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# POSITIVE SOLUTIONS FOR A CLASS OF QUASILINEAR SINGULAR EQUATIONS

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Abstract. This article concerns the existence and uniqueness of solutions to the quasilinear equation

$$-\Delta_p u = \rho(x) f(u) \quad \text{in } \mathbb{R}^N$$

with $u > 0$ and $u(x) \to 0$ as $|x| \to \infty$. Here $1 < p < \infty$, $N \geq 3$, $\Delta_p$ is the $p$-Laplacian operator, $\rho$ and $f$ are positive functions, and $f$ is singular at 0. Our approach uses fixed point arguments, the shooting method, and a lower-upper solutions argument.

## 1. Introduction

We study the existence and uniqueness of solution of the problem

$$\begin{gathered} -\Delta_p u = \rho(x) f(u) \quad \text{in } \mathbb{R}^N, \\ u > 0 \quad \text{in } \mathbb{R}^N, \quad \lim_{|x|\to\infty} u(x) = 0, \end{gathered} \tag{1.1}$$

where $1 < p < \infty$, $N \geq 3$ and $\Delta_p$ is the $p$-Laplacian operator while $\rho : \mathbb{R}^N \to [0, \infty)$ is continuous and $f : (0, \infty) \to (0, \infty)$ is a $C^1$-function, singular at zero, for instance, in the sense that $\lim_{s\to 0} f(s) = \infty$.

The case $p = 2$ has been studied by several authors. Under additional assumptions on $\rho$, Edelson [3] studied (1.1) with $f(s) = s^{-\lambda}$, $\lambda \in (0, 1)$. A solution was shown to exist provided

$$\int_1^\infty r^{(N-1)+\lambda(N-2)} \widetilde{\rho}(r)\, dr < \infty,$$

where $\widetilde{\rho}(r) := \max_{|x|=r} \rho(x)$. That result was extended for all $\lambda > 0$, by Shaker [6]. Later, Lair & Shaker [5] showed existence of a solution under the condition

$$\int_0^\infty r\widetilde{\rho}(r) dr < \infty.$$

Zhang [7] showed that (1.1) has a solution provided that $f' < 0$ and $\lim_{s\to 0} f(s) = \infty$. Yet in the case $p = 2$, Cirstea & Radulescu [1] showed that (1.1) is solvable

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under the conditions: $f$ is bounded from above near $+\infty$, $\lim_{s\to 0} f(s)/s = \infty$, and $\frac{f(s)}{s+b}$ is decreasing for some positive constant $b$.

In the present paper we shall assume that $\rho$ is radially symmetric and

$$\frac{f(s)}{s^{p-1}} \quad \text{is nonincreasing in } (0,\infty), \tag{1.2}$$

$$\liminf_{s\to 0} f(s) > 0, \quad \lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = 0. \tag{1.3}$$

Our main result is as follows.

**Theorem 1.1.** *Assume* (1.2), (1.3) *and*

$$\begin{gathered} 0 < \int_1^\infty r^{\frac{1}{p-1}} \rho(r)^{\frac{1}{p-1}} dr < \infty, \quad \text{if } 1 < p \le 2, \\ 0 < \int_1^\infty r^{\frac{(p-2)N+1}{p-1}} \rho(r) dr < \infty, \quad \text{if } p \ge 2. \end{gathered} \tag{1.4}$$

*Then* (1.1) *has:*

(ii) *A radially symmetric solution $u$ in $C^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \backslash \{0\})$ if $p < N$,*
(ii) *No radially symmetric solution in $C^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \backslash \{0\})$ if $p \ge N$.*

**Remark 1.2.** Regarding case (i), it will be shown that $u \in C^2(\mathbb{R}^N)$ if and only if $p \le 2$. Additionally, the solution is uniquely determined if $f(s)/(s+b)^{p-1}$ is nonincreasing for some $b > 0$. See Section 7 .

Theorem 1.1 improves the main existence result in Cirstea & Radulesco [1] in the sense that we allow both a broader class of nonlinear operators as well as nonlinear singular terms $f$. Our theorem applies to the class of functions

$$f(s) = s^{-\lambda} + s^\gamma, \quad \text{where } \lambda \ge 0,\ 0 \le \gamma < p-1.$$

The results below will be used in the proof of Theorem 1.1. The first result is about solving the problem

$$\begin{gathered} -\Delta_p u = \rho(x) f(u) \quad \text{in } B_R, \\ u > 0 \quad \text{in } B_R, \quad u = 0 \quad \text{in } \partial B_R, \end{gathered} \tag{1.5}$$

where $B_R$ is the ball of radius $R$.

**Theorem 1.3.** *Assume* (1.2), (1.3) *and $p < N$. Then for each sufficiently large $R$,* (1.5) *has a radially symmetric solution in $C(\overline{B}_R) \cap C^1(B_R) \cap C^2(B_R \backslash \{0\})$.*

**Theorem 1.4.** *Assume* (1.2)–(1.4) *and $p < N$. Then there is a radially symmetric function $v \in C^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \backslash \{0\})$ such that*

$$\begin{gathered} -\Delta_p v \ge \rho(x) f(v) \quad \text{in } \mathbb{R}^N \backslash \{0\}, \\ v > 0 \quad \text{in } \mathbb{R}^N, \quad \lim_{|x|\to\infty} v(x) = 0. \end{gathered} \tag{1.6}$$

The proof of Theorem 1.1 will be accomplished, by at first, using Theorem 1.4 to pick a solution $v$ of (1.6), (which will be referred to as an upper-solution of (1.1)), secondly, by choosing a sufficiently large integer $j$ and applying Theorem 1.3 to find for each integer $k > 1$, a solution say, $u_k$ of $(1.5)_{j+k}$, which after extended as zero outside $B_{j+k}$, will be shown to satisfy,

$$0 \le u_1 \le u_2 \le \dots \le u_k \le \dots \le v.$$

Then we pass to the limit as $k \to \infty$, getting to a solution of (1.1) as asserted in our main result. This kind of argument is motivated by reading Zhang [7] and Cirstea & Radulescu [1].

## 2. Some Technical Lemmas

At first we state and prove some preliminary results, crucial in the proof of Theorem 1.3. As a first step in this direction, consider the initial-value problem,

$$\begin{gathered} -\Big(r^{N-1}|u'|^{p-2}u'\Big)' = r^{N-1}\rho(r)f(u(r)) \quad \text{in } (0,\infty), \\ u(0)=a, \quad u'(0)=0, \end{gathered} \tag{2.1}$$

where $a>0$ is a parameter and note that this equation is equivalent to the integral equation,

$$u(r) = a - \int_0^r \Big[t^{1-N}\int_0^t s^{N-1}\rho(s)f(u(s))ds\Big]^{\frac{1}{p-1}}dt. \tag{2.2}$$

Moreover, a solution of (2.2) is a fixed point of the operator,

$$\mathcal{F}u(r) = a - \int_0^r \Big[t^{1-N}\int_0^t s^{N-1}\rho(s)f(u(s))ds\Big]^{\frac{1}{p-1}}dt. \tag{2.3}$$

**Lemma 2.1.** *Assume* (1.2). *Then for each $a>0$ there is $T(a)\in(0,\infty]$ and a unique solution of* (2.1), *$u := u(\cdot,a) \in C^1([0,T(a)))\cap C^2((0,T(a)))$ such that $u(r)\to 0$ as $r\to T(a)$ provided $T(a)<\infty$.*

Given $T,h>0$ set

$$X := \big\{w\in C^1([0,T]) | w\geq h\big\}.$$

If $w_1,w_2\in X$ let $H:[0,T]\to\mathbb{R}$ be the continuous function

$$H(s) := s^{N-1}\Big[|(w_2^{1/p})'|^{p-2}(w_2^{1/p})'w_2^{\frac{1-p}{p}} - |(w_1^{1/p})'|^{p-2}(w_1^{1/p})'w_1^{\frac{1-p}{p}}\Big](w_1-w_2)(s).$$

**Lemma 2.2.** *If $w_1,w_2\in X$ and $0\leq S\leq U\leq T$, then*

$$\begin{aligned} &H(U)-H(S) \\ &\leq \int_S^U \Big[\frac{(r^{N-1}|(w_2^{1/p})'|^{p-2}(w_2^{1/p})')'}{w_2^{\frac{p-1}{p}}} - \frac{(r^{N-1}|(w_1^{1/p})'|^{p-2}(w_1^{1/p})')'}{w_1^{\frac{p-1}{p}}}\Big](w_1-w_2)dr. \end{aligned}$$

**Lemma 2.3.** *Assume $a<b$ and let $u(\cdot,a),u(\cdot,b)$ be the corresponding solutions given by Lemma 2.1. Then $u(\cdot,a)<u(\cdot,b)$ in $[0,T(a))$ and moreover $T(a)\leq T(b)$.*

**Lemma 2.4.** *Assume* (1.2). *Let $\{a_n\}$ be a sequence in $(0,\infty)$ such that $a_n \nearrow a$ or $a_n\searrow a$ for some $a>0$ and let $u(\cdot,a_n),u(\cdot,a)$ be the solutions given by Lemma 2.1. If $K\in(0,\min\{T(a),\sup_n T(a_n)\})$ then*

$$\lim_{n\to\infty}\|u(\cdot,a_n)-u(\cdot,a)\|_{C([0,K])}=0 \quad \text{and} \quad \lim_{n\to\infty}|u'(r,a_n)-u'(r,a)|=0, \quad r\in[0,K].$$

Next we prove results established above. The proof of Lemma 2.1 is fairly standard and is based on Banach's Fixed Point Theorem. However we present it in detail because several related notation will be used in the rest of the paper.

## 3. Proofs of the Lemmas

*Proof of Lemma 2.1.* Let $a > 0$. Since $f \in C^1$ choose $\kappa_a > 1$ such that $f$ is Lipschitz continuous on $[a/\kappa_a, a]$. Pick $\epsilon > 0$ small enough, set

$$X_{a,\epsilon} := \left\{u \in C([0,\epsilon]) : u(0) = a, a/\kappa_a \le u(r) \le a, r \in [0,\epsilon]\right\}.$$

Note that $(X_{a,\epsilon}, \|\cdot\|_\infty)$ is a complete metric space. We claim that

$$\mathcal{F}(X_{a,\epsilon}) \subset X_{a,\epsilon}, \quad \|\mathcal{F}(u_1) - \mathcal{F}(u_2)\|_{C([0,\epsilon])} \le k\|u_1 - u_2\|_{C([0,\epsilon])} \tag{3.1}$$

for all $u_1, u_2 \in X_{a,\epsilon}$ and for some $k \in (0,1)$. The proof of (3.1) is left to an Appendix. Assuming (3.1), $\mathcal{F}$ has an only fixed point $u \in X_{a,\epsilon}$ and so (2.1) has a unique local solution. Setting

$$T(a) := \sup\left\{r > 0 : (2.1) \quad \text{has an only solution in } [0,r]\right\}$$

and letting $u(\cdot,a) : [0,T(a)) \to \mathbb{R}$ be the solution of (2.1), notice that by (2.2), $u(\cdot,a) \in C^1([0,T(a)))$ and

$$u'(r,a) = -\Big[r^{1-N}\int_0^r s^{N-1}\rho(s)f(u(s,a))ds\Big]^{\frac{1}{p-1}}, \quad 0 < r < T(a). \tag{3.2}$$

Differentiating once more, one finds that $u \in C^2((0,T(a)))$. Assuming $T(a) < \infty$, we claim that $u(T(a),a) = 0$. Indeed, if $u(T(a),a) := \tilde{a} > 0$, then $u(r,a) \ge \tilde{a}$ for $r \in [0,T(a))$. Estimating the integral in (3.2) and using (1.2),

$$\begin{aligned}\int_0^r s^{N-1}\rho(s)f(u(s,a))ds &\le \frac{f(\tilde{a})}{\tilde{a}^{p-1}}a^{p-1}T(a)^{N-1}\int_0^r s^{N-1}\rho(s)ds\\ &\le \frac{f(\tilde{a})}{\tilde{a}^{p-1}}a^{p-1}\int_0^{T(a)}\rho(s)ds.\end{aligned} \tag{3.3}$$

Using (3.2) and (3.3), $\nu := \lim_{r\nearrow T(a)} u'(r,a)$ is defined and $\nu \in (-\infty,0]$. Consider the problem,

$$\begin{gathered}-\big(r^{N-1}|u'|^{p-2}u'\big)' = r^{N-1}\rho(r)f(u) \quad \text{in } (T(a),\infty),\\ u(T(a)) = \tilde{a}, \quad u'(T(a)) = \nu,\end{gathered} \tag{3.4}$$

whose solutions are the fixed points of,

$$\widetilde{\mathcal{F}}u(r) = \tilde{a} - \int_{T(a)}^r \Big\{t^{1-N}\Big[T(a)^{N-1}|\nu|^{p-1} + \int_{T(a)}^t s^{N-1}\rho(s)f(u(s))ds\Big]\Big\}^{\frac{1}{p-1}}dt.$$

Setting,

$$X_{\tilde{a},\epsilon} := \big\{u \in C([T(a),T(a)+\epsilon]) | u(T(a)) = \tilde{a}, \tilde{a}/\kappa_{\tilde{a}} \le u(r) \le \tilde{a}, r \in [T(a),T(a)+\epsilon]\big\},$$

we infer that, (see Appendix),

$$\widetilde{\mathcal{F}}(X_{\tilde{a},\epsilon}) \subset X_{\tilde{a},\epsilon}, \quad \|\widetilde{\mathcal{F}}(u_1) - \widetilde{\mathcal{F}}(u_2)\|_\infty \le k\|u_1 - u_2\|_\infty \tag{3.5}$$

where $u_1, u_2 \in X_{\tilde{a},\epsilon}$ and $k \in (0,1)$. By standard fixed point arguments again, one infers the existence of a unique solution of (2.1) on some interval $[0, T(a)+\epsilon)$ contradicting the definition of $T(a)$. Hence, $u(\cdot,a) \in C([0,T(a)])$ and $u(a,T(a)) = 0$. □

*Proof of Lemma 2.2.* Motivated by Díaz & Saa [2] let $J : L^1([0,T]) \to \mathbb{R} \cup \{\infty\}$,

$$J(w) := \begin{cases} \frac{1}{p}\int_S^U s^{N-1}\big|(w^{1/p})'\big|^p ds, & w \in X \\ \infty, & w \notin X, \end{cases}$$

where $0 \le S \le U \le T$. It is straightforward to check that $X$ and $J$ are both convex. Letting $w_1, w_2 \in X$, $\eta = w_1 - w_2$, remarking that $w_2 + t\eta$, $w_1 - t\eta$ are in $X$, $(0 \le t \le 1)$, and denoting by $\langle J'(w), \zeta\rangle$, the directional derivative of $J$ at $w$ in the direction $\zeta$, we claim that,

$$\begin{aligned} \langle J'(w_1), -\eta\rangle = &-\frac{1}{p}U^{N-1}|(w_1^{1/p}(U))'|^{p-2}(w_1^{1/p}(U))' w_1^{\frac{1-p}{p}}(U)\eta(U) \\ &+ \frac{1}{p}S^{N-1}|(w_1^{1/p}(S))'|^{p-2}(w_1^{1/p}(S))' w_1^{\frac{1-p}{p}}(S)\eta(S) \\ &+ \frac{1}{p}\int_S^U \frac{\big(s^{N-1}|(w_1^{1/p})'|^{p-2}(w_1^{1/p})'\big)'}{w_1^{\frac{p-1}{p}}}\eta(s)ds \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \langle J'(w_2), \eta\rangle = &\frac{1}{p}U^{N-1}|(w_2^{1/p}(U))'|^{p-2}(w_2^{1/p}(U))' w_2^{\frac{1-p}{p}}(U)\eta(U) \\ &- \frac{1}{p}S^{N-1}|(w_2^{1/p}(S))'|^{p-2}(w_2^{1/p}(S))' w_2^{\frac{1-p}{p}}(S)\eta(S) \\ &- \frac{1}{p}\int_S^U \frac{\big(s^{N-1}|(w_2^{1/p})'|^{p-2}(w_2^{1/p})'\big)'}{w_2^{\frac{p-1}{p}}}\eta(s)ds. \end{aligned} \tag{3.7}$$

We show (3.6) next. Note that,

$$\langle J'(w_1), -\eta\rangle = \frac{1}{p}\lim_{s\to 0}\int_S^U s^{N-1}\Big[\frac{\big|\big((w_1 - s\eta)^{1/p}\big)'\big|^p - \big|(w_1^{1/p})'\big|^p}{s}\Big]ds.$$

By computing we find

$$\langle J'(w_1), -\eta\rangle = \lim_{s\to 0}\int_S^U s^{N-1}|\theta_s|^{p-2}\theta_s\Big[\frac{((w_1 - s\eta)^{1/p})' - (w_1^{1/p})'}{s}\Big]ds, \tag{3.8}$$

where

$$\min\Big\{((w_1 - s\eta)^{1/p})', (w_1^{1/p})'\Big\} \le \theta_s \le \max\Big\{((w_1 - s\eta)^{1/p})', (w_1^{1/p})'\Big\}.$$

Applying Lebesgue's Theorem to (3.8) we infer that,

$$\langle J'(w_1), -\eta\rangle = -\frac{1}{p}\int_S^U s^{N-1}|(w_1^{1/p})'|^{p-2}(w_1^{1/p})'(w_1^{\frac{1-p}{p}}\eta)'ds.$$

Computing this integral we get to (3.6). The verification of (3.7) follows by similar arguments. From (3.6) and (3.7),

$$\begin{aligned} &\langle J'(w_2), \eta\rangle - \langle J'(w_1), \eta\rangle \\ &= \frac{1}{p}[H(U) - H(S)] \\ &- \frac{1}{p}\int_S^U \Big[\frac{(s^{N-1}|(w_2^{1/p})'|^{p-2}(w_2^{1/p})')'}{w_2^{\frac{p-1}{p}}} - \frac{(s^{N-1}|(w_1^{1/p})'|^{p-2}(w_1^{1/p})')'}{w_1^{\frac{p-1}{p}}}\Big](w_1 - w_2)ds. \end{aligned}$$

Since $J$ is convex, $\langle J'(w_1) - J'(w_2), w_1 - w_2\rangle \ge 0$ and Lemma 2.2 follows. $\square$

*Proof of Lemma 2.3.* Assume, on the contrary, $u(r,a) < u(r,b), r \in [0,T)$ and $u(T,a) = u(T,b)$, for some $T < T(a)$. Taking $r \in (0,T)$ and using Lemma 2.2 and (1.2),

$$
\begin{aligned}
&r^{N-1}\Big[\frac{|u'(r,b)|^{p-2}u'(r,b)}{u(r,b)^{p-1}} - \frac{|u'(r,a)|^{p-2}u'(r,a)}{u(r,a)^{p-1}}\Big](u(r,a)^p - u(r,b)^p) \\
&\leq \int_0^r \Big[\frac{(t^{N-1}|u'(t,b)|^{p-2}u'(t,b))'}{u(t,b)^{p-1}} - \frac{(t^{N-1}|u'(t,a)|^{p-2}u'(t,a))'}{u(t,a)^{p-1}}\Big](u(t,a)^p - u(t,b)^p)dt \\
&= \int_0^r t^{N-1}\rho(t)\Big[\frac{f(u(t,a))}{u(t,a)^{p-1}} - \frac{f(u(t,b))}{u(t,b)^{p-1}}\Big](u(t,a)^p - u(t,b)^p)dr \leq 0.
\end{aligned}
$$

As a consequence,

$$
\frac{|u'(r,b)|^{p-2}u'(r,b)}{u(r,b)^{p-1}} - \frac{|u'(r,a)|^{p-2}u'(r,a)}{u(r,a)^{p-1}} \geq 0.
$$

Recalling that $u'(\cdot,a), u'(\cdot,b)$ are both non positive, we get, $\frac{u(\cdot,b)}{u(\cdot,a)}$ is nondecreasing in $[0,T]$, so that,

$$
1 < \frac{u(0,b)}{u(0,a)} \leq \frac{u(T,b)}{u(T,a)} = 1,
$$

which is impossible. Hence $u(r,a) < u(r,b)$ for $r \in [0,T(a))$ and Lemma 2.3 is proved. □

*Proof of Lemma 2.4.* Assume $a_n \nearrow a$. By Lemma 2.3, $K \in (0, \sup_n T(a_n))$. Take an integer $n_K \geq 1$ such that $T(a_{n_K}) > K$. By Lemma 2.3 again,

$$
T(a_{n_K}) \leq T(a_n) \leq T(a) \quad \text{and} \quad u(\cdot,a_{n_K}) \leq u(\cdot,a_n) \leq u(\cdot,a) \leq a,
$$

for $n \geq n_K$, showing that $\{u(\cdot,a_n)\}_{n=1}^\infty$ is equibounded. We claim that it is also equicontinuous in $C([0,K])$. Indeed, estimating as in (3.3) we find

$$
|u'(r,a_n)|^{p-1} \leq \frac{f(u(K,a_{n_K}))}{u(K,a_{n_K})^{p-1}} a^{p-1} \int_0^K \rho(s)ds := \widehat{K}.
$$

Let $\theta_n \in (0,K)$ such that,

$$
|u(r,a_n) - u(s,a_n)| = |u'(\theta_n,a_n)||r-s| \leq \widehat{K}^{\frac{1}{p-1}}|r-s|.
$$

Then $\{u(\cdot,a_n)\}_{n=1}^\infty$ is equicontinuous. By the Arzéla-Àscoli theorem there is some $\widetilde{u} \in C([0,K])$ such that, up to a subsequence, $u(\cdot,a_n) \to \widetilde{u}$ uniformly in $[0,K]$. We remark that

$$
s^{N-1}\rho(s)f(u(s,a_n)) \to s^{N-1}\rho(s)f(\widetilde{u}(s))
$$

and

$$
s^{N-1}\rho(s)f(u(s,a_n)) \leq \frac{f(u(K,a_{n_K}))}{u(K,a_{n_K})^{p-1}} a^{p-1} s^{N-1}\rho(s)
$$

for $t \in [0,K]$. By Lebesgue's theorem,

$$
\int_0^r s^{N-1}\rho(s)f(u(s,a_n))ds \to \int_0^r s^{N-1}\rho(s)f(\widetilde{u}(s))ds
$$

for each $r \in [0,K]$. This and (3.2) amount

$$
u'(r,a_n) \to -\Big(r^{1-N}\int_0^r s^{N-1}\rho(s)f(\widetilde{u}(s))ds\Big)^{\frac{1}{p-1}} := \overline{u}(r)
$$

so that $\int_0^r u'(t,a_n)dt \to \int_0^r \overline{u}(t)dt$ and hence

$$\widetilde{u}(r) - a = \int_0^r \overline{u}(t)dt.$$

As a consequence,

$$|\widetilde{u}'(r)|^{p-2}\widetilde{u}'(r) = -r^{1-N}\int_0^r s^{N-1}\rho(s)f(\widetilde{u}(s))ds.$$

Hence $\widetilde{u}$ is a solution of (2.1) and by uniqueness, provided by Lemma 2.1, $\widetilde{u} := u(\cdot,a)$.

It has finally been shown that $u(\cdot,a_n) \to u(\cdot,a)$ in $C([0,K])$ and $u'(\cdot,a_n) \to u'(\cdot,a)$ pointwise in $[0,K]$. The case $a_n \searrow a$ follows by similar arguments. Lemma 2.4 is proved. $\square$

## 4. Proof of Theorem 1.3

By (1.4) pick $S > 0$ such that $\int_0^S s^{N-1}\rho(s)ds > 0$. Take $R \geq 2S$ and consider the set,

$$\mathcal{A} := \{a > 0 : T(a) \geq R\}.$$

We claim that $\mathcal{A} \neq \phi$. Indeed, if $T(a) < R$ for all $a > 0$, by Lemma 2.1, $\lim_{r\to T(a)} u(r,a) = 0$ so that $u(r_a,a) = \frac{a}{2}$ for some $r_a \in (0,T(a))$. Estimating in (2.2) and using (1.2),

$$\begin{aligned} \frac{1}{2} &\leq \int_0^{r_a}\Big[t^{1-N}\int_0^t s^{N-1}\rho(s)\frac{f(u(s,a))}{u(s,a)^{p-1}}ds\Big]^{\frac{1}{p-1}}dt \\ &\leq \Big(\frac{f(\frac{a}{2})}{(\frac{a}{2})^{p-1}}\Big)^{\frac{1}{p-1}}\int_0^R\Big[t^{1-N}\int_0^t s^{N-1}\rho(s)ds\Big]^{\frac{1}{p-1}}dt. \end{aligned} \tag{4.1}$$

Making $a \to \infty$ leads to a contradiction by (1.3)(ii), showing that $\mathcal{A} \neq \phi$. We claim that $A := \inf\mathcal{A}$ is positive. Indeed, if $A = 0$, it follows by Lemma 2.3 that $u(R,a) > 0$ for all $a > 0$. Since,

$$2(u(R,a) - u(\frac{R}{2},a)) = u'(\theta_a,a), \quad \text{for some } \theta_a \in (\frac{R}{2},R),$$

and $u(R,a) \leq u(\frac{R}{2},a) \leq a$ it follows using,

$$(\theta_a)^{N-1}|u'(\theta_a,a)|^{p-2}u'(\theta_a,a) = -\int_0^{\theta_a} s^{N-1}\rho(s)f(u(s,a))ds$$

that

$$\lim_{a\to 0}\int_0^{\theta_a} s^{N-1}\rho(s)f(u(s,a))ds = 0.$$

Using Fatou's lemma and (1.3)

$$0 = \liminf_{a\to 0}\int_0^{\theta_a} s^{N-1}\rho(s)f(u(s,a))ds \geq \int_0^{R/2} s^{N-1}\rho(s)\liminf_{a\to 0} f(u(s,a))ds > 0,$$

which is impossible, showing that $A > 0$. To finish the proof of Theorem 1.3 it suffices to show that $T(A) = R$. If $T(A) < R$, pick both $\epsilon > 0$ such that $T(A)+\epsilon < R$ and a sequence $a_n \in \mathcal{A}$ with $a_n \searrow A$. Consider further, the sequence $u(T(A) + \frac{\epsilon}{2}, a_n)$ which by Lemma 2.3 is decreasing and set $T_{\epsilon,A} := \inf_n\{u(T(A) + \frac{\epsilon}{2}, a_n)\}$.

We claim that $T_{\epsilon,A} > 0$. Otherwise, it follows remarking that $u(T(A)+\epsilon, a_n) \le u(T(A)+\frac{\epsilon}{2}, a_n)$ and,

$$2\Big[u(T(A)+\epsilon, a_n) - u(T(A)+\frac{\epsilon}{2}, a_n)\Big] = u'(\theta_n, a_n)\epsilon$$

for some $\theta_n \in (T(A)+\frac{\epsilon}{2}, T(A)+\epsilon)$ that $\lim_n u'(\theta_n, a_n) = 0$. Now, by arguments as above,

$$\lim_n \int_0^{T(A)} s^{N-1}\rho(s) f(u(s,a_n))ds = 0.$$

On the other hand, by Lemmas 2.3 and 2.4 we have, for each $K \in (0, T(a))$,

$$\int_0^K s^{N-1}\rho(s) f(u(s,a_n))ds \longrightarrow \int_0^K s^{N-1}\rho(s) f(u(s,A))ds,$$

showing that $\rho = 0$ a.e. in $(0, T(A))$. So, by (3.2), $u(r,A) = A$ for $r \in [0, T(A)]$, impossible, because we are assuming $T(A) < R$ and by Lemma 2.1 $u(T(A), A) = 0$. Therefore $T_{\epsilon,A} > 0$.

Choose $\delta_0 > 0$ such that $u(r,A) < \frac{T_{\epsilon,A}}{4}$ for $r \in [T(A)-\delta_0, T(A)-\frac{\delta_0}{2}]$. By Lemma 2.4,

$$\lim_n \|u(\cdot, a_n) - u(\cdot, A)\|_{C([0,T(A)-\frac{\delta_o}{2}])} = 0$$

and so there is $n_0 > 1$ such that

$$|u(r, a_{n_0}) - u(r,A)| < \frac{T_{\epsilon,A}}{4}, \quad r \in [0, T(A) - \frac{\delta_0}{2}].$$

Thus,

$$u(r, a_{n_0}) \le |u(r, a_{n_0}) - u(r,A)| + u(r,A) < \frac{T_{\epsilon,A}}{2}, \quad r \in [T(A)-\delta_0, T(A)-\frac{\delta_0}{2}].$$

Since $u(r, a_n) \ge T_{\epsilon,A}$ for all $n > 1$ and $r \in [0, T(A)]$, it follows that

$$u(T(A)-\delta_0, a_{n_0}) < \frac{T_{\epsilon,A}}{2} < T_{\epsilon,A} \le u(T(A), a_{n_0}),$$

which is impossible. Therefore $A \in \mathcal{A}$.

Now we claim that

$$T(A) = R. \tag{4.2}$$

Indeed, pick a sequence $a_n \nearrow A$, $a_n \in \mathcal{A}^c$. By Lemma 2.3, $T(a_n) \le T(a_{n+1}) \le R$ and in fact $T(a_n) \nearrow T$ for some $T > 0$. Using Lemma 2.3 again, $T \le T(A)$. It will be shown that $T = T(A)$. Indeed, assume by the contrary, $T < T(A)$. Setting $T_A := u(T,A)$ it follows that $T_A > 0$. So, for each $n$ large take $s_n \in (0,T)$ satisfying $u(s_n, a_n) = \frac{T_A}{4}$.

Since $u(\cdot, a_n)$ is nonincreasing, consider $\tilde{s}_n \in (0, s_n)$ such that $u(\tilde{s}_n, a_n) = \frac{T_A}{2}$. We will show next that $\tilde{s}_n \to T$. Indeed, by Lemma 2.3, $\tilde{s}_n$ is monotone so that $\tilde{s}_n \to \tilde{T} \le T$.

If $\tilde{T} < T$ there is $n_0 > 1$ such that $T(a_{n_0}) > \tilde{T}$. Hence $u(r, a_n) \le \frac{T_A}{2}$ for $n \ge n_0$ and $r \in [\tilde{T}, T(a_{n_0})]$, because otherwise, there would be some $r_{n_1} \in [\tilde{T}, T(a_{n_0})]$ with $\frac{T_A}{2} < u(r_{n_1}, a_{n_1}) \le u(\tilde{s}_{n_1}, a_{n_1}) = \frac{T_A}{2}$, which is impossible.

We infer that $|u(r,a_n) - u(r,A)| \ge \frac{T_A}{2}$ for all $n \ge n_0$, $r \in [\tilde{T}, \tilde{T}+\delta)$ and for some $\delta > 0$ such that $\tilde{T}+\delta < T(a_{n_0})$. But this is impossible again, because by Lemma 2.4,

$$\lim_n \|u(\cdot, a_n) - u(\cdot, A)\|_{C([0,\tilde{T}+\delta])} = 0.$$

Therefore, $\tilde{T} = T$. Now, noticing that,

$$u(s_n, a_n) - u(\tilde{s}_n, a_n) = u'(\theta_n, a_n)(s_n - \tilde{s}_n), \quad \tilde{s}_n < \theta_n < s_n,$$

we find,

$$\lim_n |u'(\theta_n, a_n)| = \frac{T_A}{4|s_n - \tilde{s}_n|} = \infty,$$

which is impossible, because estimating in (3.2) as in (3.3), we get,

$$|u'(\theta_n, a_n)|^{p-1} \leq \frac{f(\frac{T_A}{4})}{(\frac{T_A}{4})^{p-1}} A^{p-1} \int_0^T \rho(s) ds.$$

So, $T = T(A) = R$ showing (4.2). By Lemma 2.1, $u(R, A) = 0$. As a consequence, $u(\cdot, A) \in C([0, R])$. Further on, by (3.2), $u(\cdot, A) \in C^1([0, R)) \cap C^2((0, R))$. The arguments above give a radially symmetric solution $u$ of (1.5). This proves Theorem 1.3.

## 5. Proof of Theorem 1.4

Let $C_1, C_2, \dots$ denote several positive constants. Next, given $a > 0$,

$$w(r) = a - \int_0^r \Big[t^{1-N} \int_0^t s^{N-1} \rho(s) ds\Big]^{\frac{1}{p-1}} dt, \tag{5.1}$$

is the unique solution of the problem

$$\begin{gathered} -\big(r^{N-1} |w'|^{p-2} w'\big)' = r^{N-1} \rho(r) \quad \text{in } (0, \infty), \\ w(0) = a, \quad w'(0) = 0, \quad w > 0 \quad \text{in } [0, \infty). \end{gathered} \tag{5.2}$$

It will be shown that

$$I(r) := \int_0^r \Big[t^{1-N} \int_0^t s^{N-1} \rho(s) ds\Big]^{\frac{1}{p-1}} dt, \tag{5.3}$$

has a finite limit as $r \to \infty$. Indeed, if $1 < p \leq 2$, by estimating the integral in (5.3),

$$I(r) \leq C_1 + \int_1^r t^{\frac{1-N}{p-1}} \Big[\int_0^t s^{N-1} \rho(s) ds\Big]^{\frac{1}{p-1}} dt.$$

Using the assumption $N \geq 3$ in the computation of the first integral above and Jensen's inequality to estimate the last one,

$$I(r) \leq C_2 + C_3 \int_1^r t^{\frac{3-N-p}{p-1}} \int_1^t s^{\frac{N-1}{p-1}} \rho(s)^{\frac{1}{p-1}} ds dt.$$

Computing the above integral above, we obtain

$$I(r) \leq C_2 + C_4 \int_1^r t^{\frac{1}{p-1}} \rho(t)^{\frac{1}{p-1}} \, dt.$$

Applying (1.4) in the integral above we infer that $I(r)$ has a finite limit as $r \to \infty$. On the other hand, if $p \geq 2$, set

$$H(t) := \int_0^t s^{N-1} \rho(s) ds$$

and note that either, $H(t) \leq 1$ for $t > 0$ or $H(t_0) = 1$ for some $t_0 > 0$. In the first case, $H(t)^{\frac{1}{p-1}} \leq 1$, and hence,

$$I(r) = \int_0^r t^{\frac{1-N}{p-1}} H(t)^{\frac{1}{p-1}} dt \leq C_5 + \int_1^r t^{\frac{1-N}{p-1}} dt$$

so that $I(r)$ has a finite limit because $p < N$. In the second case, $H(s)^{\frac{1}{p-1}} \le H(s)$ for $s \ge s_0$ and hence,

$$I(r) \le C_6 + \int_1^r t^{\frac{1-N}{p-1}} \int_0^t s^{N-1}\rho(s)dsdt.$$

Estimating and integrating by parts, we obtain

$$\begin{aligned} I(r) &\le C_6 + C_7\int_1^r t^{\frac{1-N}{p-1}}dt + \frac{p-1}{N-p}\Big[\int_1^r t^{\frac{(p-2)N+1}{p-1}}\rho(t)dt - r^{\frac{p-N}{p-1}}\int_0^r t^{N-1}\rho(t)dt\Big] \\ &\le C_8 + C_9\int_1^r t^{\frac{(p-2)N+1}{p-1}}\rho(t)dt. \end{aligned}$$

By (1.4) (part 2), $I(r)$ converges to some real number. Taking in (5.2),

$$a := \int_0^\infty \Big[t^{1-N}\int_0^t s^{1-N}\rho(s)ds\Big]^{\frac{1}{p-1}}dt = \lim_{r\to\infty} I(r),$$

gives, $\lim_{r\to infty} w(r) = 0$. In what follows, an upper-solution to (1.1) will be constructed. First, consider the function

$$\tilde{f}_p(t) := (f(t)+1)^{\frac{1}{p-1}}, \quad t > 0, \tag{5.4}$$

and note that the items below hold true,

$$\begin{gathered} \tilde{f}_p(t) \ge f(t)^{\frac{1}{p-1}} > 0, \\ \frac{\tilde{f}_p(t)}{t^{p-1}} \quad \text{is decreasing}, \\ \lim_{t\to\infty}\frac{\tilde{f}_p(t)}{t} = 0. \end{gathered} \tag{5.5}$$

We claim that

$$C_p a \le \int_0^{C_p^{\frac{1}{p-1}}} \frac{t^{p-1}}{\tilde{f}_p(t)}dt, \tag{5.6}$$

for some $C_p > 0$. Indeed, by (5.5)(iii),

$$\lim_{r\to\infty}\int_0^r \frac{t^{p-1}}{\tilde{f}_p(t)}dt = \infty,$$

and thus,

$$\lim_{r\to\infty}\frac{\int_0^r \frac{t^{p-1}}{\tilde{f}_p(t)}dt}{r^{p-1}} = \frac{1}{p-1}\lim_{r\to\infty}\frac{r}{\tilde{f}_p(r)} = \infty,$$

showing (5.6). Now set, for $s > 0$,

$$F_p(s) := \frac{1}{C_p}\int_0^s \frac{t^{p-1}}{\tilde{f}_p(t)}dt,$$

and notice that, $F_p(0) = 0$ and $F_p$ is increasing. Using (5.5)(iii) it follows that, $F(s) \overset{s\to\infty}{\to} \infty$. Applying the Implicit Function Theorem,

$$w(r) := \frac{1}{C_p}\int_0^{v(r)} \frac{t^{p-1}}{\tilde{f}_p(t)}dt \tag{5.7}$$

for some $C^2((0,\infty))\cap C^1([0,\infty))$-function $v$. It will be shown next that $v$ is an upper-solution to (1.1). Indeed, since $v$ is nonincreasing, it follows by (5.8), (5.7) and $w(0)=a$ that,

$$\int_0^{v(r)}\frac{t^{p-1}}{\tilde f_p(t)}dt\le\int_0^{v(0)}\frac{t^{p-1}}{\tilde f_p(t)}dt=C_pw(0)=C_pa\le\int_0^{C_p^{\frac{1}{p-1}}}\frac{t^{p-1}}{\tilde f_p(t)}dt,$$

so that,

$$v(r)\le C_p^{\frac{1}{p-1}},\quad t\ge 0. \tag{5.8}$$

Differentiating in (5.7) and computing, we get to

$$\begin{aligned}\big(r^{N-1}|w'(r)|^{p-2}w'(r)\big)'&=\big(\frac{1}{C_p}\big)^{p-1}\big(\frac{v^{p-1}}{\tilde f_p(v)}\big)^{p-1}\big(r^{N-1}|v'(r)|^{p-2}v'(r)\big)'\\&\quad+(p-1)\big(\frac{1}{C_p}\big)^{p-1}\big(\frac{v^{p-1}}{\tilde f_p(v)}\big)^{p-2}\big(\frac{d}{dv}\big(\frac{v^{p-1}}{\tilde f_p(v)}\big)\big)r^{N-1}|v'|^p.\end{aligned}$$

Now, using (5.5)(iii), (5.8) and (5.5)(i), it follows that

$$\big(r^{N-1}|v'(r)|^{p-2}v'(r)\big)'\le-\big(\frac{C_p}{v^{p-1}}\big)^{p-1}\tilde f_p(v)^{p-1}r^{N-1}\rho(r)\le -r^{N-1}\rho(r)f(v(r)).$$

Remarking that by (5.7) $v'(0)=0$ and $\lim_{r\to\infty}v(r)=0$ it follows that $v$ is a radially symmetric solution of (1.6). This ends the proof of Theorem 1.4.

## 6. Proof of Theorem 1.1

To show (i), pick an integer $j$ sufficiently large such that (1.5) with $R=j+k$ has, by Theorem 1.3, a radially symmetric solution, say $u_k\in C^1([0,j+k))\cap C([0,j+k])$ for each integer $k\ge 1$. Consider the extension to $[0,\infty)$ of $u_k$, given by $u_k(r)=0$, if $r\ge j+k$. We claim that,

$$0\le u_1\le u_2\le\dots\le u_k\le\dots\le v. \tag{6.1}$$

We will show first that $u_k\le u_{k+1}$. Indeed, we claim that $u_k(0)\le u_{k+1}(0)$. Otherwise, both $u_k(r)>u_{k+1}(r)$ for $r\in[0,T)$ and $u_k(T)=u_{k+1}(T)$ for some $T\in(0,j+k)$. Arguing as in the proof of Lemma 2.3 with the use of Lemma 2.2 we get to,

$$\frac{|u_k'|^{p-2}u_k'}{u_k^{p-1}}-\frac{|u_{k+1}'|^{p-2}u_{k+1}'}{u_{k+1}^{p-1}}\ge 0,$$

which gives, $\frac{u_k}{u_{k+1}}$ is nondecreasing in $(0,T)$, and as a consequence,

$$1<\frac{u_k(0)}{u_{k+1}(0)}\le\frac{u_k(T)}{u_{k+1}(T)}=1,$$

which is impossible. Hence, $u_k(0)\le u_{k+1}(0)$. Now, if $u_k(r)>u_{k+1}(r)$ for $r\in(S,U)$, for some $S,U\in(0,j+k)$ with $S<U$, $u_k(S)=u_{k+1}(S)$ and $u_k(U)=u_{k+1}(U)$.

Arguing as earlier again, we find,

$$1=\frac{u_k(S)}{u_{k+1}(S)}\le\frac{u_k(r)}{u_{k+1}(r)}\le\frac{u_k(U)}{u_{k+1}(U)}=1,\quad r\in[S,U],$$

so that, $u_k(r)=u_{k+1}(r)$, $r\in[S,U]$ which, impossible. This shows that $u_k\le u_{k+1}$.

To complete to proof of (6.1), it remains to show that $u_k \leq v$. This follows by arguments similar to the ones used to show that $u_k \leq u_{k+1}$, recalling that $v$ satisfies (1.7). The proof of (6.1) is complete.

Now, by (6.1), $u_k \to u$ pointwise, for some $u \leq v$ and, by the proof of Theorem 1.3,

$$u_k(r) = u_k(0) - \int_0^r \Big[s^{1-N} \int_0^s t^{N-1}\rho(t)f(u_k(t))dt\Big]^{\frac{1}{p-1}} ds, \quad r \geq 0. \tag{6.2}$$

Set $r > 0$, pick $k_0$ such that $j + k_0 \geq r + 1$ and notice that by (6.1), $u_k \geq u_{k_0}$ for $k \geq k_0$. Recalling that $u_k'$ and $v'$ are nonpositive and using (1.2) and (6.1),

$$t^{N-1}\rho(t)f(u_k(t)) \leq v(0)^{p-1} \frac{f(u_{k_0}(s))}{u_{k_0}(s)^{p-1}} t^{N-1}\rho(t), \quad t \in [0, s].$$

Since the last function above belongs to $L^1((0, s))$, by Lebegue's theorem,

$$\int_0^s t^{N-1}\rho(t)f(u_k(t))dt \to \int_0^s t^{N-1}\rho(t)f(u(t))dt, \quad s \in [0, r],$$

and employing, once more, arguments as above,

$$\int_0^r \Big[s^{1-N} \int_0^s t^{N-1}\rho(t)f(u_k(t))dt\Big]^{\frac{1}{p-1}} ds \to \int_0^r \Big[s^{1-N} \int_0^s t^{N-1}\rho(t)f(u(t))dt\Big]^{\frac{1}{p-1}} ds.$$

Passing to the limit in (6.2) we infer that,

$$u(r) = u(0) - \int_0^r \Big[s^{1-N} \int_0^s t^{N-1}\rho(t)f(u(t))dt\Big]^{\frac{1}{p-1}} ds.$$

Remark that

$$u''(r) = -\frac{h(r)}{p-1} \Big[r^{1-N} \int_0^r t^{N-1}\rho(t)f(u(t))dt\Big]^{\frac{2-p}{p-1}}, \tag{6.3}$$

where

$$h(r) := \rho(r)f(u(r)) + (1-N)r^{-N} \int_0^r t^{N-1}\rho(t)f(u(t))dt. \tag{6.4}$$

Hence, $u \in C^1([0, \infty)) \cap C^2((0, \infty))$. This together with the fact that $u \leq v$, shows (i), that is, $u$ is radially symetric solution of (1.1).

To show (ii), assume, on the contrary, that (1.1) has a solution $u$, so that

$$r^{N-1}|u'(r)|^{p-2}u'(r) \leq -C \text{ for } r \geq M,$$

where $C, M > 0$ are suitable constants. As a consequence,

$$u'(r) \leq -Cr^{\frac{1-N}{p-1}}, \quad r \geq M. \tag{6.5}$$

Integrating from $M$ to $r$ in (6.5) and taking into account the cases $N < p$ and $N = p - 1$ and at last making $r \to \infty$ we arrive at a contradiction. This finishes the proof of Theorem 1.1.

## 7. Comments on Remark 1.2

At this point we justify the claim in Remark 1.2. By (6.4), we get

$$\lim_{r\to 0} h(r) = \frac{1}{N}\rho(0)f(u(0)).$$

On the other hand,

$$\lim_{r\to 0}\Big[r^{1-N}\int_0^r t^{N-1}\rho(t)f(u(t))dt\Big] = 0.$$

Hence, by (6.3) $\lim_{r\to 0} u''(r)$ exists if and only if $p \le 2$, that is $u \in C^2([0,\infty))$ if and only if $p \le 2$.

Now let $u, v$ be solutions of (1.1). By Lemma 2.3 we can assume $u \ge v$. Let $w_1 := (v+b)^p$ and $w_2 := (u+b)^p$ and notice that $w_1, w_2 \in X$. Taking $r > 0$ and using Lemma 2.2 and (1.2), as in the proof of Lemma 2.3, we find,

$$\frac{|u'|^{p-2}u'}{(u+b)^{p-1}} - \frac{|v'|^{p-2}v'}{(v+b)^{p-1}} \ge 0,$$

and since $u', v' \le 0$, we infer that, $\frac{u+b}{v+b}$ is nondecreasing in $(0,\infty)$, so that,

$$\begin{aligned}
&\int_0^r \Big[t^{1-N}\int_0^t s^{N-1}\rho(s)f(u(s))ds\Big]^{\frac{1}{p-1}}dt\\
&\le \frac{u(r)+b}{v(r)+b}\int_0^r \Big[t^{1-N}\int_0^t s^{N-1}\rho(s)f(v(s))ds\Big]^{\frac{1}{p-1}}dt
\end{aligned}$$

By (2.2), the above inequality and the fact that $\lim_{r\to\infty} u(r) = 0$, we find

$$1 \le \frac{u(0)}{v(0)} = \lim_{r\to\infty}\frac{\int_0^r \Big[t^{1-N}\int_0^t s^{N-1}\rho(s)f(u(s))ds\Big]^{\frac{1}{p-1}}dt}{\int_0^r \Big[t^{1-N}\int_0^t s^{N-1}\rho(s)f(v(s))ds\Big]^{\frac{1}{p-1}}dt} \le 1,$$

so that, by Lemma 2.1, $u = v$.

## 8. Appendix

Recall that $\epsilon$ represents a sufficiently small positive number. Let's proof (3.1)(i) first. Pick $u \in C([0,\epsilon])$. Using (1.2) to estimate the integral expression in $\mathcal{F}(u)$, we obtain for $r \in [0,\epsilon]$,

$$\begin{aligned}
\widehat{I}(r) &:= \int_0^r \Big[t^{1-N}\int_0^t s^{N-1}\rho(s)f(u(s))ds\Big]^{\frac{1}{p-1}}dt\\
&\le a\Big(\frac{f(\frac{a}{\kappa_a})}{(\frac{a}{\kappa_a})^{p-1}}\Big)^{\frac{1}{p-1}} r\Big(\int_0^r \rho(t)dt\Big)^{\frac{1}{p-1}}.
\end{aligned}$$

Therefore, $\mathcal{F}(u) \in C([0,\epsilon])$ and $\widehat{I}(\epsilon) < \frac{\kappa_a-1}{\kappa_a}a$ and as a consequence, $\frac{a}{\kappa_a} \le \mathcal{F}(u)(r) \le a$, showing (3.1)(i). Next we show (3.1)(ii). Taking $u_j \in C([0,\epsilon])$, $j = 1,2$, we find,

$$|\mathcal{F}u_1(r) - \mathcal{F}u_2(r)| \le \int_0^r \Big|\big[X_{u_1}(t)\big]^{\frac{1}{p-1}} - \big[X_{u_2}(t)\big]^{\frac{1}{p-1}}\Big|dt,$$

where

$$X_{u_j}(t) := t^{1-N}\int_0^t s^{N-1}\rho(s)f(u_j(s))ds.$$

Using the inequality,

$$\big||x|^\sigma x - |y|^\sigma y\big| \le C_\sigma(|x|^\sigma + |y|^\sigma)|x-y| \quad x, y \in \mathbb{R} \tag{8.1}$$

where $\sigma > -1$ and $C_\sigma > 0$ are constants, we find,

$$|\mathcal{F}u_1(r) - \mathcal{F}u_2(r)| \le C_\sigma \int_0^r (|X_{u_1}(t)|^\sigma + |X_{u_2}(t)|^\sigma)|X_{u_1}(t) - X_{u_2}(t)|dt, \tag{8.2}$$

where $\sigma = (2-p)/(p-1)$. We point out that

$$\begin{aligned} |X_{u_1}(t) - X_{u_2}(t)| &\le t^{1-N} \int_0^t s^{N-1}\rho(s)|f(u_1(s)) - f(u_2(s))|ds \\ &\le K\|u_1 - u_2\|_{C([0,\epsilon])} t^{1-N} \int_0^t s^{N-1}\rho(s)ds. \end{aligned} \tag{8.3}$$

where $K$ is the Lipschitz constant of $f$ on $[\frac{a}{\kappa_a}, a]$. If $1 < p \le 2$, using (1.2),

$$|X_{u_j}(t)|^\sigma \le a^{2-p}\Big[\frac{f(\frac{a}{\kappa_a})}{(\frac{a}{\kappa_a})^{p-1}}\Big]^\sigma \Big(t^{1-N}\int_0^t s^{N-1}\rho(s)ds\Big)^\sigma. \tag{8.4}$$

From (8.2), (8.3), and (8.4) we find, for constant a $\widehat{K} > 0$,

$$|\mathcal{F}u_1(r) - \mathcal{F}u_2(r)| \le \widehat{K}\epsilon\Big(\int_0^\epsilon \rho(s)ds\Big)^{\frac{1}{p-1}} \|u_1 - u_2\|_{C([0,\epsilon])},$$

and so (3.1)(ii) follows. The case $p > 2$ is treated as the earlier one, replacing (8.4) by,

$$|X_{u_j}(t)|^\sigma \le \big(\frac{a}{\kappa_a}\big)^{2-p}\Big[\frac{f(a)}{a^{p-1}}\Big]^\sigma \Big(t^{1-N}\int_0^t s^{N-1}\rho(s)ds\Big)^\sigma.$$

This shows (3.1). To prove (3.5), we show (3.5)(i) first. To that end let $u \in X_{\tilde{a},\epsilon}$. Using (1.2) we estimate the integral below,

$$\begin{aligned} &\int_{T(a)}^r \Big\{t^{1-N}\Big[T(a)^{N-1}|\nu|^{p-1} + \int_{T(a)}^t s^{N-1}\rho(s)f(u(s))ds\Big]\Big\}^{\frac{1}{p-1}} dt \\ &\le \int_{T(a)}^\epsilon \Big\{t^{1-N}\Big[T(a)^{N-1}|\nu|^{p-1} + \kappa_{\tilde{a}}^{p-1} f(\frac{\tilde{a}}{\kappa_{\tilde{a}}})\int_{T(a)}^t s^{N-1}\rho(s)ds\Big]\Big\}^{\frac{1}{p-1}} dt \\ &\le \frac{\kappa_{\tilde{a}} - 1}{\tilde{a}}. \end{aligned}$$

Hence, $\frac{\tilde{a}}{\kappa_{\tilde{a}}} \le \widetilde{\mathcal{F}}u(r) \le \tilde{a}$, for $r \in [T(a), T(a)+\epsilon]$. This shows (3.5)(i). In order to prove (3.5)(ii), letting $u_j \in X_{\tilde{a},\epsilon}$ $(j = 1, 2)$ and using (8.1),

$$|\widetilde{\mathcal{F}}u_1(r) - \widetilde{\mathcal{F}}u_2(r)| \le C_\sigma \int_{T(a)}^r (|\widetilde{X}_{u_1}(t)|^\sigma + |\widetilde{X}_{u_2}(t)|^\sigma)|\widetilde{X}_{u_1}(t) - \widetilde{X}_{u_2}(t)|dt, \tag{8.5}$$

where

$$\widetilde{X}_{u_j}(t) := t^{1-N}\Big[T(a)^{N-1}|\nu|^{p-1} + \int_{T(a)}^t s^{N-1}\rho(s)f(u_j(s))ds\Big].$$

We remark that since $f \in C^1$,

$$|\widetilde{X}_{u_1}(t) - \widetilde{X}_{u_2}(t)| \le K_0\|u_1 - u_2\|_{C([T(a),T(a)+\epsilon])} t^{1-N}\int_{T(a)}^t s^{N-1}\rho(s)ds, \tag{8.6}$$

where $K_0$ is the Lipschitz constant of $f$ on the interval $[\frac{\tilde{a}}{\kappa_{\tilde{a}}}, \tilde{a}]$. Now, two further cases are considered. The first one is $\nu = 0$. If $1 < p \leq 2$, using (1.2),

$$|\widetilde{X}_{u_j}(t)|^\sigma \leq \kappa_{\tilde{a}}^{2-p} f(\frac{\tilde{a}}{\kappa_{\tilde{a}}})^\sigma \Big[t^{1-N} \int_{T(a)}^t s^{N-1}\rho(s)ds\Big]^\sigma. \tag{8.7}$$

By (8.5),(8.6) and (8.7), it follows that, for some constant $C > 0$,

$$\begin{aligned} &|\widetilde{\mathcal{F}}u_1(r) - \widetilde{\mathcal{F}}u_2(r)| \\ &\leq C\|u_1 - u_2\|_{C([T(a),T(a)+\epsilon])} \int_{T(a)}^{T(a)+\epsilon} \Big[t^{1-N} \int_{T(a)}^t s^{N-1}\rho(s)ds\Big]^{\frac{1}{p-1}} \\ &\leq K_1\|u_1 - u_2\|_{C([T(a),T(a)+\epsilon])}, \end{aligned}$$

where $K_1 \in (0, 1)$, showing (3.5)(ii). If $p \geq 2$, using (1.2) again, one obtains,

$$|\widetilde{X}_{u_j}(t)|^\sigma \leq (\frac{1}{\kappa_{\tilde{a}}})^{2-p} f(\tilde{a})^\sigma \Big[t^{1-N} \int_{T(a)}^t s^{N-1}\rho(s)ds\Big]^\sigma. \tag{8.8}$$

Argueing as before, with (8.8) instead (8.7) we show (3.5)(ii). The second case is $\nu < 0$. If $1 < p \leq 2$, we get by using (1.2),

$$|\widetilde{X}_{u_j}(t)|^\sigma \leq \Big[T(a)^{-1}|\nu|^{p-1} + T(a)^{-1} f(\frac{\tilde{a}}{\kappa_{\tilde{a}}}) \int_{T(a)}^{T(a)+\epsilon} \rho(s)ds\Big]^\sigma t^{\frac{2-p}{p-1}}. \tag{8.9}$$

On the other hand, if $p \geq 2$, we get

$$|\widetilde{X}_{u_j}(t)|^\sigma \leq \Big[\frac{T(a)^{N-1}|\nu|^{p-1}}{(T(a)+\epsilon)^N}\Big]^\sigma t^{\frac{2-p}{p-1}}. \tag{8.10}$$

Proceeding as above, by replacing respectively (8.7) and (8.8) by (8.9) and (8.10), we show (3.5)(ii). This completes the verification of (3.5).

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