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QUALITATIVE PROPERTIES OF SOLUTIONS TO SEMILINEAR HEAT EQUATIONS WITH SINGULAR INITIAL DATA

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Abstract. This article concerns the nonnegative solutions to the Cauchy problem

$$u_t - \Delta u + b(x,t)|u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty),$$
$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N.$$

We investigate how the comparison principle, extinction in finite time, instantaneous shrinking of support, and existence of solutions depend on the behaviour of the coefficient $b(x,t)$.

1. Introduction

In this paper we investigate the qualitative properties of the nonnegative solutions to the Cauchy problem

$$Lu := u_t - \Delta u + b(x,t)|u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty), \tag{1.1}$$
$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where $0 < p < 1$, $b(x,t) \geq 0$, and $u_0(x)$ satisfies the hypothesis

(H1) $u_0(x) \in C(\mathbb{R}^N \setminus \{0\})$, $0 < u_0(x) \leq f(x) := f_0 + \frac{f_1}{|x|^{k_0}}$ in $\mathbb{R}^N$ ($f_0 \geq 0$, $f_1 > 0$, $k_0 > 0$).

Note that for any positive number q, when k_0 is large, $f(x) \notin L^q_{\rm loc}(\mathbb{R}^N)$. So to give a proper definition of the solution to (1.1), (1.2) is the first thing to be consider. Moreover, due to the singularity of the initial value, solutions to (1.1), (1.2) are in general unbounded, and the singularity at $x = 0$ cannot be "kill" for $t > 0$ even if $b(x,t)$ possesses some kind of singularity at $x = 0$; see for instance the following example:

Example 1.1. Let $b(x,t) = (k_0(k_0 + 2 - N))/(|x|^{k_0(1-p)+2})$ with $k_0 + 2 > N$ and $u(x,0) = 1/(|x|^{k_0})$. Then $u(x,t) = \frac{1}{|x|^{k_0}}$ is a classical solution to (1.1) in $(\mathbb{R}^N \setminus \{0\}) \times (0,\infty)$.

Due to these facts, we give the definition of solution to (1.1), (1.2) as follows.

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Definition 1.2. By a solution to problem (1.1), (1.2) we mean a function $u(x,t) \in C((\mathbb{R}^N \setminus \{0\}) \times [0,\infty)) \cap C^{2,1}((\mathbb{R}^N \setminus \{0\}) \times (0,\infty))$ satisfying classically (1.1) in $(\mathbb{R}^N \setminus \{0\}) \times (0,\infty))$ with $u(x,0) = u_0(x)$ in $(\mathbb{R}^N \setminus \{0\})$.

One sees that this definition allows solutions of (1.1), (1.2) take their potential singular points only at $x = 0$. Does problem (1.1), (1.2) have such a solution? Although we do not know at the moment the precise and proper conditions imposed on $b(x,t)$ under which problem (1.1), (1.2) is global solvable in the sense of Definition 1.2, we give in Section 4 a positive answer when $b(x,t)$ satisfies certain conditions. One also believe that if $b(x,t)$ is only nonnegative such phenomena as comparison principle, extinction in finite time and instantaneous shrinking of support for solutions cease to hold. Our aim in this paper is to give suitable conditions on $b(x,t)$ under which the above mentioned phenomena are valid, i.e., we are mainly interested in the following: What conditions we add on $b(x,t)$ so that

(a) Comparison principles for subsolutions and supersolutions of (1.1) hold.
(b) The solution $u(x,t)$ of (1.1) has the property of instantaneous shrinking of the support (the support of $u(x,t)$ is bounded for $t > 0$ although the initial value $u_0(x)$ is positive every where).
(c) The solution of (1.1) becomes extinct in finite time.
(d) Problem (1.1), (1.2) has a global solution.

There are many results on (a)–(d) when initial value $u_0(x)$ does not possess singular points; see [3, 5, 7, 9, 11] for (a); [1, 6, 7, 8, 10, 11] for (b) and (c); and [3, 7, 9, 11] for (d). The reader can find further references therein. However, when initial value $u_0(x)$ is subject to (H1), to our knowledge there are few developments in these direction. As is known to all, the comparison principle is one of the cornerstones in dealing with phenomena (b) and (c). We establish, in the next section, a comparison principle when $b(x,t)$ is under some conditions. We also state a negative result on comparison principles. From this negative result one can see that the comparison principle is not valid when the singularity of $b(x,t)$ is not very "strong". These results and their proofs are of interest in themselves. The study of phenomena (b) and (c) is the subject of Section 3. Section 4 is devoted to global existence problems.

2. Comparison Principle

In the sequel we use ϵ_0, R_0 and b_i $(i = 0, 1)$ to denote different positive constants, their values may change from one place to the next. The statement that a constant depends only on the data means that this constant can be determined in terms of N, p, f_0, f_1 and k_0. We also use $B_r(x_0)$ to denote the ball in $\mathbb{R}^N$ of radius r and centered at x_0.

Definition 2.1. For $M_0 \geq f_0$, $0 < T_0 \leq \infty$, $\mathbb{F}_{M_0}(T_0)$ is the set of all nonnegative functions $u(x,t)$ in $C((\mathbb{R}^N \setminus \{0\}) \times [0,T_0)) \cap C^{2,1}((\mathbb{R}^N \setminus \{0\}) \times (0,T_0))$ satisfying

$$u(x,t) \leq M_0 + \frac{f_1}{|x|^{k_0}} \quad \text{in } \mathbb{R}^N \setminus \{0\}) \times [0,T_0). \tag{2.1}$$

Theorem 2.2 (Comparison Principle). *Assume* $0 < p \leq 1$, *and let* $u^{\pm}(x,t) \in \mathbb{F}_{M_0}(T_0)$ *satisfy*

$$\begin{gathered} \pm Lu^{\pm} \geq 0 \quad \textit{in } (\mathbb{R}^N \setminus \{0\}) \times (0,T_0), \\ u^-(x,0) \leq u^+(x,0) \quad \textit{in } \mathbb{R}^N \setminus \{0\}. \end{gathered} \tag{2.2}$$

(a) *If $k_0 < N-2$ and $b(x,t) \geq 0$ in $(\mathbb{R}^N \setminus \{0\}) \times [0,T_0)$, then*
$$u^-(x,t) \leq u^+(x,t) \quad \text{in } (\mathbb{R}^N \setminus \{0\}) \times [0,T_0);$$
(b) *if $k_0 \geq N-2$, and for $k > k_0$, $R_0 > 0$,*
$$b(x,t) \geq \begin{cases} \dfrac{k_0(k+2-N)f_1^{1-p}}{p|x|^{k_0(1-p)+2}} & \text{in } (B_{R_0}(0) \setminus \{0\}) \times [0,T_0), \\ 0 & \text{in } (\mathbb{R}^N \setminus B_{R_0}(0)) \times [0,T_0), \end{cases} \tag{2.3}$$
then $u^-(x,t) \leq u^+(x,t)$ in $(\mathbb{R}^N \setminus \{0\}) \times [0,T_0)$.

Proof. Suppose the contrary, then there exists a point $(x^0,t^0) \in (\mathbb{R}^N \setminus \{0\}) \times (0,T_0)$ such that
$$u^-(x^0,t^0) - u^+(x^0,t^0) = 2a > 0. \tag{2.4}$$
Let $\theta_0 \in (0,1)$ and $\lambda_0 \geq 1$ be fixed, and set
$$v^\pm(x,t) = \begin{cases} \dfrac{|x|^{k_0+\theta_0}}{(1+|x|^2)^{\frac{k_0+2\theta_0}{2}}} u^\pm(x,t), & x \neq 0,\ t \in [0,T_0), \\ 0, & x = 0,\ t \in [0,T_0), \end{cases}$$
$$w(x,t) = e^{\lambda_0(t^0-t)}(v^-(x,t) - v^+(x,t)), \quad E = \{(x,t) \in \mathbb{R}^N \times (0,t^0]; w(x,t) > 0\}.$$
From the hypotheses, one easily see that $v^\pm(x,t) \in C(\mathbb{R}^N \times [0,T_0)) \cap C^{2,1}((\mathbb{R}^N \setminus \{0\}) \times (0,T_0))$ and
$$\lim_{|x|\to\infty} v^\pm(x,t) = 0 \quad \text{uniformly in } t \in [0,t^0]. \tag{2.5}$$
A straightforward calculation yields
$$\begin{aligned} \pm\big(v_t^\pm - \Delta v^\pm + 2b_i(x)v_{x_i}^\pm\big) &= \pm \frac{|x|^{k_0+\theta_0}}{(1+|x|^2)^{\frac{k_0+2\theta_0}{2}}} Lu^\pm \\ &\quad \pm \Big\{c(x)v^\pm + \frac{(k_0+\theta_0)(k_0+\theta_0+2-N)}{|x|^2}v^\pm \\ &\quad - \Big(\frac{|x|^{k_0+\theta_0}}{(1+|x|^2)^{\frac{k_0+2\theta_0}{2}}}\Big)^{1-p} b(x,t)(v^\pm)^p\Big\}, \\ &\geq \pm\Big\{-\Big(\frac{|x|^{k_0+\theta_0}}{(1+|x|^2)^{\frac{k_0+2\theta_0}{2}}}\Big)^{1-p} b(x,t)(v^\pm)^p + c(x)v^\pm \\ &\quad + \frac{(k_0+\theta_0)(k_0+\theta_0+2-N)}{|x|^2}v^\pm\Big\} \end{aligned}$$
in $(\mathbb{R}^N \setminus \{0\}) \times (0,T_0)$, where
$$b_i(x) = \frac{(k_0+\theta_0)x_i}{|x|^2} - \frac{(k_0+2\theta_0)x_i}{1+|x|^2},$$
$$c(x) = \frac{(k_0+2\theta_0)(N-2k_0-2\theta_0)}{1+|x|^2} - \frac{(k_0+2\theta_0)(2-k_0-2\theta_0)|x|^2}{(1+|x|^2)^2}.$$
Therefore,
$$\begin{aligned} &w_t - \Delta w + 2b_i(x)w_{x_i} \\ &\leq -\lambda_o w - \frac{pb(x,t)w}{(u^-)^{1-p}} + N(k_0+2\theta_0)w + \frac{(k_0+\theta_0)(k_0+\theta_0+2-N)}{|x|^2}w \end{aligned} \tag{2.6}$$

in E, where we have used the elementary inequality:

$$a^p - b^p \geq \frac{p(a-b)}{a^{1-p}} \quad \text{for } a > b \geq 0.$$

Case i) $k_0 \geq N-2$. In view of (2.3), (2.6) and $u^- \in \mathbb{F}_{M_0}(T_0)$, we first fix θ_0 and $\bar{R} \in (0, R_0)$ small enough such that

$$\frac{pb(x,t)}{(u^-(x,t))^{1-p}} \geq \frac{k_0(k+2-N)f_1^{1-p}}{(M_0 + \frac{f_1}{|x|^{k_0}})^{1-p}|x|^{k_0(1-p)+2}} \geq \frac{(k_0+\theta_0)(k_0+\theta_0+2-N)}{|x|^2} \tag{2.7}$$

in $(B_{\bar{R}}(0) \setminus \{0\}) \times (0, t^0]$, and then take λ_0 satisfying

$$\lambda_0 \geq \frac{(k_0+\theta_0)(k_0+\theta_0+2-N)}{\bar{R}^2} + (k_0+2\theta_0)(N+3k_0+4\theta_0+2), \tag{2.8}$$

to obtain $w_t - \Delta w + 2b_i(x)w_{x_i} \leq 0$ in E.

Case ii) $k_0 < N-2$. This time we choose λ_0 as in (2.8) and θ_0 small enough such that $k_0 + \theta_0 + 2 - N \leq 0$. Then from (2.6) we get

$$w_t - \Delta w + 2b_i(x)w_{x_i} \leq 0 \quad \text{in } E.$$

Thus, we have for small θ_0 and large λ_0,

$$w_t - \Delta w + 2b_i(x)w \leq 0 \quad \text{in } E. \tag{2.9}$$

For any $t \in (0, t^o]$, if , $w(x,t)$ attains its positive maximum at $x(t)$ ((2.5) implies that $w(x,t)$ cannot attain its positive maximum at infinity), then by (2.9) we deduce

$$w_t(x(t),t) \leq 0. \tag{2.10}$$

We introduce now the function

$$W(t) = \sup_{x \in \mathbb{R}^N} w(x,t).$$

From $w(x,0) \leq 0$ and (2.4),

$$W(t^0) \geq 2a, \quad W(0) \leq 0. \tag{2.11}$$

Set

$$t^* = \inf\{t \in [0,t^0]; W(\tau) \geq a, \tau \in (t,t^0]\}. \tag{2.12}$$

By (2.11) and the continuity of $W(t)$ on $\{t \in [0,t^0]; W(t) \geq \frac{a}{2}\}$, we have

$$0 < t^* < t^0, \quad \text{ and } W(t^*) = a. \tag{2.13}$$

From the proof of [4, Theorem 4.5], one can easily see that $W(t)$ is Lipschitz continuous on $[t^*, t^0]$ and

$$W'(t) \leq w_t(x(t),t) \quad \text{a.e. in } [t^*, t^0], \tag{2.14}$$

where $x(t)$ is a point $\in \mathbb{R}^N \setminus \{0\}$ satisfying $w(x(t),t) = W(t)$. Therefore, from (2.10) and (2.14),

$$W'(t) \leq 0 \quad \text{a.e. in } [t^*, t^0],$$

which implies $a = W(t^*) \geq W(t^0) \geq 2a$, a contradiction. $\square$

For the case $k_0 \geq N-2$ we do not know at the moment in a certain sense a precise or a sharp condition on $b(x,t)$ under which the comparison principle remains valid. However, we have the following negative result.

Theorem 2.3. *Assume $k_0 \ge N-2$, $N > 4$ and $0 < p < 1$. Suppose that for positive constants ϵ_0, b_0, b_1, and R_0,*

$$b_0(1+\frac{1}{|x|^{N-2}})^{1-p} \le b(x,t) \le b_1(1+\frac{1}{|x|^{N-2}})^{1-p} \tag{2.15}$$

in $(B_{R_0}(0)\setminus\{0\})\times[0,T_0)$, and

$$b(x,t) \ge \epsilon_0 \quad \text{in } (\mathbb{R}^N\setminus B_{R_0}(0))\times[0,T_0). \tag{2.16}$$

Then there exist $u^\pm \in \mathbb{F}_{M_0}(T_0)$ satisfying (2.2) and a point $(x^0,t^0) \in (\mathbb{R}^N\setminus\{0\})\times(0,T_0)$ such that

$$u^-(x^0,t^0) > u^+(x^0,t^0), \tag{2.17}$$

where $M_0 = f_0$ if $f_0 > 0$; $M_0 > 0$ if $f_0 = 0$.

Proof. If not, then we have for any pair $u^\pm \in \mathbb{F}_{M_0}(T_0)$ satisfying (2.2),

$$u^-(x,t) \le u^+(x,t) \quad \text{in } (\mathbb{R}^N\setminus\{0\})\times[0,T_0). \tag{2.18}$$

Let α,β and $\bar f \in (0,1)$ be fixed, denote $a_+ = \max\{a,0\}$ and set

$$V^*(x) = \frac{\bar f}{\alpha^{2\omega}|x|^{N-2}}(\alpha^2-|x|^2)_+^\omega, \quad U^*(x,t) = \bar f(1+\frac{1}{|x|^{N-2}})(1-\beta t)_+^\sigma, \tag{2.19}$$

where $\omega = 2/(1-p)$, $\sigma = 1/(1-p)$.

We may choose $\bar f$ small enough such that V^*, U^* are in $\mathbb{F}_{M_0}(\infty)$. One can easily verify

$$\begin{aligned} LV^* &= (\alpha^2-|x|^2)_+^{\omega p}(\frac{\bar f}{|x|^{N-2}})^p\Big[\frac{b(x,t)}{\alpha^{2\omega p}} - \frac{4\omega(\omega-1)|x|^2}{\alpha^{2\omega}}(\frac{\bar f}{|x|^{N-2}})^{1-p} \\ &\quad - \frac{2\omega(N-4)}{\alpha^{2\omega}}(\frac{\bar f}{|x|^{N-2}})^{1-p}(\alpha^2-|x|^2)_+\Big] \\ &\le (\alpha^2-|x|^2)_+^{\omega p}(\frac{\bar f}{|x|^{N-2}})^{1-p}\Big[\frac{b(x,t)}{\alpha^{2\omega p}} - \frac{\bar\omega}{\alpha^{2(\omega-1)}}(\frac{\bar f}{|x|^{N-2}})^{1-p}\Big], \end{aligned} \tag{2.20}$$

and

$$LU^* = (1-\beta t)_+^{\sigma p}\big[\bar f(1+\frac{1}{|x|^{N-2}})\big]^p\big[b(x,t) - \beta\sigma(\bar f(1+\frac{1}{|x|^{N-2}}))^{1-p}\big], \tag{2.21}$$

where $\bar\omega = 2\omega\min\{2\omega-2, N-4\}$. From (2.15), (2.16), (2.20) and (2.21), it easily follows

$$\begin{gathered} LV^* \le 0 \quad \text{in } (\mathbb{R}^N\setminus\{0\})\times(0,\infty), \\ LU^* \ge 0 \quad \text{in } (\mathbb{R}^N\setminus\{0\})\times(0,\infty), \end{gathered}$$

provided that α and β are small enough. Hence by $V^*(x) \le U^*(x,0)$ in $\mathbb{R}^N\setminus\{0\}$ and (2.18),

$$V^*(x) \le U^*(x,t) \quad \text{in } (\mathbb{R}^N\setminus\{0\})\times[0,t^*], \tag{2.22}$$

where $t^* = \min\{1, \frac{T_0}{2}\}$. One can establish step by step on t that

$$V^*(x) \le U^*(x,t) \quad \text{in } (\mathbb{R}^N\setminus\{0\})\times[0,\infty), \tag{2.23}$$

in particular,

$$\frac{\bar f}{\alpha^{2\omega}|x|^{N-2}}(\alpha^2-|x|^2)_+^\omega = V^*(x) \le U^*(x,t) = 0, \quad t > \frac{1}{\beta},\ x \ne 0,$$

this yields a contradiction. $\square$

3. Instantaneous shrinking of the support: Extinction properties

For an arbitrary nonnegative continuous function $v(x,t)$ defined in $(\mathbb{R}^N \setminus \{0\}) \times [0,T_0)(0 < T_0 \le \infty)$, we set

$$\xi(t;v) = \sup\{|x|; v(x,t) > 0\}, \quad t \in [0,T_0). \tag{3.1}$$

Definition 3.1. Assume that $0 \le v(x,t) \in C((\mathbb{R}^N \setminus \{0\}) \times [0,T_0))$, and $\xi(0;v) = \infty$. We say that instantaneous shrinking of support occurs for v if there exists $\tau > 0$ such that $\xi(t;v) < \infty$ for all $t \in (0,\tau]$.

Definition 3.2. Assume that $0 \le v(x,t) \in C((\mathbb{R}^N \setminus \{0\}) \times [0,\infty))$, we say that extinction in finite time occurs for v if there exists $T_0 > 0$ such that $v(x,t) \equiv 0$ in $(\mathbb{R}^N \setminus \{0\}) \times [T_0,\infty)$.

We begin with a theorem on the instantaneous shrinking of support property.

Theorem 3.3. *Assume that $0 < p < 1$ and (H_1) hold, and let $u(x,t) \in \mathbb{F}_{M_0}(T_0)$ be a solution of* (1.1), (1.2)*. If $b(x,t)$ satisfies, in addition to $b(x,t) \ge 0$ in $(\mathbb{R}^N \setminus \{0\}) \times [0,T_0)$,*

$$b(x,t) \ge h(|x|) f^{1-p}(x) \quad \text{in } (\mathbb{R}^N \setminus B_{R_0}(0)) \times [0,T_0), \tag{3.2}$$

where $R_0 > 0, h(r)$ is a positive non-decreasing C^2-function in $[R_0,\infty)$ satisfying for some positive constant h_0,

$$\begin{gathered} \lim_{r\to\infty} h(r) = \infty, \\ h'(r) + |h''(r)| \le h_0 h(r), \quad r \in [R_0,\infty). \end{gathered} \tag{3.3}$$

Then instantaneous shrinking of support occurs for u. Further,

$$\xi(t;u) \le h^{-1}(\frac{1}{\beta t}), \quad \forall\, t \in (0,\tau],$$

where β is a positive constant depending only on p; τ is small enough, and $h^{-1}(a) = \sup\{r; h(r) = a\}$.

Proof. Denote $\omega = \frac{2}{1-p}$, and let $\beta \in (0,1)$ and $l_0 \ge R_0 + 1$ be fixed. We introduce the function

$$U(x,t) = 2f(x)(1 - \beta t h(|x|))_+^{\omega} := 2f(x) Z^{\omega}(x,t).$$

We wish to prove that for small $\tau > 0$ and large l_0,

$$u(x,t) \le U(x,t) \quad \text{in } \{|x| \ge l_0\} \times [0,\tau]. \tag{3.4}$$

From the definition of $h^{-1}(\cdot)$, one can see that $h^{-1}(\frac{1}{\beta t}) > l_0$ for $t \in (0,\tau]$, provided that τ is small enough. Hence the assertion of the theorem easily follows from (3.4). Set

$$Q^+ = \{(x,t) \in (\mathbb{R}^N \setminus B_{l_0}(0)) \times (0,T_0);\ Z(x,t) > 0\}.$$

It is obvious that $U(x,t) \in C^{2,1}((\mathbb{R}^N \setminus \{0\}) \times [0,T_0))$, $0 < Z(x,t) \le 1$ in Q^+ and $LU = 0$ in $(\mathbb{R}^N \setminus B_{l_0}(0)) \times (0,T_0) \setminus Q^+$. A straightforward calculation yields

$$\begin{aligned} LU = & -2\omega\beta h(|x|) f(x) Z^{\omega-1} - \frac{2k_0(k_0+2-N)f_1}{|x|^{2+k_0}} Z^{\omega} \\ & - \frac{4\omega k_0 f_1}{|x|^{1+k_0}} \beta t h' Z^{\omega-1} - 2\omega(\omega-1)(\beta t h')^2 f Z^{\omega p} \\ & + 2\beta t \omega h'' f Z^{\omega-1} + \frac{2\omega(N-1)}{|x|} \beta t h' f Z^{\omega-1} + 2^p b f^p Z^{\omega p}. \end{aligned} \tag{3.5}$$

By (3.2), (3.3) and (3.5), we get

$$
\begin{aligned}
LU &\geq 2fZ^{\omega p}\big[-\omega\beta h(|x|) - \frac{k_0(k_0+2)}{l_0^2} - \frac{2\omega k_0 h_0}{l_0} - \omega(\omega-1)h_0^2 - h_0\omega + \frac{h(|x|)}{2}\big] \\
&\geq 0 \quad \text{in } Q^+,
\end{aligned}
\tag{3.6}
$$

provided that β is small and l_0 is large. ¿From $U(x,0) > u(x,0)$ on $|x| = l_0$ and the continuity of U and u, there exists $\tau = \tau(l_0) > 0$ such that

$$
u(x,t) \leq U(x,t),\ |x| = l_0,\ t \in [0,\tau]. \tag{3.7}
$$

From (3.6), (3.7) and $U(x,0) > u(x,0)$ in $\mathbb{R}^N \setminus B_{l_0}(0)$, an application of the standard comparison principle immediately leads to the desired estimate (3.4). □

The next theorem demonstrates that the hypothesis $\lim_{r\to\infty} h(r) = \infty$ in Theorem 3.3 is in a certain sense sharp.

Theorem 3.4. *Assume $0 < p < 1$. Let $u \in \mathbb{F}_{M_0}(T_0)$ be a solution of* (1.1), (1.2) *with $u_0(x) = f(x)$, and suppose that for a positive constant b_1,*

$$
0 \leq b(x,t) \leq b_1 f^{1-p}(x) \quad \text{in } (\mathbb{R}^N \setminus B_{R_0}(0)) \times [0,T_0). \tag{3.8}
$$

Then there exist positive constants H and τ (H depends only on the data, R_0 and b_1; τ is small enough) such that

$$
u(x,t) \geq \frac{f(x)}{2}(1-Ht)_+^{\frac{1}{1-p}} \quad \text{in } (\mathbb{R}^N \setminus B_{R_0}(0)) \times [0,\tau]. \tag{3.9}
$$

In particular, instantaneous shrinking of support does not occur for u.

Proof. . Let $H > 0$ be fixed , consider the function

$$
V(x,t) = \frac{f(x)}{2}(1-Ht)_+^{\sigma} := \frac{f(x)}{2}Z^{\sigma}(t) \quad (\sigma = \frac{1}{1-p}).
$$

In view of (3.8), a direct calculation gives

$$
\begin{aligned}
LV &= -\frac{H\sigma f}{2}Z^{\sigma p} - \frac{k_0(k_0+2-N)f_1}{2|x|^{2+k_0}}Z^{\sigma} + (\frac{f}{2})^p bZ^{\sigma p} \\
&\leq fZ^{\sigma p}(-\frac{H\sigma}{2} + \frac{k_0 N}{2R_0^2} + \frac{b_1}{2^p}) < 0
\end{aligned}
\tag{3.10}
$$

in $(\mathbb{R}^N \setminus B_{R_0}(0)) \times (0,T_0)$, provided H is large enough. Since $u(x,0) > V(x,0)$ when $|x| = R_0$, arguing similarly as in the proof of Theorem 3.3, one can easily see the validity of (3.9) for small $\tau > 0$. □

We pass now to the extinction in finite time phenomenon. We distinguish two cases: $k_0 < N-2$ and $k_0 \geq N-2$.

Theorem 3.5 (Case $k_0 < N-2$)**.** *Assume $0 < p < 1$ and that (H1) holds, and let $u \in \mathbb{F}_{M_0}(\infty)$ be a solution to* (1.1), (1.2). *If*

$$
b(x,t) \geq b_0 f^{1-p}(x) \quad \text{in } (\mathbb{R}^N \setminus \{0\}) \times [0,\infty)\ (b_0 > 0), \tag{3.11}
$$

then extinction in finite time occurs for u.

Proof. Let β be fixed, and consider the function

$$U(x,t)=f(x)(1-\beta t)_+^{\sigma} \quad (\sigma=\frac{1}{1-p}). \tag{3.12}$$

A quick calculation gives

$$LU=(1-\beta t)_+^{\sigma p}\big[-\beta\sigma f(x)-\frac{f_1k_0(k_0+2-N)}{|x|^{k_0+2}}(1-\beta t)_+ + b(x,t)f^p(x)\big]. \tag{3.13}$$

Then by the hypotheses, we have

$$LU\ge (1-\beta t)_+^{\sigma p} f(x)(-\beta\sigma+b_0)\ge 0,$$

provided β is small enough. Hence by Theorem 2.2,

$$u(x,t)\le U(x,t) \quad \text{in } (\mathbb{R}^N\setminus\{0\})\times[0,\infty),$$

which implies $u(x,t)\equiv 0$ in $(\mathbb{R}^N\setminus\{0\})\times[\frac{1}{\beta},\infty)$. □

Theorem 3.6 (Case $k_0\ge N-2$)**.** *Assume p, $u_0(x)$ and $u(x,t)$ are as in Theorem 3.5. Suppose that for positive constants $k>k_0$, R_0 and ϵ_0,*

$$b(x,t)\ge\begin{cases}\frac{k_0(k+2-N)f_1^{1-p}}{p|x|^{k_0(1-p)+2}} & \text{in } (B_{R_0}(0))\setminus\{0\})\times[0,\infty),\\ \epsilon_0 & \text{in } (\mathbb{R}^N\setminus B_{R_0}(0))\times[0,\infty).\end{cases} \tag{3.14}$$

Then extinction in finite time occurs for u.

Proof. This time we introduce the function

$$U(x,t)=(\tilde f_0+\frac{f_1}{|x|^{k_o}})(1-\beta t)_+^{\sigma} \quad (\sigma=\frac{1}{1-p}),$$

where

$$\tilde f_0=f_0+1+(\frac{2f_1k_0(k_0+2-N)}{\epsilon_0R_0^{k_0+2}})^{1/p}. \tag{3.15}$$

First, by the hypotheses and (3.15), one can see in $(\mathbb{R}^N\setminus B_{R_0}(0))\times(0,\infty)$,

$$\begin{aligned}LU&=(1-\beta t)_+^{\sigma p}\big\{-\beta\sigma(\tilde f_0+\frac{f_1}{|x|^{k_0}})-\frac{f_1k_0(k_0+2-N)}{|x|^{k_0+2}}(1-\beta t)_+\\&\quad+b(x,t)(\tilde f_0+\frac{f_1}{|x|^{k_0}})^p\big\}\\&\ge(1-\beta t)_+^{\sigma p}\big\{-\beta\sigma(\tilde f_0+\frac{f_1}{R_0^{k_0}})-\frac{f_1k_0(k_0+2-N)}{R_0^{k_0+2}}+\epsilon_0\tilde f_0^p\big\}\\&\ge(1-\beta t)_+^{\sigma p}\big\{-\beta\sigma(\tilde f_0+\frac{f_1}{R_0^{k_0}})+\frac{\epsilon_0\tilde f_0^p}{2}\big\}.\end{aligned} \tag{3.16}$$

Next, in $(B_{R_0}(0))\times(0,\infty)$,

$$LU\ge(1-\beta t)_+^{\sigma p}\big\{-\beta\sigma(\tilde f_0+\frac{f_1}{|x|^{k_0}})+(\frac{1}{p}-1)\frac{k_0(k+2-N)f_1^{1-p}}{|x|^{k_0(1-p)+2}}(\tilde f_0+\frac{f_1}{|x|^{k_0}})^p\big\}. \tag{3.17}$$

From (3.16) and (3.17), one can easily conclude

$$LU\ge 0 \quad \text{in } (\mathbb{R}^N\setminus\{0\})\times(0,\infty),$$

provided β is small enough. Thus, by comparison

$$u(x,t)\le U(x,t) \quad \text{in } (\mathbb{R}^N\setminus\{0\})\times[0,\infty). \tag{3.18}$$

The assertion of the theorem follows immediately from (3.18) and the definition of $U(x,t)$. $\square$

The following negative result shows that in a certain sense the condition $b(x,t) \geq \epsilon_0 > 0$ in $(\mathbb{R}^N \setminus B_{R_0}(0)) \times [0,\infty)$ in (3.14) is necessary.

Theorem 3.7. *Assume $k_0 \geq N-2$ and $\frac{1}{2} < p < 1$, and let u in $\mathbb{F}_{M_0}(\infty)$ be a solution of (1.1), (1.2) with $u_0(x) = f(x)$. If*

$$b(x,t) \geq \frac{k_0(k+2-N)f_1^{1-p}}{p|x|^{k_0(1-p)+2}} \quad \text{in } (B_{R_0}(0) \setminus \{0\}) \times [0,\infty), \tag{3.19}$$

and

$$0 \leq b(x,t) \leq g(|x|)f^{1-p}(x) \quad \text{in } (\mathbb{R}^N \setminus B_{R_o}(0)) \times [0,\infty), \tag{3.20}$$

where $g(r)$ is a positive non-increasing C^2-function in $[R_0,\infty)$ satisfying

$$\begin{gathered} \lim_{r\to\infty} g(r) = 0, \\ -\frac{g'(r)}{r} + |g''(r)| \leq g_0 g^2(r) \quad \text{in } [R_0,\infty) \ (g_0 > 0). \end{gathered} \tag{3.21}$$

Then extinction in finite time does not occur for u.

Proof. Let γ be large enough such that $(1-\gamma g(|x|))_+ > 0$ implies $|x| \geq R_0$. Consider the function

$$V(x,t) = f(x)(1-\gamma(t+1)g(|x|))_+^\sigma := f(x)Z^\sigma(x,t) \quad (\sigma = \frac{1}{1-p}).$$

Since $\frac{1}{2} < p < 1$, $V(x,t) \in C^{2,1}((\mathbb{R}^N \setminus \{0\}) \times [0,\infty))$. Set

$$Q^+ = \{(x,t) \in (\mathbb{R}^N \setminus \{0\}) \times (0,\infty); Z(x,t) > 0\}.$$

It is obvious that $LV = 0$ in $(\mathbb{R}^N \setminus \{0\}) \times (0,\infty) \setminus Q^+$ and $Q^+ \subseteq (\mathbb{R}^N \setminus B_{R_o}(0)) \times (0,\infty)$. Now, we compute in Q^+,

$$\begin{aligned} LV = f(x)Z^{\sigma p}(x,t)\big[&-\sigma\gamma g(|x|) - \frac{f_1 k_0(k_0+2-N)}{|x|^{k_0+2}f(x)}Z(x,t) - \frac{2\sigma k_0 f_1\gamma(1+t)g'(|x|)}{|x|^{k_0+1}f(x)} \\ &- \sigma(\sigma-1)(\gamma(1+t)g'(|x|))^2 Z^{-1}(x,t) + \sigma(N-1)\frac{\gamma(1+t)g'(|x|)}{|x|} \\ &+ \sigma\gamma(1+t)g''(|x|) + \frac{b(x,t)}{f^{1-p}(x)}\big]. \end{aligned} \tag{3.22}$$

Then by (3.19)–(3.22), we have

$$LV \leq f(x)Z^{\sigma p}(x,t)[-\sigma\gamma g(|x|) + \sigma g_0(2k_0+1)g(|x|) + g(|x|)] \leq 0,$$

provided γ is large enough. Hence from (3.19) and (3.20), an application of Theorem 2.2 yields

$$u(x,t) \geq V(x,t) \quad \text{in } (\mathbb{R}^N \setminus \{0\}) \times [0,\infty). \tag{3.23}$$

From the definition of $V(x,t)$ and $\lim_{r\to\infty} g(r) = 0$, we see that for any $T > 0$, there exists a point $x = x(T) \in \mathbb{R}^N \setminus \{0\}$ such that $V(x,T) > 0$, whence by (3.23) $u(x,T) > 0$. $\square$

For the case $k_0 < N-2$ we have the following result.

Theorem 3.8. *Assume $k_0 < N-2$ and $\frac{1}{2} < p < 1$. Let $u(x,t) \in \mathbb{F}_{M_0}(\infty)$ be a solution to the problem* (1.1), (1.2) *with $u_0(x) = f(x)$. If*
$$0 \le b(x,t) \le \frac{b_1 f^{1-p}(x)}{|x|^2} \quad \text{in } (\mathbb{R}^N \setminus \{0\}) \times [0,\infty) \quad (b_1 > 0). \tag{3.24}$$
Then extinction in finite time does not occur for u.

Proof. We introduce the function
$$V(x,t) = f(x)(1 - \frac{\gamma t}{|x|^2})_+^\sigma := f(x) Z^\sigma(x,t) \quad (\sigma = \frac{1}{1-p}),$$
where γ is a large constant to be chosen. A simple calculation yields
$$\begin{aligned} &LV \\ &= -\frac{\sigma\gamma}{|x|^2} f(x) Z^{\sigma p}(x,t) - \frac{k_0(k_0+2-N) f_1}{|x|^{k_o+2}} Z^\sigma(x,t) - \frac{2\sigma(N-4)\gamma t}{|x|^4} f(x) Z^{\sigma p}(x,t) \\ &\quad - \frac{4\sigma(\sigma-1)}{|x|^6}(\gamma t)^2 f(x) Z^{\sigma-2}(x,t) + \frac{4\sigma k_0 f_1}{|x|^{k_0+4}} \gamma t Z^{\sigma p}(x,t) + b(x,t) f^p(x) Z^{\sigma p}(x,t). \end{aligned}$$
Then by (3.24), we have for large γ
$$LV \le f(x) Z^{\sigma p}(x,t)\{-\frac{\sigma\gamma}{|x|^2} + \frac{k_o(N+4\sigma)+8\sigma}{|x|^2} + \frac{b_1}{|x|^2}\} \le 0,$$
and hence by comparison, $u(x,t) \ge V(x,t)$ in $(\mathbb{R}^N \setminus \{0\}) \times [0,\infty)$. The remaining of the proof is as before. $\square$

4. Existence

We begin with a global existence result.

Theorem 4.1. *Assume that $0 < p \le 1$ and (H1) hold, and suppose that $b(x,t) \in C^{\alpha_0, \frac{\alpha_0}{2}}((\mathbb{R}^N \setminus \{0\}) \times [0,\infty))(\alpha_0 \in (0,1))$ satisfies*
$$b(x,t) \ge \begin{cases} \frac{k_0(k_0+2-N)_+ f_1^{1-p}}{|x|^{k_0(1-p)+2}} & \text{in } (B_{R_0}(0) \setminus \{0\}) \times [0,\infty), \\ 0 & \text{in } (\mathbb{R}^N \setminus B_{R_0}(0)) \times [0,\infty). \end{cases} \tag{4.1}$$
Then there exists a solution $u(x,t) \in \mathbb{F}_{M_0}(\infty)$ to (1.1), (1.2) *for some constant M_o depending only on the data and R_0.*

Proof. . For any large n, let $u_n(x,t)$ be the unique solution of the approximated problem
$$(u_n)_t - \Delta u_n + b_n(x,t)|u_n|^{p-1} u_n = 0 \quad \text{in } B_n(0) \times (0, n^2], \tag{4.2}$$
$$u_n(x,t) = 0 \quad \text{on } \partial B_n(0) \times (0, n^2], \tag{4.3}$$
$$u_n(x,0) = u_{o,n}(x)\psi_n(x) \quad \text{in } B_n(0). \tag{4.4}$$
Here $\psi_n(x)$ is a smooth cutoff function in $B_n(0)$ satisfying $0 \le \psi_n \le 1$, $\psi_n(x) = 0$ on $\partial B_n(0)$ and $\psi_n(x) = 1$ in $B_{n-1}(0)$; $u_{0,n}(x)$ and $b_n(x,t)$ are, respectively, the smooth approximation of $u_0(x)$ and $b(x,t)$ satisfying
$$\begin{gathered} \lim_{n\to\infty} u_{0,n}(x) = u_0(x) \quad \text{in } \mathbb{R}^N \setminus \{0\}, \\ 0 \le u_{0,n}(x) \le f_0 + \frac{f_1}{(|x|^2 + n^{-2})^{\frac{k_0}{2}}} \quad \text{in } \mathbb{R}^N, \end{gathered} \tag{4.5}$$

$$\lim_{n\to\infty} b_n(x,t) = b(x,t) \quad \text{in } (\mathbb{R}^N \setminus \{0\}) \times [0,\infty) \tag{4.6}$$

and

$$b_n(x,t) \geq \begin{cases} \frac{k_0(k_0+2-N)_+ f_1^{1-p}}{(|x|^2+n^{-2})^{\frac{k_0(1-p)+2}{2}}}, & \text{in } B_{R_o}(0)\setminus\{0\})\times[0,\infty), \\ 0 & \text{in } (\mathbb{R}^N \setminus B_{R_0}(0)) \times [0,\infty). \end{cases} \tag{4.7}$$

Clearly, $u_n(x,t) \geq 0$ in $B_n(0)\times[0,n^2]$. To estimate the upper bound of $u_n(x,t)$, we introduce the function

$$U_n(x,t) = \tilde{M} + f_1 g(\sqrt{|x|^2+n^{-2}}), \tag{4.8}$$

where $\tilde{M} \geq f_0$ is a large constant to be fixed, and

$$g(r) = \begin{cases} \frac{1}{r^{k_0}}, & 0 < r \leq r_0 := \frac{k_0+1}{2(k_0+3)} R_0, \\ -\frac{k_0(k_0+1)^2}{12 r_0^{k_0+3}} (r - \frac{R_0}{2})^3 + \frac{k_0+3}{3(k_0+1) r_0^{k_0}}, & r_0 < r \leq \frac{R_0}{2}, \\ \frac{k_0+3}{3(k_0+1) r_0^{k_0}}, & r > \frac{R_0}{2}. \end{cases} \tag{4.9}$$

It is readily to verify that $g(r) \in C^2(0,\infty)$. From the definition of $g(r)$, we compute

$$rg''(r) + (N-1)g'(r) \leq \begin{cases} \frac{k_0(k_0+2-N)_+}{r} g(r), & 0 < r \leq r_0, \\ C, & r_0 < r \leq \frac{R_0}{2}, \\ 0, & r > \frac{R_0}{2}, \end{cases} \tag{4.10}$$

where C depends only on N, k_0, and R_0; and

$$\begin{aligned} LU_n &= b_n(x,t)U_n^p(x,t) - f_1\big\{ \frac{|x|^2}{|x|^2+n^{-2}} g''(\sqrt{|x|^2+n^{-2}}) \\ &\quad + (\frac{N}{\sqrt{|x|^2+n^{-2}}} - \frac{|x|^2}{(|x|^2+n^{-2})^{\frac{3}{2}}}) g'(\sqrt{|x|^2+n^{-2}})\big\} \\ &\geq b_n(x,t)U_n^p(x,t) - \frac{f_1}{\sqrt{|x|^2+n^{-2}}}\big[\sqrt{|x|^2+n^{-2}}\, g''(\sqrt{|x|^2+n^{-2}}) \\ &\quad + (N-1)g'(\sqrt{|x|^2+n^{-2}})\big]. \end{aligned} \tag{4.11}$$

By (4.7)–(4.11) we have for large $\tilde{M}$,

$$LU_n \geq 0 \quad \text{in } B_n(0)\times(0,n^2],$$

and hence by comparison,

$$u_n(x,t) \leq U_n(x,t) \leq 2\tilde{M} + \frac{f_1}{|x|^{k_o}} \quad \text{in } (B_n(0)\setminus\{0\})\times[0,n^2]. \tag{4.12}$$

From this equation, according to the well-known interior estimates of solutions and their continuity module (see for instance [2, 3, 9]), it follows that for any $l \geq 2$ and $n \geq l+4$, we have

$$\|u_n\|_{C^{2+\alpha_1, 1+\frac{\alpha_1}{2}}(Q_l)} \leq C_l, \tag{4.13}$$

$$|u_n(x,t) - u_n(x',t')| \leq \omega_l(|x-x'| + |t-t'|^{\frac{1}{2}}) \tag{4.14}$$

for all $(x,t),(x',t') \in \widehat{Q_l}$, where $Q_l = (B_l(0)\setminus B_{1/l}(0)) \times [\frac{1}{l^2}, l^2]$, $\widehat{Q_l} = (B_l(0) \setminus B_{1/l}(0)) \times [0,l^2]$; $\alpha_1 (>0)$ is independent of n and l, C_l and $\omega_l(r)$ are independent of n, and $\omega_l(r)$ tends to zero as $r \downarrow 0$.

The above estimates together with Arzela's lemma and a diagonal argument imply that there exists a function $u(x,t) \in \mathbb{F}_{M_0}(\infty)$ ($M_0 = 2\tilde{M}$) such that, after extracting a subsequence if necessary,

$$\lim_{n\to\infty} u_n(x,t) = u(x,t) \quad \text{in } (\mathbb{R}^N \setminus \{0\}) \times [0,\infty),$$

and $u(x,t)$ solves (1.1), (1.2) in the classical sense. $\square$

From Theorems 2.2 and 4.1 we have the following statement.

Corollary 4.2. *Assume* $0 < p \le 1$ *and (H1) hold, and suppose that* $b(x,t) \in C^{\alpha_0, \frac{\alpha_0}{2}}((\mathbb{R}^N \setminus \{0\}) \times [0,\infty))$, $\alpha_0 \in (0,1)$, *satisfies:*

(a) *For* $k_0 < N-2$, $b(x,t) \ge 0$ *in* $(\mathbb{R}^N \setminus \{0\}) \times [0,\infty)$

(b) *For* $k_0 \ge N-2$, $k > k_0$,

$$b(x,t) \ge \begin{cases} \frac{k_0(k+2-N) f_1^{1-p}}{p|x|^{k_0(1-p)+2}} & \text{in } (B_{R_0}(0) \setminus \{0\}) \times [0,\infty), \\ 0 & \text{in } (\mathbb{R}^N \setminus B_{R_0}(0)) \times [0,\infty). \end{cases}$$

Then problem (1.1), (1.2) *is uniquely solvable in* $\mathbb{F}_{M_0}(\infty)$ *for some* $M_0 \ge f_0$.

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