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A NOTE ON A DEGENERATE ELLIPTIC EQUATION WITH APPLICATIONS FOR LAKES AND SEAS

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Abstract. In this paper, we give an intermediate regularity result on a degenerate elliptic equation with a weight blowing up on the boundary. This kind of equations is encountoured when modelling some phenomena linked to seas or lakes. We give some examples where such regularity is useful.

1. Introduction

This paper is devoted to a degenerate elliptic equation that we can find in several models in oceanography when we consider a domain with a depth vanishing on the shore. A lot of mathematical studies in oceanography assume a domain with a strictly positive depth in order to prevent the study in weighted spaces. Few papers have been devoted to coefficients with degenerated behavior.

Regularity results in weighted Sobolev spaces on degenerate elliptic equations have been studied for instance in [2, 3] with a vanishing weight on the boundary that implies no boundary condition on the unknown. Here we study a degenerate elliptic equation with a weight with a blowing up comportment on the shore. A H^2 regularity in weighted spaces is proved allowing to consider general weights. We will obtain such regularity by a careful study of the weight adapting the standard method of translation, see [8]. For example, we use and adapt some results on weighted Sobolev spaces that have been studied in [7].

Section 2 is devoted to the regularity result related to the degenerate elliptic equation. Then in Section 3, we explain why this kind of equation is important in oceanography. In the last section we describe precise examples where such regularity is used. At first we give some examples where such regularity result is used for existence result or error estimates that means the planetary-geostrophic equations and the vertical geostrophic equations. Then we give an example where it is used in a splitting projection method. We also mention that such equation is obtained from the Great Lake equation. We remark that this degenerate elliptic equation may be found in an other field such as in electromagnetism with the maxwell's system, see [18]. Similar degenerate elliptic equation may also be encountoured for a problem

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related to Saint-Venant's equations if we want to apply Babuska-Brezzi's Inf-Sup Lemma in weighted spaces, see [1].

2. The degenerate elliptic equation

Let $\mathcal{O}$ be a two-dimensional domain. This section is devoted to a regularity result on the following degenerate elliptic equation: Given $h : \mathcal{O} \to \mathbb{R}$ and $g : \theta \to \mathbb{R}$, the problem is to find $\Psi : \mathcal{O} \to \mathbb{R}$ such that

$$-\nabla_{\mathrm{x}} \cdot (\frac{1}{h}\nabla_{\mathrm{x}}\Psi) = g\text{in } \mathcal{O}, \quad \Psi = 0 \text{ on } \partial\mathcal{O}. \tag{1}$$

Here, h is a function data satisfying

$$h \in W^{1,\infty}(\mathcal{O}), \quad h > 0 \text{ in } \mathcal{O}, \tag{2}$$

$$h(\mathrm{x}) = \varphi(\delta(\mathrm{x}))\text{in a neighbourhood of } \partial\mathcal{O} \tag{3}$$

with $\delta(\mathrm{x}) = \mathrm{dist}(\mathrm{x}, \mathcal{O})$. Moreover we assume that

$$\varphi \text{ is a non decreasing Lipschitz function}, \quad \varphi(0) = 0, \tag{4}$$

$$\exists\, c > 0 \text{ such that: } \forall s > 0,\ \Big|\frac{\varphi'(s)}{\varphi(s)}\Big| \le \frac{c}{s} \tag{5}$$

and for all $c_1, c_2 > 0$, there exist $\alpha_1, \alpha_2 > 0$ such that

$$\forall s, r > 0,\ c_1 \le \frac{s}{r} \le c_2 \Longrightarrow \alpha_1 \le \frac{\varphi(s)}{\varphi(r)} \le \alpha_2. \tag{6}$$

Remark Note that $\varphi(s) = c\, s^\alpha$, $0 < \alpha < 1$ and $c > 0$, satisfies the previous hypothesis.

Now, we define the function space

$$H(\mathcal{O}) = \{\Psi \in L^2(\mathcal{O}) : \frac{\nabla_{\mathrm{x}}\Psi}{h^{1/2}} \in (L^2(\mathcal{O}))^2, \Psi = 0 \text{ on } \partial\mathcal{O}\} \tag{7}$$

endowed with the norm

$$\|\Psi\|_{H(\mathcal{O})} = \|\frac{\nabla_{\mathrm{x}}\Psi}{h^{1/2}}\|_{(L^2(\mathcal{O}))^2}.$$

where h is defined by (2)–(3). We remark that $\|\cdot\|_{H(\mathcal{O})}$ is a norm since $\frac{1}{h} \ge c > 0$ and $\Psi = 0$ on $\partial\mathcal{O}$ implies that there exists $c > 0$ such that for all $\Psi \in H(\mathcal{O})$,

$$\|\Psi\|_{L^2(\mathcal{O})} \le c\|\frac{\nabla_{\mathrm{x}}\Psi}{h^{1/2}}\|_{(L^2(\mathcal{O}))^2}.$$

Lemma 2.1. *Let $H(\mathcal{O})$ be defined by (7) with h satisfying (2)–(6). Then $\mathcal{D}(\mathcal{O})$ is dense in $H(\mathcal{O})$.*

The proof of this lemma is similar to the proof of [7, Theorem 11.2]. Therefore, we omit it. The main result of this paper is the following

Theorem 2.2. *Let $\mathcal{O}$ be a (two-dimensional) bounded domain of class $\mathcal{C}^{\ni}$. Let g be such that $\delta h^{1/2} g \in L^2(\mathcal{O})$ with h satisfying (2)–(6). There exists a unique solution Ψ of (1) such that $\Psi \in H(\mathcal{O})$ and*

$$\|\Psi\|_{H(\mathcal{O})} \le c\|\delta h^{1/2} g\|_{L^2(\mathcal{O})}.$$

Moreover, if

$$h^{1/2} g \in L^2(\mathcal{O}) \tag{8}$$

then

$$h^{1/2}\nabla_{\mathrm{x}}(\frac{1}{h}\nabla_{\mathrm{x}}\Psi) \in (L^2(\mathcal{O}))^4, \tag{9}$$

$$\|h^{1/2}\nabla_{\mathrm{x}}(\frac{1}{h}\nabla_{\mathrm{x}}\Psi)\|_{(L^2(\mathcal{O}))^4} \leq c\|h^{1/2}g\|_{L^2(\mathcal{O})} \tag{10}$$

with c a constant depending only on the domain.

Proof. **Weak solutions:** The existence and uniqueness of weak solutions of (1) follows from the Lax-Milgram theorem, since

$$\begin{aligned}\|\Psi\|^2_{H(\mathcal{O})} = \int_{\mathcal{O}} g\Psi &\leq \|\delta h^{1/2} g\|_{L^2(\mathcal{O})}\|\frac{\Psi}{\delta h^{1/2}}\|_{L^2(\mathcal{O})}\\ &\leq c\|\delta h^{1/2}g\|_{L^2(\mathcal{O})}\|\frac{\nabla_{\mathrm{x}}\Psi}{h^{1/2}}\|_{(L^2(\mathcal{O}))^2}.\end{aligned}$$

In the previous estimate we have used Hardy's inequality in weighted space which will be proved in Lemma 2.3.

Regularity: We use the usual difference quotients (cf. Brezis [8]). The interior regularity is well known since $h \geq c(\omega) > 0$ in each $\omega \Subset \mathcal{O}$. To obtain the regularity result up to the boundary, we define a local diffeomorphism T which preserves the normal direction. More precisely we define the local diffeomorphism $T : Q \to V$ by $T(x^*, r) = (x^*, \alpha(x^*)) + r\, n(x^*, \alpha(x^*))$ for all $(x^*, r) \in Q$ where $\partial\mathcal{O}$ is locally the graph of a $\mathcal{C}^3$ function α (see Figure 1). This property, combined with the hypothesis (3) of h, will be strongly used in the sequel. We also define a cut-off function θ such that

$$\begin{gathered}\theta \in \mathcal{C}^2 \text{ in } V,\\ \theta = 0 \text{ on } \mathbb{R}^2\backslash V, \quad \theta = 1 \text{ in } \mathcal{V}_1,\\ \frac{\partial\theta}{\partial\widetilde{n}} = 0 \text{ on a neighbourhood of } \partial\mathcal{O}\cap V,\end{gathered}$$

where $\mathcal{V}_1 \Subset V$ and the extension $\widetilde{n}$ of the normal n is defined for all $(x, y) \in V$ by

$$\widetilde{n}(x, y) = n(T(x^*, 0))$$

where $(x, y) = T(x^*, r)$, $(x^*, r) \in Q$.

remark Due to the $\mathcal{C}^3$ regularity of $\partial\mathcal{O}$, T is a $\mathcal{C}^2$ diffeormorphism from Q onto a neighbourhood of (x_0, y_0) denoted by V.

Multiplying (1) by θ, and denoting $\xi = \theta\Psi$, it follows that ξ is the (unique) solution in $H(V \cap \mathcal{O})$ of

$$\begin{gathered}-\nabla_{\mathrm{x}} \cdot (\frac{1}{h}\nabla_{\mathrm{x}}\xi) = f \text{ in } V\cap\mathcal{O},\\ \xi|_{\partial(V\cap\mathcal{O})} = 0,\end{gathered} \tag{11}$$

where $f = \theta g + \frac{\nabla_{\mathrm{x}}h\cdot\nabla_{\mathrm{x}}\theta}{h^2}\Psi - \frac{1}{h}\Delta_{\mathrm{x}}\theta\Psi - \frac{2}{h}\nabla_{\mathrm{x}}\theta\cdot\nabla_{\mathrm{x}}\Psi$ (since $h^{1/2}f \in L^2(V\cap\mathcal{O})$). For this, it suffices to check that

$$\frac{\nabla_{\mathrm{x}}h\cdot\nabla_{\mathrm{x}}\theta}{h^{3/2}}\Psi \in L^2(V\cap\mathcal{O}).$$

Indeed, since $\partial\theta/\partial\widetilde{n} = 0$ on a neighbourhood of $\partial\mathcal{O}\cap V$, then

$$\nabla_{\mathrm{x}}h\cdot\nabla_{\mathrm{x}}\theta = \frac{\partial h}{\partial\widetilde{\tau}}\frac{\partial\theta}{\partial\widetilde{\tau}} = 0$$

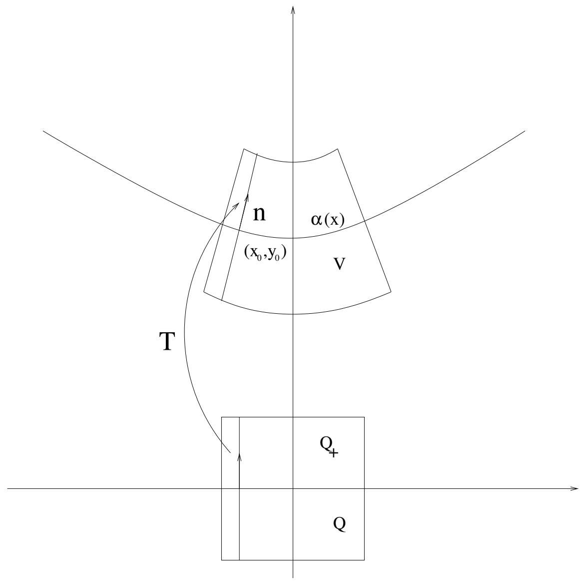

FIGURE 1. The local diffeomorphism T

on a neighbourhood of $\partial\mathcal{O}\cap V$. We recall that h doesn't depend on $\widetilde{\tau}$ (where $\widetilde{\tau}$ is defined as $\widetilde{n}$) using that h is given by (3) and using the definition of T.

Now we use the difference quotient technic on (11) to deduce the weight regularity announced in the theorem. Let $\varphi:\mathcal{O}\to\mathbb{R}^n$ ($n\geq 1$) be a function. We denote, for all $(x^*,r)\in Q_+$,

$$\widetilde{\varphi}(x^*,r)=\varphi(T(x^*,r)),$$

$$a_{kl}=\sum_{j=1}^{2}\partial_j T_k^{-1}(T(x^*,r))\partial_j T_l^{-1}(T(x^*,r))|\operatorname{Jac}T(x^*,r)|,\quad k,l=1,2,$$

$$\hat{k}(x^*,r)=\widetilde{f}(x^*,r)|\operatorname{Jac}T(x^*,r)|.$$

Then we get that $\widetilde{\xi}$ is the unique solution in $\widetilde{H}(Q_+)$ of:

$$\int_{Q_+}\sum_{k,l}\frac{a_{kl}}{\widetilde{h}}\,\partial_k\widetilde{\xi}\,\partial_l\widetilde{\varphi}dx^*dr=\int_{Q_+}\hat{k}\widetilde{\varphi}dx^*dr \tag{12}$$

for all $\widetilde{\varphi}\in\widetilde{H}(Q_+)$ where

$$\widetilde{H}(Q_+)=\{\widetilde{\varphi}\in L^2(Q_+):\widetilde{h}^{-1/2}\nabla_{\mathrm{x}}\widetilde{\varphi}\in(L^2(Q_+))^2,\widetilde{\varphi}=0\text{ on }\partial Q_+\}.$$

We choose $\widetilde{\varphi}=D_{-\tau}(D_\tau\widetilde{\xi})$ with $\tau=|\tau|e_1$, $D_\tau\widetilde{\xi}=\big(\widetilde{\xi}(x+\tau)-\widetilde{\xi}(x)\big)/|\tau|$ and $|\tau|$ small enough in order to obtain $\widetilde{\varphi}\in H(Q_+)$. Using

$$\|\frac{1}{\widetilde{h}^{1/2}}D_{-\tau}(D_\tau\widetilde{\xi})\|_{L^2(Q_+)}\leq c\|\frac{1}{\widetilde{h}^{1/2}}\nabla_{\mathrm{x}}(D_\tau\widetilde{\xi})\|_{L^2(Q_+)}$$

and

$$\|\widetilde{h}^{1/2}\hat{k}\|_{L^2(Q_+)}\leq c\|h^{1/2}g\|_{L^2(\mathcal{O})}$$

we get

$$\|\hat{k}D_{-\tau}(D_\tau\widetilde{\xi})\|_{L^1(Q_+)} \le c\|h^{1/2}g\|_{L^2(\mathcal{O})}\|\frac{1}{\widetilde{h}^{1/2}}\nabla_{\rm x}(D_\tau\widetilde{\xi})\|_{L^2(Q_+)}. \tag{13}$$

Moreover, denoting

$$I = \sum_{k,l}\int_{Q_+} D_\tau(\frac{1}{\widetilde{h}}a_{kl}\partial_k\widetilde{\xi})\partial_l(D_\tau\widetilde{\xi})$$

since $D_\tau(a_{kl}/\widetilde{h}) = D_\tau(a_{kl})/\widetilde{h}$, (recall that $\widetilde{h}$ does not depend on τ) and $T \in \mathcal{C}^2$, we have

$$\begin{aligned} I &\ge c\|\frac{1}{\widetilde{h}^{1/2}}\nabla_{\rm x}(D_\tau\widetilde{\xi})\|^2_{(L^2(Q_+))^2} - c\|\frac{1}{\widetilde{h}^{1/2}}\nabla_{\rm x}\widetilde{\xi}\|_{L^2}\|\frac{1}{\widetilde{h}^{1/2}}\nabla_{\rm x}(D_\tau\widetilde{\xi})\|_{(L^2(Q_+))^2} \\ &\ge c\|\frac{1}{\widetilde{h}^{1/2}}\nabla_{\rm x}(D_\tau\widetilde{\xi})\|^2_{(L^2(Q_+))^2} - c\|h^{1/2}g\|_{L^2(\mathcal{O})}\|\frac{1}{\widetilde{h}^{1/2}}\nabla_{\rm x}(D_\tau\widetilde{\xi})\|_{(L^2(Q_+))^2} \end{aligned} \tag{14}$$

Using the variational formulation satisfied by $\widetilde{\xi}$, (13) and (14) we get

$$\|\frac{1}{\widetilde{h}^{1/2}}\nabla_{\rm x}(D_\tau\widetilde{\xi})\|_{(L^2(Q_+))^2} \le c\|h^{1/2}g\|_{L^2(\mathcal{O})}. \tag{15}$$

Thus, by classical arguments,

$$\frac{\partial_1^2\widetilde{\xi}}{\widetilde{h}^{1/2}} \in L^2(Q_+) \quad \text{and} \quad \frac{\partial_2\partial_1\widetilde{\xi}}{\widetilde{h}^{1/2}} \in L^2(Q_+), \tag{16}$$

and their respective norms are bounded by $c\|h^{1/2}g\|_{L^2(\mathcal{O})}$. In particular,

$$\widetilde{h}^{1/2}\partial_1(\widetilde{h}^{-1}\partial_1\widetilde{\xi}) \in L^2(Q_+), \quad \widetilde{h}^{1/2}\partial_1(\widetilde{h}^{-1}\partial_2\widetilde{\xi}) \in L^2(Q_+) \tag{17}$$

and their respective norms depend continuously on $h^{1/2}g$ in $L^2(\mathcal{O})$.

We remark that contrary to the homogeneous case, that means the standard Laplacian operator, we have not yet the regularity $\widetilde{h}^{1/2}\partial_2(\widetilde{h}^{-1}\partial_1\widetilde{\xi}) \in L^2(Q_+)$. We will obtain such regularity using the hypothesis (5) on h. Indeed, in the distribution sense,

$$\widetilde{h}^{1/2}\partial_2(\frac{1}{\widetilde{h}}\partial_1\widetilde{\xi}) = \frac{-\partial_2\widetilde{h}}{\widetilde{h}^{3/2}}\partial_1\widetilde{\xi} + \frac{1}{\widetilde{h}^{1/2}}\partial_2\partial_1\widetilde{\xi}. \tag{18}$$

Since $\partial_1\widetilde{\xi} = 0$ on ∂Q_+ then (16) yields $\partial_1\widetilde{\xi} \in H(Q_+)$. Using the Hardy's inequality (21) and Hypothesis (5), we get

$$\int_{Q_+}\Big|\frac{\partial_2\widetilde{h}}{\widetilde{h}^{3/2}}\partial_1\widetilde{\xi}\Big|^2 \le c\int_{Q_+}\frac{|\partial_1\widetilde{\xi}|^2}{\widetilde{\delta}^2\widetilde{h}} \le c\int_{Q_+}\frac{|\partial_2\partial_1\widetilde{\xi}|^2}{\widetilde{h}}.$$

Thus, using the regularity (16), we get from (18)

$$\begin{gathered} \widetilde{h}^{1/2}\partial_2(\frac{1}{\widetilde{h}}\partial_1\widetilde{\xi}) \in L^2(Q_+), \\ \|\widetilde{h}^{1/2}\partial_2(\partial_1\widetilde{\xi}/\widetilde{h})\|_{(L^2(Q_+))^2} \le c\|h^{1/2}g\|_{L^2(\mathcal{O})}. \end{gathered} \tag{19}$$

Now we use the variational formulation (12) satisfied by $\widetilde{\xi}$ to obtain the regularity on $\widetilde{h}^{1/2}\partial_2(\partial_2\widetilde{\xi}/\widetilde{h})$. We have

$$\Big|\int_{Q_+}\frac{a_{22}}{\widetilde{h}}\partial_2\widetilde{\xi}\partial_2\Phi\Big| \le c\|h^{1/2}g\|_{L^2(\mathcal{O})}\|\frac{1}{\widetilde{h}^{1/2}}\Phi\|_{L^2(Q_+)}$$

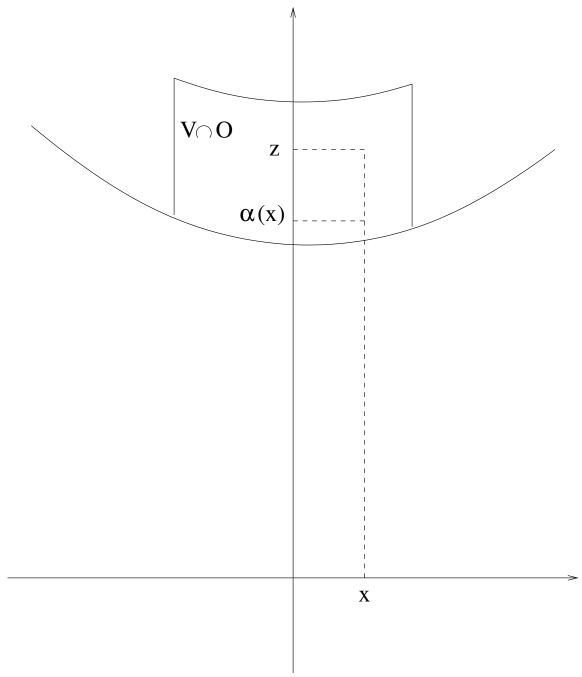

FIGURE 2. The local coordinates

for all $\Phi \in \mathcal{D}(Q_+)$. Using now the weak regularity of $\widetilde{\xi}$ and $a_{22} \geq c > 0$ in Q_+, this gives

$$\widetilde{h}^{1/2}\partial_2(\frac{1}{\widetilde{h}}\partial_2\widetilde{\xi}) \in L^2(Q_+). \tag{20}$$

Therefore, (17), (19) and (20) give the regularity (9). □

Lemma 2.3 (Hardy's inequality in weighted spaces)**.** *Let h satisfy* (2)–(6) *and let* $\Psi \in H(\mathcal{O})$*. Then*

$$\|\frac{\Psi}{\delta h^{1/2}}\|_{L^2(\mathcal{O})} \leq c\|\frac{\nabla_x \Psi}{h^{1/2}}\|_{L^2(\mathcal{O})^2} \tag{21}$$

where c depends only on $\mathcal{O}$.

Proof. The proof of this lemma is similar to the proof of the classical Hardy's inequality (see for instance [14]) introducing the corresponding weight. By density, it suffices to consider $\Psi \in \mathcal{D}(\mathcal{O})$. The interior estimate is obvious. In the local coordinates (see Figure 2), we write

$$
\begin{aligned}
&\int_{\alpha(x)}^{z} \frac{|\Psi(x,y)|^2 dy}{|y-\alpha(x)|^2\varphi(y-\alpha(x))}\\
&\le 2\int_{\alpha(x)}^{z}\Big(\int_{y}^{+\infty}\frac{dt}{|t-\alpha(x)|^2\varphi(t-\alpha(x))}\Big)\Psi(x,y)\partial_y\Psi(x,y)dy\\
&\le 2\int_{\alpha(x)}^{z}\frac{|\Psi(x,y)||\partial_y\Psi(x,y)|}{|y-\alpha(x)|\varphi(y-\alpha(x))}dy\\
&\le 2\Big(\int_{\alpha(x)}^{z}\frac{|\Psi(x,y)|^2}{|y-\alpha(x)|^2\varphi(y-\alpha(x))}dy\Big)^{1/2}\Big(\int_{\alpha(x)}^{z}\frac{|\partial_y\Psi(x,y)|^2}{\varphi(y-\alpha(x))}dy\Big)^{1/2}.
\end{aligned}
$$

Therefore,

$$
\int_{\alpha(x)}^{z}\frac{|\Psi(x,y)|^2}{|y-\alpha(x)|^2\varphi(y-\alpha(x))}\,dy \le 4\int_{\alpha(x)}^{z}\frac{|\partial_y\Psi(x,y)|^2}{\varphi(y-\alpha(x))}\,dy.
$$

Thus, integrating with respect to x, we get

$$
\int_{V\cap\mathcal{O}}\frac{|\Psi(x,z)|^2}{|z-\alpha(x)|^2\varphi(z-\alpha(x))}\,dz\,dx \le 4\int_{V\cap\mathcal{O}}\frac{|\partial_y\Psi|^2}{\varphi(\xi-\alpha(x))}\,d\xi\,dx.
$$

Since α is smooth enough, there exists $c>1$ such that, for all $(x,z)\in V\cap\mathcal{O}$,

$$
\delta(x,z)\le|z-\alpha(x)|\le c\delta(x,z).
$$

Therefore, using (5)–(6), we get

$$
\int_{V\cap\mathcal{O}}\frac{|\Psi|^2}{\delta^2\varphi(\delta)}\le c\int_{V\cap\mathcal{O}}\frac{|\partial_y\Psi|^2}{\varphi(\delta)}
$$

and the result follows. □

3. Importance of this degenerate equation

Let us introduce the three-dimensional oceanographic domain

$$
\Omega=\{(\mathrm{x},z)\in\mathbb{R}^3:\mathrm{x}=(x,y)\in\mathcal{O},-h(\mathrm{x})<z<0\}
$$

with $\mathcal{O}\subset\mathbb{R}^2$ the surface domain and $h:\overline{\mathcal{O}}\to\mathbb{R}$ with $h>0$ in $\mathcal{O}$, the bottom function. Moreover, $\Gamma_s=\overline{\mathcal{O}}\times\{0\}$ is the surface boundary and $\Gamma_b=\partial\Omega\setminus\Gamma_s$ the bottom.

We denote $\nabla=(\nabla_{\mathrm{x}},\partial_z)$ the three dimensional gradient vector (with $\nabla_{\mathrm{x}}=(\partial_x,\partial_y)$ the vectorial horizontal part) and Δ is the Laplace operator. We explain, in this section, why such degenerate equation naturally appears in different models issued from oceanography when hydrostatic pressure is assumed. In all these equations, the field $u=(v,w)$ and the pressure p satisfy the equation

$$
\begin{aligned}
Lv+\nabla_{\mathrm{x}}p=f,\quad \partial_z p=0 &\quad \text{in } \Omega,\\
\partial_z w=-\operatorname*{div}_x v &\quad \text{in } \Omega,
\end{aligned}
\tag{22}
$$

and at least one of the the boundary conditions

$$
(v,w)\cdot n_{\partial\Omega}=0,\quad \overline{v}\cdot n_{\partial\mathcal{O}}=0 \tag{23}
$$

where we use the notation $\overline{v}=\int_{-h}^{0}v\,dz$. We note that L is a certain operator (algebraic or differential), see (30), (32) or (37) for some examples.

Remark Of course Boundary conditions (23) are not necessary or sufficient to solve System (22). We have to choose other boundary conditions following the choice of the operator L.

Integrating the divergence free equation with respect to the vertical coordinate and using the boundary condition (23), part 1, we obtain

$$\nabla_{\mathrm{x}} \cdot \overline{v} = 0 \text{ in } \mathcal{O}. \tag{24}$$

If the domain is simply connected then, using (24), there exists a stream function Ψ such that

$$\overline{v} = \nabla_{\mathrm{x}}^{\perp} \Psi \text{ in } \mathcal{O}, \tag{25}$$

where $\nabla_x^{\perp}$ is the $2D$ curl operator, i.e., $(-\partial_y, \partial_x)$. The boundary condition (23), part 2, gives

$$\Psi = 0 \quad \text{on} \partial\mathcal{O}. \tag{26}$$

We assume that v maybe formally written as

$$v = A\nabla_{\mathrm{x}} p + g_1 \quad \text{ in } \Omega. \tag{27}$$

where A is a matrix function (see the examples below). The purpose is to obtain some regularity result on v. Integrating (27) with respect to z (taking into account that $\partial_z p = 0$ in Ω), we obtain

$$\overline{v} = \overline{A}\nabla_{\mathrm{x}} p + \overline{g}_1,$$

where $\overline{A} = \int_{-h}^0 A$ and $\overline{g}_1 = \int_{-h}^0 g_1 =$. Therefore, using (25), we obtain

$$\nabla_{\mathrm{x}}^{\perp} \Psi = \overline{A}\nabla_{\mathrm{x}} p + \overline{g}_1$$

and thus, assuming $\overline{A}$ invertible

$$\nabla_{\mathrm{x}} p = B(\nabla_{\mathrm{x}}^{\perp} \Psi - \overline{g}_1) \tag{28}$$

where $B = (\overline{A})^{-1}$. Taking the horizontal curl operator of (28), using that $\nabla_{\mathrm{x}}^{\perp} \cdot \nabla_{\mathrm{x}} = 0$, we get

$$\nabla_{\mathrm{x}}^{\perp} \cdot (B(\nabla_{\mathrm{x}}^{\perp} \Psi - \overline{g}_1)) = 0 \text{ in } \mathcal{O}, \quad \Psi = 0 \text{ on } \partial{=}\, \mathcal{O}. \tag{29}$$

On the other-hand, (27) and (28) yield

$$\nabla_{\mathrm{x}} v = \nabla_{\mathrm{x}} \Big(A\big(B(\nabla_{\mathrm{x}}^{\perp} \Psi - \overline{g}_1)\big) \Big) + \nabla_{\mathrm{x}} g_1.$$

Thus the regularity of $\nabla_{\mathrm{x}} v$ depends on the regularity of Ψ and g_1. Theorem 2.2 may be extended easily to more general degenerate elliptic equations including for instance (29).

Now assume that $A = \mathrm{Id}$ then we get $B = 1/h\, \mathrm{Id}$ and therefore $A\big(B(\nabla_{\mathrm{x}}^{\perp} \Psi)\big) = \nabla^{\perp} \Psi / h$. Then the regularity of $\nabla_{\mathrm{x}} v$ in $(L^2(\Omega))^4$ is given by the regularity of Ψ

$$h^{1/2} \nabla_{\mathrm{x}} (h^{-1} \nabla_{\mathrm{x}} \Psi) \in (L^2(\mathcal{O}))^4$$

deduced from Theorem 2.2.

Let us give now some applications of such regularity results on the stream function.

4. Some applications for lakes and seas

We consider again, in the three first examples, the three-dimensional domain

$$\Omega = \{(\mathrm{x}, z) \in \mathbb{R}^3 : \mathrm{x} = (x, y) \in \mathcal{O}, \ -h(\mathrm{x}) < z < 0\}.$$

4.1. **Planetary geostrophic equation.** Let us consider the hyperviscous parametrization for the planetary equations given by

$$\begin{gathered}\varepsilon_H v + f v^{\perp} + \nabla_{\mathrm{x}} p = 0, \quad \partial_z p = +T = 0, \\ \operatorname*{div}_{\mathrm{x}} v + \partial_z w = 0, \quad (v, w) \cdot n_{\partial\Omega} = 0, \quad \overline{v} \cdot n_{\partial\mathcal{O}} = 0, \\ \partial_t T - K_h \Delta_{\mathrm{x}} T - K_v \partial_z^2 T + v \cdot \nabla_{\mathrm{x}} T + w \partial_z T + \gamma\, \mathcal{D} T = Q,\end{gathered} \tag{30}$$

with $f = (1 + \beta y)$ where β is a constant, with suitable bounda= ry and initial conditions and $\mathcal{D}$ a suitable fourth order differential operator. This system has been studied in [6] using some relations between T and $u = (v, w)$ jointly with the regularity result proved in Theorem 2.2. Here, we only recall how to get the estimate on $\mathrm{div}_{\mathrm{x}} v$ which is necessary to obtain the existence result for (30) (the regularity result proved in Theorem 2.2 is used there).

Let (v, p, T) be an approximate sequence of (30) built for instance using a Galerkin method. Then we get the following estimate.

Theorem 4.1. *Let $\mathcal{O}$ be a bounded simply connected domain of class $\mathcal{C}^3$ and h satisfied the hypothesis of Theorem 2. Then*

$$\|\mathrm{div}_{\mathrm{x}} v\|_{L^2(\Omega)} \le c\Big(\|\mathcal{D}^{1/2} T\|_{L^2(\Omega)} + \|\nabla_{\mathrm{x}} T\|_{(L^2(\Omega))^2}\Big).$$

where c depends only on Ω.

This estimate allows to pass to the limit on the approximate solutions and to get an existence result of weak solutions for (30).

Proof. The hydrostatic equation $\partial_z p + T = 0$ reads $p = p_s + \int_z^0 T$ with $p_s = p_s(\mathrm{x})$. Then,

$$\widetilde{M} v + \nabla_{\mathrm{x}} p_s = \nabla_{\mathrm{x}} \int_z^0 T \tag{31}$$

with $\widetilde{M} = \begin{pmatrix} \varepsilon_H & -f \\ f & \varepsilon_H \end{pmatrix}$. Using the same procedure than in Section 3, we prove that there exists Ψ such that $\overline{v} = \nabla_{\mathrm{x}}^{\perp} \Psi$. The stream function Ψ satisfies a degenerate elliptic equations similar to (1) and therefore it is possible to prove that

$$h^{1/2} \nabla_{\mathrm{x}} (\frac{1}{h} \nabla_{\mathrm{x}} \Psi) \in (L^2(\mathcal{O}))^4$$

using Theorem 2.2 and the weak regularity satisfied by T. This regularity result implies the required estimate on $\mathrm{div}_{\mathrm{x}}\, v$ using (31) and the relation

$$-\nabla_{\mathrm{x}} p_s = \frac{\widetilde{M}}{h} \nabla_{\mathrm{x}}^{\perp} \Psi + \frac{1}{h} \int_{-h}^0 \nabla_{\mathrm{x}} (\int_z^0 T)$$

coming from the hydrostatic approximation. The suitable fourth order differential operator is obtained using the relation between v and T. □

4.2. **Vertical-geostrophic equations.** We recall here the equations studied in [5] in domains with vanishing depth. These equations may also be obtained in lubrication and thin film theory. We consider that the flow satisfies the vertical-geostrophic equations

$$\begin{gathered}
-\partial_z^2 v + k v^\perp + \nabla_{\mathrm{x}} p = 0, \quad \partial_z p = 0, \\
\operatorname*{div}_x v + \partial_z w = 0, \\
\partial_z v = g, \quad w = 0 \text{ on } \Gamma_s, \\
(v, w) = 0 \text{ on } \Gamma_b, \quad \overline{v} \cdot n_{\partial\mathcal{O}} = 0.
\end{gathered} \tag{32}$$

If $k = 0$, integrating two times with respect to z, we prove that (v, w) is given by

$$v = \frac{1}{2}(z^2 - h^2)\nabla_{\mathrm{x}} p + (z + h) g, \quad w = -\int_0^z \operatorname{div}_{\mathrm{x}} v. \tag{33}$$

Similar formula may be obtained if $k \neq 0$, see for instance [5]. Using the same lines than in Section 3, there exists Ψ such that

$$\nabla_{\mathrm{x}}^\perp \Psi = \frac{h^3}{3}\nabla_{\mathrm{x}} p + \frac{h^2}{2} g,$$

and therefore we get, from (33),

$$v = \frac{3}{2\,h^3}(z^2 - h^2)\nabla_{\mathrm{x}}^\perp \Psi + (z + h) g - \frac{3}{4h}(z^2 - h^2) g, \quad w = -\int_0^z \operatorname{div}_{\mathrm{x}} v.$$

Using the regularity of the stream function Ψ associated to $\overline{v}$, that implies

$$h^{3/2}\nabla_{\mathrm{x}}\Big(\frac{1}{h^3}\nabla_{\mathrm{x}}\Psi\Big) \in (L^2(\mathcal{O}))^4.$$

This gives a regularity result of (v, w) in $(H^1(\Omega))^3$ assuming that g is smooth enough, h satisfying the hypothesis of Theorem 2.2 and $h \in W^{2,\infty}$. We note that $\operatorname{div}_{\mathrm{x}} v$ depends only on the first derivative of Ψ.

4.3. **Splitting-projection methods for the hydrostatic Navier-Stokes equations.** In many geophysical fluids it is natural to assume hydrostatic pressure and the "rigid lid" hypothesis (see [15]), hence the 3D Navier-Stokes equations gives to the hydrostatic Navier-Stokes equations, see [12, 13]. For instance, these equations model the general circulation in lakes and oceans. For simplicity, we impose constant density, cartesian coordinates and temperature and salinity effects decoupled of the dynamic flow. This gives the hydrostatic Navier-Stokes equations:

$$\begin{gathered}
\partial_t v + (u \cdot \nabla) v - \nu\,\Delta v + f v^\perp + \nabla_{\mathrm{x}}\, p_s = \mathrm{F} \text{ in } \Omega \times (0, T), \\
w(t; \mathrm{x}, z) = \int_z^0 \operatorname{div}_{\mathrm{x}} v(t; \mathrm{x}, s)\, ds, \quad \operatorname{div}_{\mathrm{x}} \overline{v} = 0 \text{ in } \mathcal{O} \times (0, T), \\
v|_{\Gamma_b} = 0, \quad \nu \partial_z v|_{\Gamma_s} = \mathrm{g}_s, \quad v_{|t=0} = v_0 \text{ in } \Omega.
\end{gathered} \tag{34}$$

The unknowns of this problem are: $u = (v, w) : \Omega \times (0, T) \to \mathbb{R}^3$ the 3D velocity field (with $v = (v_1, v_2)$ the corresponding horizontal velocity field) and $p_s : \mathcal{O} \times (0, T) \to \mathbb{R}$ a potential function, defined only on the surface $\mathcal{O}$. The term $f v^\perp$ represents the Coriolis forces, being f a constant (depending on the angular velocity of the earth and the latitude), $\mathrm{F} : \Omega \times (0, T) \to \mathbb{R}^2$ are the horizontal external forces, $\mathrm{g}_s : \Gamma_s \times (0, T) \to \mathbb{R}^2$ models the friction effects on the surface and $v_0 : \Omega \to \mathbb{R}^2$ is the initial velocity.

(i) Description of the splitting projection scheme,[9, 10]**.** We consider a regular partition of the time interval $[0,T]$ by M subintervals of length $k = T/M$, hence we have the nodes $\{t_m = m\,k\}_{m=0,\dots,M}$.

Initialization: Given $v^0 \simeq v_0$, to compute $w^0 = \int_z^0 \mathrm{div}_{\mathrm{x}} v^0$ in Ω.

Time Step m**:**

Sub-step 1: Given $u^{m-1} = (v^{m-1}, w^{m-1})$, to find $\widetilde{v^m} : \Omega \to \mathbb{R}^2$ such that

$$\begin{gathered}\frac{1}{k}(v^m - v^{m-1}) + (u^{m-1}\cdot\nabla)\widetilde{v^m} - \nu\Delta\widetilde{v^m} + f(\widetilde{v^m})^\perp = \mathrm{f}^m \quad \text{in } \Omega,\\ \nu\partial_z\widetilde{v^m}|_{\Gamma_s} = \mathrm{g}_s^m, \quad \widetilde{v^m}|_{\Gamma_b} = 0.\end{gathered}\tag{35}$$

Sub-step 2: Given $\widetilde{v^m}$, to find $v^m : \Omega \to \mathbb{R}^2$ and $p_s^m : \mathcal{O} \to \mathbb{R}$ such that

$$\begin{gathered}\frac{1}{k}(v^m - \widetilde{v}^m) + \nabla_{\mathrm{x}}\, p_s^m = 0 \quad \text{in } \Omega,\\ \operatorname*{div}_{\mathrm{x}} \overline{v^m} = 0 \quad \text{in } \mathcal{O}, \quad \overline{v^m}\cdot n_{\partial\mathcal{O}} = 0 \text{ on } \partial\mathcal{O}.\end{gathered}\tag{36}$$

Sub-step 3: Given v^m, to compute $w^m(\mathrm{x},z) = \int_z^0 \mathrm{div}_{\mathrm{x}} v^m(\mathrm{x},s)\,ds$.

Note that sub-step 2 is equivalent to an elliptic-Neuman prob= lem in $\mathcal{O}$ for p_s^m (which degenerates when $h = 0$ on $\partial\mathcal{O}$) and the explicit relation $v_m = \widetilde{v}_m - k\nabla_x p_s^m$. Indeed, integrating with respect top z from $-h$ to 0, we get

$$\overline{v^m} - \overline{\widetilde{v}^m} + kh\nabla_x p_s^m = 0 \text{ in } \mathcal{O}.\tag{37}$$

Hence, taking div_x and multiplying by $n_{\partial\mathcal{O}}$,

$$k\operatorname*{div}_x(h\nabla_x p_s^m) = \operatorname*{div}_x \overline{\widetilde{v}^m} \text{ in } \mathcal{O}, \quad h\nabla_x p_s^m\cdot n_{\partial\mathcal{O}} = 0.$$

(ii) A Coriolis correction scheme. In [9], a variant of this scheme is considered, where one has Coriolis in an explicit way in the first sub-step (changing $(\widetilde{v}^m)^\perp$ by $(v^{m-1})^\perp$ in (35), but with a correction of the Coriolis terms in the second sub-step, changing (36) by

$$\begin{gathered}\frac{1}{k}(v^m - \widetilde{v}^m) + f(v^m - v^{m-1})^\perp + \nabla_{\mathrm{x}}\, p_s^m = 0 \quad \text{in } \Omega,\\ \nabla_{\mathrm{x}}\cdot\overline{v^m} = 0 \quad \text{in } \mathcal{O}, \quad \overline{v^m}\cdot n_{\partial\mathcal{O}} = 0 \quad \text{on } \partial\mathcal{O}.\end{gathered}\tag{38}$$

Stability of these schemes. It is well known in the Navier-Stokes framework (see for instance [16], [17]), that splitting-projection schemes are stables in the H^1-norm for both velocities $\widetilde{v}^m$ and v^m. In order to obtain stability in $H^1(\Omega)$ for v^m in the hydrotatic Navier-Stokes equations (that is fundamental to prove the convergence of the scheme) it is necessary to get the $H^2(\Omega)$ stability for the pressure p_s^m. Now, in the Primitive Equations framework, taking into account that p_s^m does not depend on z, this $H^2(\Omega)$ regularity is equivale= nt to a weight $H^2(\mathcal{O})$ regularity depending on the bathymetry h. This H^2 weighted regularity can be deduced from Theorem 2.2, see [9] following the steps described in Section 3.

4.4. **Great-Lake equations.** This example is not an illustration of Section 3 but, as we will see, we obtain the same degenerate equation on the stream function. Let us give here an equation studied in [11] in a domain with a depth with sidewalls. This system will be studied in [4] with a depth vanishing on the shore.

We assume that the mean flow (u, p) satisfies in a simply connected two-dimensional space domain $\mathcal{O}$, the following system

$$\begin{gathered}\partial_t u + u \cdot \nabla_{\mathrm{x}} u + \nabla_{\mathrm{x}} p = 0 \text{ in } \mathcal{O},\\ \nabla_{\mathrm{x}} \cdot (hu) = 0 \text{ in } \mathcal{O},\\ hu \cdot n = 0 \text{ on } \partial\mathcal{O},\\ u(0) = u_0.\end{gathered}$$

Denoting $\omega = \mathrm{curl}_{\mathrm{x}}\, u/h$, and using the equation $\nabla_{\mathrm{x}} \cdot (hu) = 0$, we can easily prove that the system is equivalent to

$$\begin{gathered}\partial_t \omega + u \cdot \nabla_{\mathrm{x}} \omega = 0, \quad u = \nabla_{\mathrm{x}}^{\perp} \Psi / h \text{ in } \mathcal{O},\\ \nabla_{\mathrm{x}} \cdot (\frac{1}{h} \nabla_{\mathrm{x}} \Psi) = h\omega \text{ in } \mathcal{O}, \quad \Psi = 0 \text{ on } \partial\mathcal{O},\\ u(0) = u_0.\end{gathered} \tag{39}$$

We obtain again the degenerate equation on Ψ that we have studied in Section 2. Now, the "H^2 regularity" obtained in Section 2 is not enough to follow the study of Youdovitch made on the standard Euler equations. An "L^r regularity" in weighted spaces is required. This result remains as an interesting open question.

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