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CONVERGENCE RESULTS FOR A CLASS OF ABSTRACT CONTINUOUS DESCENT METHODS

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Abstract. We study continuous descent methods for the minimization of Lipschitzian functions defined on a general Banach space. We establish convergence theorems for those methods which are generated by approximate solutions to evolution equations governed by regular vector fields. Since the complement of the set of regular vector fields is σ-porous, we conclude that our results apply to most vector fields in the sense of Baire's categories.

1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space, $(X^*, \|\cdot\|_*)$ its dual space, and let $f : X \to \mathbb{R}^1$ be a function which is bounded from below and Lipschitzian on bounded subsets of X. Recall that for each pair of sets $A, B \subset X^*$,

$$H(A,B) = \max\{\sup_{x\in A}\inf_{y\in B}\|x-y\|_*,\ \sup_{y\in B}\inf_{x\in A}\|x-y\|_*\}$$

is the Hausdorff distance between A and B. For each $x \in X$, let

$$f^0(x,h) = \limsup_{t\to 0^+,\ y\to x}[f(y+th)-f(y)]/t, \quad h \in X, \tag{1.1}$$

be the Clarke generalized directional derivative of f at the point x, let

$$\partial f(x) = \{l \in X^* : f^0(x,h) \ge l(h) \text{ for all } h \in X\} \tag{1.2}$$

be Clarke's generalized gradient of f at x, and set

$$\Xi(x) = \inf\{f^0(x,h) : h \in X \text{ and } \|h\| = 1\}. \tag{1.3}$$

It is well known that $\partial f(x)$ is nonempty and bounded. Set

$$\inf(f) = \inf\{f(x) : x \in X\}.$$

Denote by $\mathcal{A}$ the set of all mappings $V : X \to X$ such that V is bounded on every bounded subset of X, and for each $x \in X$, $f^0(x, Vx) \le 0$. We denote by $\mathcal{A}_c$ the set of all continuous $V \in \mathcal{A}$ and by $\mathcal{A}_b$ the set of all $V \in \mathcal{A}$ which are bounded on X. Finally, let $\mathcal{A}_{bc} = \mathcal{A}_b \cap \mathcal{A}_c$. Next we endow the set $\mathcal{A}$ with two metrics, ρ_s and ρ_w. To define ρ_s, we first set, for each $V_1, V_2 \in \mathcal{A}$,

$$\tilde\rho_s(V_1,V_2) = \sup\{\|V_1x - V_2x\| : x \in X\}$$

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and then let

$$\rho_s(V_1, V_2) = \tilde{\rho}_s(V_1, V_2)(1 + \tilde{\rho}_s(V_1, V_2))^{-1}. \tag{1.4}$$

(Here we use the convention that $\infty/\infty = 1$.) Clearly, $(\mathcal{A}, \rho_s)$ is a complete metric space. To define ρ_w, we first set, for each $V_1, V_2 \in \mathcal{A}$ and each integer $i \geq 1$,

$$\rho_i(V_1, V_2) = \sup\{\|V_1x - V_2x\| : x \in X \text{ and } \|x\| \leq i\} \tag{1.5}$$

and then let

$$\rho_w(V_1, V_2) = \sum_{i=1}^{\infty} 2^{-i}[\rho_i(V_1, V_2)(1 + \rho_i(V_1, V_2))^{-1}]. \tag{1.6}$$

Clearly, $(\mathcal{A}, \rho_w)$ is a complete metric space. It is also not difficult to see that the collection of the sets

$$E(N, \epsilon) = \{(V_1, V_2) \in \mathcal{A} \times \mathcal{A} : \|V_1x - V_2x\| \leq \epsilon,\ x \in X,\ \|x\| \leq N\},$$

where $N, \epsilon > 0$, is a base for the uniformity generated by the metric ρ_w. It is easy to see that

$$\rho_w(V_1, V_2) \leq \rho_s(V_1, V_2) \quad \text{for all } V_1, V_2 \in \mathcal{A}.$$

The metric ρ_w induces on $\mathcal{A}$ a topology which is called the weak topology and ρ_s induces a topology which is called the strong topology. Clearly, $\mathcal{A}_c$ is a closed subset of $\mathcal{A}$ with the weak topology while $\mathcal{A}_b$ and $\mathcal{A}_{bc}$ are closed subsets of $\mathcal{A}$ with the strong topology. We consider the subspaces $\mathcal{A}_c$, $\mathcal{A}_b$ and $\mathcal{A}_{bc}$ with the metrics ρ_s and ρ_w which induce the strong and the weak topologies, respectively.

The study of steepest descent and other minimization methods is a central topic in optimization theory. See, for example, [4, 9, 11, 12, 13, 14, 15]. When the function f is convex, one usually looks for a sequence $\{x_i\}_{i=1}^{\infty}$ which either tends to a minimum point of f (if such a point exists) or at least such that $\lim_{i\to\infty} f(x_i) = \inf(f)$. If f is not necessarily convex, but X is finite-dimensional, then we expect to construct a sequence which tends to a critical point z of f, namely a point z for which $0 \in \partial f(z)$. If f is not necessarily convex and X is infinite-dimensional, then the problem is more difficult and less understood because we cannot guarantee, in general, the existence of a critical point and a convergent subsequence. To partially overcome this difficulty, we have introduced the function $\Xi : X \to \mathbb{R}^1$. Evidently, a point z is a critical point of f if and only if $\Xi(z) \geq 0$. Therefore we say that z is ϵ-critical for a given $\epsilon > 0$ if $\Xi(z) \geq -\epsilon$. In [14] we looked for sequences $\{x_i\}_{i=1}^{\infty}$ such that either $\liminf_{i\to\infty} \Xi(x_i) \geq 0$ or at least $\limsup_{i\to\infty} \Xi(x_i) \geq 0$. In the first case, given $\epsilon > 0$, all the points x_i, except possibly a finite number of them, are ϵ-critical, while in the second case this holds for a subsequence of $\{x_i\}_{i=1}^{\infty}$.

In [14] we show, under certain assumptions on f, that for most (in the sense of Baire's categories) vector fields $W \in \mathcal{A}$, certain discrete iterative processes yield sequences with the desirable properties. Moreover, we show that the complement of the set of "good" vector fields is not only of the first category, but also σ-porous. Analogous results for convex functions f were obtained in [12, 13]. This approach, when a certain property is investigated not for a single point of a complete metric space, but for the whole space, has also been successfully applied in the theory of dynamical systems [5, 6], optimization [10], and optimal control [18], as well as in approximation theory [7]. Before we continue, we briefly recall the concept of porosity [2, 16, 17]. As a matter of fact, several different notions of porosity have been used in the literature. In the present paper we will use porosity with respect to a pair of metrics, a concept which was introduced in [18].

When (Y, d) is a metric space we denote by $B_d(y, r)$ the closed ball of center $y \in Y$ and radius $r > 0$. Assume that Y is a nonempty set and $d_1, d_2 : Y \times Y \to [0, \infty)$ are two metrics which satisfy $d_1(x, y) \le d_2(x, y)$ for all $x, y \in Y$.

A subset $E \subset Y$ is called porous with respect to the pair (d_1, d_2) (or just porous if the pair of metrics is fixed) if there exist $\alpha \in (0, 1)$ and $r_0 > 0$ such that for each $r \in (0, r_0]$ and each $y \in Y$, there exists $z \in Y$ for which $d_2(z, y) \le r$ and

$$B_{d_1}(z, \alpha r) \cap E = \emptyset.$$

A subset of the space Y is called σ-porous with respect to (d_1, d_2) (or just σ-porous if the pair of metrics is understood) if it is a countable union of porous (with respect to (d_1, d_2)) subsets of Y.

Note that if $d_1 = d_2$, then by Proposition 1.1 of [18] our definitions reduce to those in [6, 7, 13]. We use porosity with respect to a pair of metrics because in applications a space is usually endowed with a pair of metrics and one of them is weaker than the other. Note that the porosity of a set with respect to one of these two metrics does not imply its porosity with respect to the other metric. However, it is shown in [18, Proposition 1.2] that if a subset $E \subset Y$ is porous with respect to (d_1, d_2), then E is porous with respect to any metric which is weaker than d_2 and stronger than d_1. For each set $E \subset X$, we denote by $cl(E)$ the closure of E in the norm topology. The results of [14] were established in any Banach space and for those functions which satisfy the following two assumptions.

A(i) For each $\epsilon > 0$, there exists $\delta \in (0, \epsilon)$ such that

$$cl(\{x \in X : \Xi(x) < -\epsilon\}) \subset \{x \in X : \Xi(x) < -\delta\};$$

A(ii) for each $r > 0$, the function f is Lipschitzian on the ball $\{x \in X : \|x\| \le r\}$.

We will say that a mapping $V \in \mathcal{A}$ is regular if for any natural number n, there exists a positive number $\delta(n)$ such that for each $x \in X$ satisfying $\|x\| \le n$ and $\Xi(x) < -1/n$, we have $f^0(x, Vx) \le -\delta(n)$.

This concept of regularity is a non-convex analog of the regular vector fields introduced in [14]. We denote by $\mathcal{F}$ the set of all regular vector fields $V \in \mathcal{A}$.

The following result was established in [14].

Theorem 1.1. *Assume that both A(i) and A(ii) hold. Then $\mathcal{A} \setminus \mathcal{F}$ (respectively, $\mathcal{A}_c \setminus \mathcal{F}$, $\mathcal{A}_b \setminus \mathcal{F}$ and $\mathcal{A}_{bc} \setminus \mathcal{F}$) is a σ-porous subset of the space $\mathcal{A}$ (respectively, $\mathcal{A}_c$, $\mathcal{A}_b$ and $\mathcal{A}_{bc}$) with respect to the pair (ρ_w, ρ_s).*

In [14] two of the authors studied the convergence of discrete descent methods generated by regular vector fields. In [1] we obtained analogs of the main results of [14] for continuous descent methods generated by regular vector fields. Our purpose in the present paper is to study some continuous descent methods for the minimization of Lipschitzian functions which are generated by approximate solutions to evolution equations governed by regular vector fields. Such methods would be quite useful in practice. Section 2 contains an auxiliary result. In Section 3 we state and prove three convergence theorems. An extension of our convergence theory to Lipschitzian functions satisfying a Palais-Smale type condition is presented in Section 4. In view of Theorem 1.1, our results apply to most vector fields in the sense of Baire's categories.

2. An auxiliary result

Throughout this paper we let $x \in W^{1,1}(0,T;X)$, i.e. (see, e.g., [3]),

$$x(t) = x_0 + \int_0^t u(s)ds, \quad t \in [0,T],$$

where $T > 0$, $x_0 \in X$ and $u \in L^1(0,T;X)$. Then $x : [0,T] \to X$ is absolutely continuous and $x'(t) = u(t)$ for a.e. $t \in [0,T]$. Recall that the function $f : X \to \mathbb{R}^1$ is Lipschitzian on bounded subsets of X. Thus the restriction of f to the set $\{x(t) : t \in [0,T]\}$ is Lipschitzian. Hence the function $(f \cdot x)(t) := f(x(t))$, $t \in [0,T]$, is absolutely continuous. It follows that for almost every $t \in [0,T]$, both the derivatives $x'(t)$ and $(f \cdot x)'(t)$ exist:

$$x'(t) = \lim_{h\to 0} h^{-1}[x(t+h) - x(t)],$$

$$(f \cdot x)'(t) = \lim_{h\to \infty} h^{-1}[f(x(t+h)) - f(x(t))].$$

We need the following result proved in [1, Proposition 2.1].

Proposition 2.1. *Assume that $t \in [0,T]$ and that both the derivatives $x'(t)$ and $(f \cdot x)'(t)$ exist. Then*

$$(f \cdot x)'(t) = \lim_{h\to 0} h^{-1}[f(x(t) + hx'(t)) - f(x(t))]. \tag{2.1}$$

In the sequel we denote by $\mu(E)$ the Lebesgue measure of a Lebesgue measurable set $E \subset \mathbb{R}^1$.

Assume that $V \in \mathcal{A}$ and that $x \in W^{1,1}(0,T;X)$ satisfies

$$x'(t) = V(x(t)) \text{ a.e. } t \in [0,T].$$

Then by Proposition 2.1, $(f \cdot x)'(t) \le 0$ for a.e. $t \in [0,T]$ and $f(x(\cdot))$ is decreasing.

3. Three convergence theorems

Theorem 3.1. *Let A(ii) hold, let $V \in \mathcal{A}$ be regular, and assume that*

$$\lim_{\|x\|\to\infty} f(x) = \infty.$$

Let K_0 and ϵ be positive numbers. Then there exist $N_0 > 0$ and $\tilde{K} > 0$ such that the following property holds:

For each $T \ge N_0$, there is $\gamma > 0$ such that if $x \in W^{1,1}(0,T;X)$ satisfies

$$\|x(0)\| \le K_0 \tag{3.1}$$

and

$$\|x'(t) - V(x(t))\| \le \gamma \textit{ for a.e. } t \in [0,T], \tag{3.2}$$

then

$$\|x(t)\| \le \tilde{K},\ t \in [0,T] \tag{3.3}$$

and

$$\mu\{t \in [0,T] : \Xi(x(t)) < -\epsilon\} \le N_0. \tag{3.4}$$

Proof. We may assume that $\epsilon < 1/4$. Choose

$$K_1 > \sup\{|f(x)| : x \in X,\ \|x\| \le K_0 + 1\}. \tag{3.5}$$

Then the set

$$\{x \in X : f(x) \le K_1 + |\inf(f)| + 4\} \tag{3.6}$$

is bounded. Consequently, there exists a constant $\tilde{K} > K_0 + K_1 + 2$ such that

$$\text{if } x \in X \text{ and } f(x) \le K_1 + |\inf(f)| + 4, \text{ then } \|x\| \le \tilde{K}. \tag{3.7}$$

There also exists a constant $L > 1$ such that

$$|f(y_1) - f(y_2)| \le L\|y_1 - y_2\| \tag{3.8}$$

for all $y_1, y_2 \in X$ such that

$$\|y_1\|, \|y_2\| \le \tilde{K} + 1. \tag{3.9}$$

Since V is regular, there is $\delta_0 \in (0,1)$ such that for each $x \in X$ satisfying

$$\|x\| \le \tilde{K} + 1 \text{ and } \Xi(x) < -\epsilon, \tag{3.10}$$

we have

$$f^0(x, Vx) \le -\delta_0. \tag{3.11}$$

Choose

$$N_0 > 2\delta_0^{-1}[2 + K_1 + |\inf(f)|] + 1 \tag{3.12}$$

and let $T \ge N_0$. Choose a positive number

$$\gamma < (4LT)^{-1}. \tag{3.13}$$

Assume that $x \in W^{1,1}(0,T;X)$ satisfies (3.1) and (3.2). We will show that for each $t \in [0,T]$,

$$f(x(t)) \le f(x(0)) + tL\gamma. \tag{3.14}$$

There is $\Delta \in (0,1)$ such that

$$\|x(t) - x(0)\| \le 1/4,\ t \in [0,\Delta]. \tag{3.15}$$

Clearly,

$$\|x(t)\| \le \|x(0)\| + 1/4 \le K_0 + 1/4 < \tilde{K},\ t \in [0,\Delta]. \tag{3.16}$$

Let $s \in (0,\Delta]$. It follows from Proposition 2.1, the relation $V \in \mathcal{A}$, and the subadditivity of Clarke's generalized directional derivative that

$$\begin{aligned} f(x(s)) - f(x(0)) &= \int_0^s (f \cdot x)'(t)dt \le \int_0^s f^0(x(t), x'(t))dt \qquad (3.17)\\ &\le \int_0^s f^0(x(t), V(x(t)))dt + \int_0^s f^0(x(t), x'(t) - V(x(t)))dt \\ &\le \int_0^s f^0(x(t), x'(t) - V(x(t)))dt. \end{aligned}$$

By (3.16), (3.2), (1.1), and the definition of L (see (3.8) and (3.9)), we have for a.e. $t \in [0,s]$,

$$f^0(x(t), x'(t) - V(x(t))) \le L\|x'(t) - V(x(t))\| \le L\gamma.$$

When combined with (3.17), this inequality implies that

$$f(x(s)) - f(x(0)) \le L\gamma s.$$

Thus (3.14) holds for any $t \in [0,\Delta]$. Set

$$\Omega = \{h \in (0,T] :\ \text{inequality (3.14) holds for all } t \in [0,h]\}. \tag{3.18}$$

Clearly, $\Delta \in \Omega$. Set $h_0 = \sup \Omega$. It is not difficult to see that

$$f(x(t)) \le f(x(0)) + tL\gamma \text{ for all } t \in [0, h_0]. \tag{3.19}$$

We now show that $h_0 = T$. Let us assume the converse. Then $h_0 < T$.

By (3.19) and (3.13),

$$f(x(h_0)) \le f(x(0)) + h_0 L\gamma < f(x(0)) + TL\gamma < f(x(0)) + 4^{-1}.$$

There is $h_1 \in (h_0, T)$ such that for each $t \in [h_0, h_1]$,

$$f(x(t)) < f(x(0)) + 1/4. \tag{3.20}$$

Relations (3.20), (3.1), (3.5) and (3.7) imply that for all $t \in [h_0, h_1]$,

$$f(x(t)) < K_1 + 1/4 \text{ and } \|x(t)\| \le \tilde{K}. \tag{3.21}$$

Let $s \in (h_0, h_1]$. It follows from Proposition 2.1, the subadditivity of Clarke's directional derivative, and the relation $V \in \mathcal{A}$ that

$$\begin{aligned} f(x(s)) - f(x(h_0)) &= \int_{h_0}^{s} (f \cdot x)'(t)dt \le \int_{h_0}^{s} f^0(x(t), x'(t))dt \\ &\le \int_{h_0}^{s} f^0(x(t), V(x(t)))dt + \int_{h_0}^{s} f^0(x(t), x'(t) - V(x(t)))dt \\ &\le \int_{h_0}^{s} f^0(x(t), x'(t) - V(x(t)))dt. \end{aligned} \tag{3.22}$$

By (3.21), (3.2), and the definition of L (see (3.8) and (3.9)), we have for a.e. $t \in [h_0, s]$,

$$f^0(x(t), x'(t) - V(x(t))) \le L\|x'(t) - V(x(t))\| \le L\gamma.$$

When combined with (3.22), this inequality implies that

$$f(x(s)) - f(x(h_0)) \le (s - h_0)L\gamma.$$

This latter inequality and (3.19) lead to

$$f(x(s)) \le (s - h_0)L\gamma + f(x(h_0)) \le (s - h_0)L\gamma + f(x(0)) + h_0 L\gamma = f(x(0)) + sL\gamma.$$

Thus $f(x(s)) \le f(x(0)) + sL\gamma$ for each $s \in (h_0, h_1]$. This means that $h_1 \in \Omega$, a contradiction. The contradiction we have reached proves that $h_0 = T$ and that (3.14) is indeed true for all $t \in [0, T]$.

From (3.14), (3.13), (3.1) and (3.5) it follows that for all $t \in [0, T]$,

$$f(x(t)) \le f(x(0)) + tL\gamma \le f(x(0)) + TL\gamma < f(x(0)) + 1/4 < K_1 + 1/4.$$

When combined with (3.7), this inequality implies that $\|x(t)\| \le \tilde{K}$, $t \in [0, T]$. Thus (3.3) holds. Set

$$\Omega_0 = \{t \in [0, T] : \Xi(x(t)) < -\epsilon\}. \tag{3.23}$$

By Proposition 2.1 and the subadditivity of Clarke's directional derivative,

$$\begin{aligned} &f(x(T)) - f(x(0)) \\ &= \int_0^T (f \cdot x)'(t)dt \\ &\le \int_0^T f^0(x(t), x'(t))dt \\ &\le \int_0^T f^0(x(t), V(x(t)))dt + \int_0^T f^0(x(t), x'(t) - V(x(t)))dt. \end{aligned} \tag{3.24}$$

By the relation $V \in \mathcal{A}$, (3.23), (3.3), and the definition of δ_0 (see (3.10) and (3.11)),

$$\int_0^T f^0(x(t), V(x(t)))dt \le \int_{\Omega_0} f^0(x(t), V(x(t)))dt \le -\delta_0 \mu(\Omega_0). \tag{3.25}$$

By (3.3), the definition of L (see (3.8) and (3.9)), (3.2) and (3.13), we have

$$\begin{aligned} \int_0^T f^0(x(t), x'(t) - V(x(t)))dt &\le \int_0^T L\|x'(t) - V(x(t))\|dt \\ &\le \int_0^L L\gamma dt = TL\gamma < 1/4. \end{aligned}$$

When combined with (3.24), (3.25), (3.1) and (3.5), this inequality implies that

$$\inf(f) - K_1 \le f(x(T)) - f(x(0)) \le -\delta_0 \mu(\Omega_0) + 1,$$

which yields, together with (3.12),

$$\mu(\Omega_0) \le \delta_0^{-1}(1 + K_1 - \inf(f)) < N_0.$$

Theorem 3.1 is proved. □

Theorem 3.2. *Let A(ii) hold, let $V \in \mathcal{A}$ be regular, and assume that*

$$\lim_{\|x\| \to \infty} f(x) = \infty.$$

Let $\gamma : [0, \infty) \to [0, 1]$ be such that $\lim_{t\to\infty} \gamma(t) = 0$. If $x \in W^{1,1}_{\rm loc}([0, \infty); X)$ is bounded and satisfies

$$\|x'(t) - V(x(t))\| \le \gamma(t) \quad \text{a.e. } t \in [0, \infty), \tag{3.26}$$

then for each $\epsilon > 0$, there exists $N_\epsilon > 0$ such that the following property holds:
For each $\Delta \ge N_\epsilon$, there is $t_\Delta > 0$ such that if $s \ge t_\Delta$, then

$$\mu\{t \in [s, s + \Delta] : \Xi(x(t)) < -\epsilon\} \le N_\epsilon.$$

Proof. Let $\epsilon > 0$. There is $K_0 > 0$ such that

$$\|x(t)\| \le K_0, \quad t \in [0, \infty). \tag{3.27}$$

By Theorem 3.1, there is $N_\epsilon > 0$ such that the following property holds:
(P1) For each $\Delta \ge N_\epsilon$, there is $\gamma_\Delta > 0$ such that if $y \in W^{1,1}(0, \Delta; X)$ satisfies

$$\|y(0)\| \le K_0 \tag{3.28}$$

and

$$\|y'(t) - V(y(t))\| \le \gamma_\Delta \quad \text{for a.e. } t \in [0, \Delta], \tag{3.29}$$

then

$$\mu\{t \in [0, \Delta] : \Xi(x(t)) < -\epsilon\} \le N_\epsilon. \tag{3.30}$$

Let $\Delta \geq N_\epsilon$ and let γ_Δ be as guaranteed by (P1). There exists $t_\Delta > 0$ such that

$$\gamma(t) \leq \gamma_\Delta \quad \text{for all } t \geq t_\Delta. \tag{3.31}$$

Assume that $s \geq t_\Delta$ and set

$$y(t) = x(t+s), \quad t \in [0, \Delta]. \tag{3.32}$$

Clearly, by (3.27),

$$\|y(0)\| = \|x(s)\| \leq K_0$$

so that (3.28) holds. It follows from (3.32), the relation $s \geq t_\Delta$, (3.26) and (3.31) that for a.e. $t \in [0, \Delta]$,

$$\|y'(t) - V(y(t))\| = \|x'(t+s) - V(x(t+s))\| \leq \gamma(t+s) \leq \gamma_\Delta.$$

Thus (3.29) holds too. By property (P1),

$$N_\epsilon \geq \mu\{t \in [0, \Delta] : \Xi(y(t)) < -\epsilon\} = \mu\{t \in [s, s+\Delta] : \Xi(x(t)) < -\epsilon\}.$$

Theorem 3.2 is proved. □

Theorem 3.3. *Let A(ii) hold, let $V \in \mathcal{A}$ be regular, and assume that*

$$\lim_{\|x\| \to \infty} f(x) = \infty.$$

Let a function $\gamma : [0, \infty) \to [0, 1]$ satisfy $\lim_{t\to\infty} \gamma(t) = 0$. If $x \in W^{1,1}_{\text{loc}}([0, \infty); X)$ is bounded and satisfies (3.26)*, then for each $\epsilon > 0$,*

$$\lim_{T\to\infty} \mu\{t \in [0, T] : \Xi(x(t)) < -\epsilon\}/T = 0.$$

Proof. Let $\epsilon > 0$ and $\delta \in (0, 1)$. Let $N_\epsilon > 0$ be as guaranteed by Theorem 3.2 and choose a number Δ such that

$$\Delta > 4(N_\epsilon + 1)/\delta. \tag{3.33}$$

By Theorem 3.2, there is $t_\Delta > 0$ such that for each $s \geq t_\Delta$,

$$\mu\{t \in [s, s+\Delta] : \Xi(x(t)) < -\epsilon\} \leq N_\epsilon. \tag{3.34}$$

Choose

$$T_0 > (t_\Delta + 2\Delta)(4/\delta). \tag{3.35}$$

Let $T \geq T_0$. There is a natural number n such that

$$T - n\Delta \geq t_\Delta > T - (n+1)\Delta. \tag{3.36}$$

This implies that

$$n \leq (T - t_\Delta)/\Delta < n + 1 \leq 2n. \tag{3.37}$$

It follows from (3.34) that for each integer $i = 0, 1, \dots, n-1$,

$$\mu\big\{t \in [t_\Delta + i\Delta, t_\Delta + (i+1)\Delta] : \Xi(x(t)) < -\epsilon\big\} \leq N_\epsilon. \tag{3.38}$$

Then by (3.36), (3.37) and (3.38),

$$\begin{aligned}
&\mu\big\{t \in [0,T] : \Xi(x(t)) < -\epsilon\big\}\\
&= \mu\big\{t \in [0,t_\Delta] : \Xi(x(t)) < -\epsilon\big\}\\
&+ \mu\big\{t \in [t_\Delta, t_\Delta + n\Delta] : \Xi(x(t)) < -\epsilon\big\} + \mu\big\{t \in [t_\Delta + n\Delta, T] : \Xi(x(t)) < -\epsilon\big\}\\
&\le t_\Delta + \mu\big\{t \in [t_\Delta, t_\Delta + n\Delta] : \Xi(x(t)) < -\epsilon\big\} + \Delta\\
&\le t_\Delta + \Delta + \sum_{i=0}^{n-1} \mu\big\{t \in [t_\Delta + i\Delta, t_\Delta + (i+1)\Delta] : \Xi(x(t)) < -\epsilon\big\}\\
&\le t_\Delta + \Delta + nN_\epsilon.
\end{aligned}$$

By this inequality, (3.35), (3.37) and (3.33),

$$\begin{aligned}
\mu\big\{t \in [0,T] : \Xi(x(t)) < -\epsilon\big\}/T &\le [t_\Delta + \Delta + nN_\epsilon]/T\\
&\le (t_\Delta + \Delta)/T_0 + (nN_\epsilon)/T < \delta/4 + (nN_\epsilon)/T\\
&\le \delta/4 + (T/\Delta)N_\epsilon/T\\
&\le \delta/4 + N_\epsilon/\Delta < \delta/4 + \delta/4 < \delta\,.
\end{aligned}$$

Thus

$$\mu\{t \in [0,T] : \Xi(x(t)) < -\epsilon\}/T < \delta$$

for all $T \ge T_0$. This concludes the proof of Theorem 3.3. □

4. Lipschitzian functions satisfying the Palais-Smale condition

We start this section by recalling several results which were established in our previous paper [1].

Proposition 4.1 ([1]). *For each $\epsilon > 0$, there exists $x_\epsilon \in X$ such that*

$$f(x_\epsilon) \le \inf(f) + \epsilon \text{ and } \Xi(x_\epsilon) \ge -\epsilon.$$

This proposition follows from Ekeland's variational principle [8].

We say that the function f satisfies the Palais-Smale (P-S) condition if each sequence $\{x_n\}_{n=1}^\infty \subset X$ for which

$$\sup\{|f(x_n)| : n = 1, 2, \dots\} < \infty$$

and $\limsup_{n\to\infty} \Xi(x_n) \ge 0$, has a norm convergent subsequence.

Denote

$$\mathrm{Cr}(f) = \{x \in X : \Xi(x) \ge 0\}.$$

Proposition 4.2 ([1]). *If $\{x_n\}_{n=1}^\infty \subset X$, $\lim_{n\to\infty} x_n = x$, and $\liminf_{n\to\infty} \Xi(x_n) \ge 0$, then $\Xi(x) \ge 0$.*

Propositions 4.1 and 4.2 imply the next three propositions which can also be found in [1].

Proposition 4.3 ([1]). *If f satisfies the (P-S) condition, then $\mathrm{Cr}(f) \ne \emptyset$.*

Proposition 4.4 ([1]). *If the function f satisfies the (P-S) condition, then for each $r > 0$, the set*

$$\{x \in X : \|x\| \le r\} \cap \mathrm{Cr}(f)$$

is compact in the norm topology.

For each $x \in X$ and $A \subset X$ set

$$d(x, A) = \inf\{\|x - y\| : y \in A\}.$$

Proposition 4.5 ([1]). *Let $r, \epsilon > 0$, and let f satisfy the (P-S) condition. Then there is $\delta > 0$ such that if $x \in X$ satisfies $\|x\| \le r$ and $\Xi(x) \ge -\delta$, then $d(x, \mathrm{Cr}(f)) \le \epsilon$.*

We are now ready to present and prove our three convergence results regarding functions satisfying the Palais-Smale condition.

Theorem 4.6. *Let A(ii) hold, and let $V \in \mathcal{A}$ be regular. Assume that*

$$\lim_{\|x\| \to \infty} f(x) = \infty,$$

and that f satisfies the (P-S) condition. Let K_0 and ϵ be positive numbers. Then there exist $N_*, \tilde{K} > 0$ such that the following property holds:

for each $T \ge N_*$, there is $\gamma > 0$ such that if $x \in W^{1,1}(0, T; X)$ satisfies

$$\|x(0)\| \le K_0 \tag{4.1}$$

and

$$\|x'(t) - V(x(t))\| \le \gamma \text{ for a.e. } t \in [0, T], \tag{4.2}$$

then

$$\|x(t)\| \le \tilde{K},\ t \in [0, T], \tag{4.3}$$

and

$$\mu\{t \in [0, T] : d(x(t), Cr(f)) > \epsilon\} \le N_*. \tag{4.4}$$

Proof. By Theorem 3.1 (with $\epsilon = 1/2$), there are $N_0, \tilde{K} > 0$ such that the following property holds:

(P2) For each $T \ge N_0$, there is $\gamma > 0$ such that if $x \in W^{1,1}(0, T; X)$ satisfies (4.1) and (4.2), then (4.3) holds.

By Proposition 4.5, there is $\delta > 0$ such that

$$\text{if } z \in X,\ \|z\| \le \tilde{K} \text{ and } \Xi(z) \ge -\delta, \text{ then } d(z, \mathrm{Cr}(f)) \le \epsilon. \tag{4.5}$$

By Theorem 3.1 (with $\epsilon = \delta$), there exists $N_1 > 0$ such that:

(P3) For each $T \ge N_1$, there is $\gamma > 0$ such that if $x \in W^{1,1}(0, T; X)$ satisfies (4.1) and (4.2), then

$$\mu(\{t \in [0, T] : \Xi(x(t)) < -\delta\} \le N_1. \tag{4.6}$$

Set

$$N_* = N_0 + N_1. \tag{4.7}$$

Let $T \ge N_*$. By virtue of (P2), there is $\gamma_1 > 0$ such that the following property holds:

(P4) If $x \in W^{1,1}(0, T; X)$ satisfies (4.1) and

$$\|x'(t) - V(x(t))\| \le \gamma_1 \text{ for a.e. } t \in [0, T], \tag{4.8}$$

then (4.3) is true.

Also, by (P3) there exists $\gamma_2 > 0$ such that the following property holds:

(P5) If $x \in W^{1,1}(0, T; X)$ satisfies (4.1) and

$$\|x'(t) - V(x(t))\| \le \gamma_2 \text{ for a.e. } t \in [0, T], \tag{4.9}$$

then (4.6) holds.

Set
$$\gamma = \min\{\gamma_0, \gamma_1\}. \tag{4.10}$$
Assume that $x \in W^{1,1}(0,T;X)$ satisfies (4.1) and (4.2). Then by property (P4), (4.1), (4.2), and (4.10), inequality (4.3) holds as well. By property (P5), (4.1), (4.2), and (4.10), inequality (4.6) holds. Let
$$t \in [0,T] \text{ and } d(x(t), \mathrm{Cr}(f)) > \epsilon.$$
Then by (4.3) and (4.5), $\Xi(x(t)) < -\delta$, so that
$$\{t \in [0,T] : d(x(t), \mathrm{Cr}(f)) > \epsilon\} \subset \{t \in [0,T] : \Xi(x(t)) < -\delta\},$$
while by (4.6) and (4.7),
$$\mu\{t \in [0,T] : d(x(t), \mathrm{Cr(f)}) > \epsilon\} \le \mu\{t \in [0,T] : \Xi(x(t)) < -\delta\} \le N_1 \le N_*.$$
The proof of Theorem 4.6 is complete. □

Theorem 4.7. *Let A(ii) hold, let $V \in \mathcal{A}$ be regular. Assume that f satisfies the (P-S) condition and that*
$$\lim_{\|x\| \to \infty} f(x) = \infty.$$
Let $\gamma : [0,\infty) \to [0,\infty)$ be such that $\lim_{t\to\infty} \gamma(t) = 0$. If $x \in W^{1,1}_{\mathrm{loc}}([0,\infty);X)$ is bounded and satisfies
$$\|x'(t) - V(x(t))\| \le \gamma(t) \quad \textit{for a.e. } t \in [0,\infty),$$
then for each $\delta > 0$, there exists $N_0 > 0$ such that the following property holds:
For each $\Delta \ge N_0$, there is $t_\Delta > 0$ such that if $s \ge t_\Delta$, then
$$\mu\{t \in [s.\, s+\Delta] : d(x(t), Cr(f)) > \delta\} \le N_0.$$

Proof. Let $\delta > 0$. Since x is assumed to be bounded, there is a constant
$$K_0 > \sup\{\|x(t)\| : t \in [0,\infty)\}. \tag{4.11}$$
By Proposition 4.5, there exists $\epsilon > 0$ such that
$$\text{if } z \in X,\ \|z\| \le K_0, \text{ and } \Xi(z) \ge -\epsilon, \text{ then } d(x, \mathrm{Cr}(f)) \le \delta. \tag{4.12}$$
Let $N_\epsilon > 0$ be as guaranteed by Theorem 3.2. Put $N_0 = N_\epsilon$ and let $\Delta \ge N_\epsilon$. By the choice of N_ϵ and Theorem 3.2, the following property holds:
(P6) There is $t_\Delta > 0$ such that for each $s \ge t_\Delta$,
$$\mu\{t \in [s, s+\Delta] : \Xi(x(t)) < -\epsilon\} \le N_\epsilon. \tag{4.13}$$
Assume that $s \ge t_\Delta$. By (4.11) and (4.12),
$$\text{if } t \in [s, s+\Delta] \text{ and } d(x(t), \mathrm{Cr}(f)) > \delta, \text{ then } \Xi(x(t)) < -\epsilon,$$
and
$$\{t \in [s, s+\Delta] : d(x(t), \mathrm{Cr}(f)) > \delta\} \subset \{t \in [s, s+\Delta] : \Xi(x(t)) < -\epsilon\}.$$
When combined with (4.13), this inclusion implies that
$$\mu\{t \in [s, s+\Delta] : d(x(t), \mathrm{Cr}(f)) > \delta\} \le N_\epsilon = N_0,$$
as claimed. Theorem 4.7 is proved. □

Theorem 4.8. *Let A(ii) hold, let $V \in \mathcal{A}$ be regular, and assume that*

$$\lim_{\|x\|\to\infty} f(x) = \infty.$$

Let $\gamma : [0,\infty) \to [0,1]$ be such that $\lim_{t\to\infty} \gamma(t) = 0$. If $x \in W^{1,1}_{\text{loc}}([0,\infty); X)$ is bounded and satisfies

$$\|x'(t) - V(x(t))\| \le \gamma(t) \textit{ for a.e. } t \in [0,\infty),$$

then for each $\delta > 0$,

$$\lim_{T\to\infty} \mu\{t \in [0,T] : d(x(t), \mathrm{Cr}(f)) > \delta\}/T = 0.$$

Proof. Since x is assumed to be bounded, there is a constant K_0 such that

$$K_0 > \sup\{\|x(t)\| : t \in [0,\infty)\}. \tag{4.14}$$

Let $\delta > 0$. By Proposition 4.5, there exists $\epsilon > 0$ such that

$$\text{if } x \in X,\ \|z\| \le K_0,\ \text{ and } \Xi(z) \ge -\epsilon, \ \text{ then } d(x, \mathrm{Cr}(f)) \le \delta. \tag{4.15}$$

By Theorem 3.3,

$$\lim_{T\to\infty} \mu\{t \in [0,T] : \Xi(x(t)) < -\epsilon\}/T = 0. \tag{4.16}$$

By (4.14) and (4.15), for each $T > 0$,

$$\{t \in [0,T] : d(x(t), \mathrm{Cr}(f)) > \delta\} \subset \{t \in [0,T] : \Xi(x(t)) < -\epsilon\}.$$

When combined with (4.16), this inclusion implies that

$$\lim_{T\to\infty} \mu\{t \in [0,T] : d(x(t), \mathrm{Cr}(f)) > \delta\}/T = 0.$$

Theorem 4.8 is proved. □

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