s^x is locally measurable on $\mathbb{R}_+$. If there exists a real valued, locally measurable function a on $\mathbb{R}_+$ for which $\inf_{r \geq 0} a(r) > 0$ and $\lim_{t\to\infty} \int_t^{t+\mu} a(r)dr = \infty$ for some positive μ. If, in addition, for each $x \in X$,*

$$\sup_{s \geq 0} \Big[\sup_{t \geq s} \int_t^{t+\mu} a(r) U_s^x(r) dr \Big] < \infty$$

then the evolution family $\mathcal{U}$ is exponentially stable.

Proof. It suffices to apply Theorem 2.2 for the solid space E consisting by all real valued, locally measurable functions f defined on $\mathbb{R}_+$ for which

$$\rho(f) := \sup_{t \geq 0} \int_t^{t+\mu} a(r)|f(r)|dr < \infty.$$

$\square$

With the above notation, let us consider the real-valued map

$$V_s^x(r) := \|U(r+s,s)x\|, \quad r \geq 0.$$

It is interesting to see what happens if we put V_s^x instead of U_s^x in Theorem 2.2. A result in this spirit was shown in [15], where the exponential stability property of the evolution family $\mathcal{U}$ was obtained under the following two assumptions:

(1) The normed solid space E satisfies (H1).
(2) There exists a strictly increasing unbounded sequence (t_n) of positive real numbers such that:

$$\sup_{n \in \mathbf{N}} (t_{n+1} - t_n) < \infty \text{ and } \inf_{n \in \mathbf{N}} |\chi_{[t_n,t_{n+1}]}|_E > 0.$$

Next, we obtain same conclusion without using the second assumption above.

Let f be a X-valued function defined on $\mathbb{R}_+$. Then the map

$$t \mapsto \|f(t)\| : \mathbb{R}_+ \to \mathbb{R}_+$$

will be denoted by the symbol $\|f\|$. Let $E(\mathbb{R}_+, X)$ be the linear space of all X-valued functions defined on $\mathbb{R}_+$ for which $\|f\|$ lies in the space E. We will endow the space $E(\mathbb{R}_+, X)$ with the norm $|f|_{E(\mathbb{R}_+,X)} := |\|f\||_E$.

Theorem 2.4. *Let $\mathcal{U}$ be an evolution family with exponential growth and E be a solid space over $\mathbb{R}_+$ which satisfies (H1). If for each $s \geq 0$ and each $x \in X$ the map V_s^x belongs to the space E and*

$$\sup_{s\geq 0} |V_s^x|_E = K(x) < \infty$$

then $\mathcal{U}$ is uniformly exponentially stable.

Before proving this theorem, we recall the following known Lemma, see ([21, Lemma 8.12.3']) or [11] for the case of reversible evolution families.

Lemma 2.5. *Let $\mathcal{U} = \{U(t,s), t \geq s \geq 0\}$ be an evolution family with exponential growth. If $\mathcal{U}$ is not uniformly exponentially stable then for all $T > 0$ and all $0 < q < 1$ there exist $r_0 \geq 0$ and $x \in X$, such that*

$$\|U(r_0 + \tau, r_0)x\| > q\|x\| \text{ for all } T \geq \tau \geq 0. \tag{2.2}$$

Lemma 2.6. *Under the hypotheses of Theorem 2.4, it follows that there is a positive constant K such that*

$$\sup_{s\geq 0} |V_s^x|_E \leq K\|x\| \text{ for all } x \in X. \tag{2.3}$$

Proof. For each $s \geq 0$ let us consider the linear and bounded operator $V_s : X \to E(\mathbb{R}_+, X)$ given by

$$(V_s x)(t) := U(s+t, s)x, \quad t \in \mathbb{R}_+,\ x \in X.$$

Then for each $x \in X$, we have

$$|V_s x|_{E(\mathbb{R}_+,X)} = |\|U(s+\cdot, s)\||_E = |V_s^x|_E \leq K(x).$$

The assertion of Lemma 2.6 follows by the Uniform Boundedness Principle applied to the family $\mathcal{V} := \{V_s : s \geq 0\}$. □

Proof of Theorem 2.4. Suppose that $\mathcal{U}$ is not uniformly exponentially stable. Then from (2.2) and (2.3) follows that

$$K \geq q|\chi_{[0,T]}|_E$$

for all positive real number T, which is a contradiction. □

To the best of our knowledge the result in Theorem 2.4 is new and generalizes to the non-autonomous case some recently obtained autonomous or periodic versions in literature; see ([16, Theorem 4.2]) or ([6, Theorem 4.5]).

Using the method developed by Schnaubelt ([23]), see also [9], we can prove the following generalization of the L^1-version of Datko theorem.

Theorem 2.7. *Let $\mathcal{U} := \{U(t,s) : t \geq s \geq 0\}$ be an evolution family with exponential growth on a Banach space X. We suppose that for each $x \in X$ the map*

$$(t,s) \mapsto U(t,s)x : \{(t,s) : t \geq s \geq 0\}$$

is measurable. Then $\mathcal{U}$ is uniformly exponentially stable if and only if

$$\sup_{s\geq 0} \int_s^\infty \|U(t,s)x\| dt < \infty \tag{2.4}$$

for all $x \in X$.

Proof. As in the proof of Lemma 2.6, there exists a positive constant K, (independent of x and s), such that

$$\|U_s^x\|_{L^1(\mathbb{R}_+)} \leq K\|x\|. \tag{2.5}$$

Let us consider the evolution semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ associated with $\mathcal{U}$ on $L^1(\mathbb{R}_+, X)$. Recall that for each $t \geq 0$ and each $f \in L^1(\mathbb{R}_+, X)$ the map $T(t)f$ is given by

$$(T(t)f)(s) = \begin{cases} U(s, s-t)f(s-t), & s \geq t \\ 0, & 0 \leq s < t. \end{cases}$$

¿From the hypothesis on the measurability and using the fact that the evolution family $\mathcal{U}$ has exponential growth it follows that the map $T(t)f$ belongs to $L^1(\mathbb{R}_+, X)$ for all $t \geq 0$ and all $f \in L^1(\mathbb{R}_+, X)$. Moreover it is easy to see that the evolution semigroup $\mathbf{T}$ has exponential growth. Thus for each $f \in L^1(\mathbb{R}_+, X)$, the map $t \mapsto \|T(t)f\|_{L^1(\mathbb{R}_+,X)}$ is measurable, see e.g. ([16, Remark 4.3]). From (2.5) using the Fubini theorem follows

$$\begin{aligned}
\int_0^\infty \|T(t)f\|_{L^1(\mathbb{R}_+,X)} dt &= \int_0^\infty \int_0^\infty \chi_{[t,\infty)}(s)\|U(s,s-t)f(s-t)\|\, ds\, dt \\
&= \int_0^\infty \int_0^s \|U(s,\xi)f(\xi)\|\, d\xi\, ds \\
&= \int_0^\infty \int_0^\infty \chi_{[0,s]}(\xi)\|U(s,\xi)f(\xi)\|\, ds\, d\xi \\
&= \int_0^\infty \int_\xi^\infty \|U(s,\xi)f(\xi)\|\, ds\, d\xi \\
&\leq K\|f\|_{L^1(\mathbb{R}_+,X)}.
\end{aligned}$$

Now we apply the Datko-Pazy theorem for $p = 1$ (see the beginning of our paper) and use the well-known fact that if the semigoup $\mathbf{T}$ is exponentially stable then the evolution family $\mathcal{U}$ is uniformly exponentially stable as well, see [10, Theorem 2.2]. □

Remark 2.8. (1) The result contained in the above theorem may be known. It follows, for example, from ([8, Corollary 3.2]), for $\phi(t) = t$, $t \geq 0$. However, the main hypothesis of this Corollary is the boundedness of the function $(s,x) \mapsto \int_s^\infty \phi(\|U(t,s)x\|)dt$ on $\mathbb{R}_+ \times \overline{B}(0,1)$, where $\overline{B}(0,1)$ is the closed unit ball in X and ϕ is a nondecreasing function such that $\phi(t) > 0$ for every $t > 0$, which seems to be a more strongly require than the similar one from Theorem 2.7.
(2) The result stated in Theorem 2.7 holds under the general hypothesis that for each $x \in X$ and some real-valued, strictly increasing (or nondecreasing and positive

on $(0,\infty)$) and convex function Φ on $\mathbb{R}_+$, one has

$$\sup_{s\geq 0}\int_s^\infty \Phi(\|U(t,s)x\|)dt<\infty. \tag{2.6}$$

Proof of 2. For every $k=1,2,3,\cdots$ let us consider the set

$$X_k=\big\{x\in X:\sup_{s\geq 0}\int_s^\infty \Phi(\|U(t,s)x\|)\leq k.\big\}$$

By the assumption (2.5) follows that $X=\cup_{k\geq 1}X_k$. Using the well-known Fatou Lemma it is easily to see that each X_k is closed. Then there is a natural number k_0 such that X_{k_0} has nonempty interior. Let $x_0\in X$ and $\delta>0$ such that X_{k_0} contains the open ball with the centre in x_0 and radius δ. We will prove that the open ball which the centre in origin and radius $\frac{\delta}{2}$ is also contained in X_{k_0}. Indeed for each positive s and each $x\in X$ with $\|x\|\leq\delta$, one has

$$\begin{aligned}
\int_0^\infty \Phi(\|U(t,s)(\frac{1}{2}x)\|)dt &\leq \int_s^\infty \Phi(\frac{\|U(t,s)(x+x_0)\|+\|U(t,s)x_0\|}{2})dt\\
&\leq \frac{1}{2}(\int_s^\infty \Phi(\|U(t,s)(x+x_0)\|)dt+\int_s^\infty \Phi(\|U(t,s)x_0\|)dt)\\
&\leq k_0.
\end{aligned}$$

Now we can apply [8, Corollary 3.2]. We remark that in this proof only the strong measurability of the maps $t\mapsto U(t,s)$ $(s\geq 0, t\geq s)$ were used.

The "if" part can be obtained in the following way. Upon replacing Φ be a some multiple of itself we may assume that $\Phi(1)=1$. It is clear that $\Phi(0)=0$. Let N and ν two positive constants such that

$$\|U(t,s)\|\leq Ne^{-\nu(t-s)}\quad\text{for all } t\geq s\geq 0.$$

Then for a sufficiently large and positive h, (independent of s), we have

$$\int_s^\infty \Phi(\|U(t,s)x\|)\leq\int_0^h \Phi(Ne^{-\nu u})du+\int_h^\infty Ne^{-\nu u}du<\infty.$$

Finally we remark that the result holds even if the set of all $x\in X$ for which (2.5) holds is a second category in X. □

Another result of this type can be formulate as follows.

Theorem 2.9. *Let E be a solid Banach function space over $\mathbb{R}_+$ which satisfies (H1) and $\mathcal{U}$ be an evolution family such that for each positive s the map $t\mapsto U(t,s)$ is strongly measurable on $[s,\infty)$. If the norm of E has the Fatou property* [18] *and if the set of all $x\in X$ for which*

$$\sup_{s\geq 0}|||U(\cdot+s,s)x|||_E<\infty \tag{2.7}$$

is of the second category then $\mathcal{U}$ is uniformly exponentially stable.

Proof. As above, (see also [18] for the semigroup case), using the triangle inequality in the space E instead of convexity it follows that (2.7) holds for every $x\in X$. Then we apply Theorem 2.4 above to complete the proof. □

The following result shows that the hypothesis on the convexity of Φ from Remark 2.8 may be removed. However the converse statement of the Theorem 2.10 below does not hold without the convexity of Φ, see [21, Example 8.12.1].

Theorem 2.10. *Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function such that $\phi(t) > 0$ for all $t > 0$ and $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ be an evolution family such that for each $s \geq 0$ the map $t \mapsto U(t,s)$ is strongly measurable. If the set of all $x \in X$ for which*

$$M_\phi(x) := \sup_{s\geq 0} \int_s^\infty \phi(\|U(t,s)x\|)dt < \infty \tag{2.8}$$

is of second category in X then $\mathcal{U}$ is uniformly exponentially stable.

Proof. First we prove that the family $\mathcal{U}$ is uniformly bounded. Indeed for each $x \in X$ satisfying (2.8) there exists a real number $C(x)$ such that

$$\sup_{t\geq s\geq 0} \|U(t,s)x\| \leq C(x), \tag{2.9}$$

see [7, Lemma1]. It is clear that (2.9) holds for every $x \in X$, because it holds for each x in a set of second category in X. Then we apply the Uniform Boundedness Theorem to obtain the uniform boundedness of $\mathcal{U}$. On the other hand (2.8) can be written as

$$M_\phi(x) = \sup_{s\geq 0} \int_0^\infty \phi(\|U(t+s,s)x\|)dt < \infty. \tag{2.10}$$

From [17, Lemma 3.2.1] follows that there exists an Orlicz's space E which satisfies (H1) and such that for each x which satisfies (2.10), the map $t \mapsto \|U(t+s,s)x\|$ belongs to E. Using (2.10) we can derive (2.7). Now we apply Theorem 2.9 to complete the proof. $\square$

We conclude by stating another related result.

Proposition 2.11. *Let $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\}$ be an evolution family with exponential growth on a Banach space X and $x \in X$ be fixed. If for each $s \geq 0$, the map U_s^x (or the map V_s^x) belongs to a rearrangement invariant solid space E which verifies the hypothesis (H1), then the trajectory $U(s+\cdot,s)x$ of the evolution family $\mathcal{U}$ is asymptotically stable, that is, for each $s \geq 0$, one has:*

$$\lim_{t\to\infty} U(s+t,s)x = 0.$$

The proof of this proposition follows the arguments in [3, Theorem 2.1], and we omit it.

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