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# EXISTENCE AND COMPARISON RESULTS FOR QUASILINEAR EVOLUTION HEMIVARIATIONAL INEQUALITIES

SIEGFRIED CARL, VY K. LE, & DUMITRU MOTREANU

ABSTRACT. We generalize the sub-supersolution method known for weak solutions of single and multivalued nonlinear parabolic problems to quasilinear evolution hemivariational inequalities. To this end we first introduce our basic notion of sub- and supersolutions on the basis of which we then prove existence, comparison, compactness and extremality results for the hemivariational inequalities under considerations.

## 1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$, $Q = \Omega \times (0, \tau)$, and $\Gamma = \partial\Omega \times (0, \tau)$, with $\tau > 0$. In this paper, we study the following quasilinear evolution hemivariational inequality:

$$\begin{gathered} u \in W_0, \ u(\cdot, 0) = 0 \quad \text{in } \Omega \\ \langle \frac{\partial u}{\partial t} + Au - f, v - u \rangle + \int_Q j^o(u; v - u)\, dx\, dt \geq 0, \quad \forall\, v \in V_0, \end{gathered} \tag{1.1}$$

where $V_0 = L^p(0, \tau; W_0^{1,p}(\Omega))$, $2 \leq p < \infty$, with the dual $V_0^* = L^q(0, \tau; W^{-1,q}(\Omega))$, $W_0 = \{w \in V_0 : \partial w/\partial t \in V_0^*\}$, and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $V_0^*$ and $V_0$. The real $q$ is the conjugate to $p$ satisfying $1/p + 1/q = 1$. By $j^o(s; r)$ we denote the generalized directional derivative of the locally Lipschitz function $j : \mathbb{R} \to \mathbb{R}$ at $s$ in the direction $r$ given by

$$j^o(s; r) = \limsup_{y \to s,\ t \downarrow 0} \frac{j(y + t\, r) - j(y)}{t}, \tag{1.2}$$

cf., e.g., [4, Chap. 2]. The operator $A : V \to V_0^*$ is assumed to be a second order quasilinear differential operator in divergence form of Leray-Lions type

$$Au(x, t) = -\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, t, u(x, t), \nabla u(x, t)), \tag{1.3}$$

where $\nabla u = (\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N})$.

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Let $\partial j : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ denote Clarke's generalized gradient of $j$ defined by

$$\partial j(s) := \{\zeta \in \mathbb{R} : j^o(s;r) \geq \zeta\, r,\ \forall r \in \mathbb{R}\}. \tag{1.4}$$

A method of super-subsolutions has been established recently in [2] for quasilinear parabolic differential inclusion problems in the form

$$\frac{\partial u}{\partial t} + Au + \partial j(u) \ni f, \quad \text{in} \quad Q, \quad u = 0 \quad \text{on} \quad \Gamma, \quad u(\cdot, 0) = 0 \quad \text{in} \quad \Omega. \tag{1.5}$$

One can show that any solution of (1.5) is a solution of the hemivariational inequality (1.1). The reverse is true only if the function $j$ is regular in the sense of Clarke which means that the one-sided directional derivative and the generalized directional derivative coincide, cf. [4, Chap. 2.3].

The main goal of this paper is to generalize the sub-supersolution method to the general case of evolution hemivariational inequalities (1.1). This extension is by no means a straightforward generalization of the theory developed for the multivalued problems (1.5) because of the intrinsic asymmetry of hemivariational inequalities compared with the symmetric structure of the multivalued equation (1.5). In this paper we introduce our basic notion of sub- and supersolutions for inequalities in the form (1.1) in a unified and coherent way which is inspired by recent papers on the sub-supersolution method for variational inequalities, see [6, 7].

The plan of the paper is as follows: In Section 2 we introduce the notion of sub-supersolution, and in Section 3 we provide some preliminary results used later. In Section 4 we prove an existence and comparison result in terms of sub- and supersolutions. Topological and extremality results of the solution set within the interval formed by sub- and supersolutions are given in Section 5.

The theory developed in this paper can be extended to evolution hemivariational inequalities involving even more general quasilinear operators of Leray-Lions type and functions $j : Q \times \mathbb{R} \to \mathbb{R}$ depending, in addition, on the space-time variables $(x,t)$. Moreover, without loss of generality homogeneous initial and boundary data have been assumed.

## 2. Notation and hypotheses

Let $W^{1,p}(\Omega)$ denote the usual Sobolev space and $(W^{1,p}(\Omega))^*$ its dual space, and let us assume $2 \leq p < \infty$. Then $W^{1,p}(\Omega) \subset L^2(\Omega) \subset (W^{1,p}(\Omega))^*$ forms an evolution triple with all the embeddings being continuous, dense and compact, cf. [9].

We set $V = L^p(0,\tau; W^{1,p}(\Omega))$ , whose dual space is $V^* = L^q(0,\tau; (W^{1,p}(\Omega))^*)$, and define a function space

$$W = \{u \in V : u_t \in V^*\}\,,$$

where the derivative $u' := u_t = \partial u/\partial t$ is understood in the sense of vector-valued distributions, cf. [9], which is characterized by

$$\int_0^\tau u'(t)\phi(t)\,dt = -\int_0^\tau u(t)\phi'(t)\,dt, \quad \forall\ \phi \in C_0^\infty(0,\tau).$$

The space $W$ endowed with the graph norm

$$\|u\|_W = \|u\|_V + \|u_t\|_{V^*}$$

is a Banach space which is separable and reflexive due to the separability and reflexivity of $V$ and $V^*$, respectively. Furthermore it is well known that the embedding $W \subset C([0,\tau], L^2(\Omega))$ is continuous, cf. [9]. Finally, because $W^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$, we have by Aubin's lemma a compact embedding of $W \subset L^p(Q)$, cf. [9].

By $W_0^{1,p}(\Omega)$ we denote the subspace of $W^{1,p}(\Omega)$ whose elements have generalized homogeneous boundary values. Let $W^{-1,q}(\Omega)$ denote the dual space of $W_0^{1,p}(\Omega)$. Then obviously $W_0^{1,p}(\Omega) \subset L^2(\Omega) \subset W^{-1,q}(\Omega)$ forms an evolution triple and all statements made above remain true also in this situation when setting $V_0 = L^p(0,\tau; W_0^{1,p}(\Omega))$, $V_0^* = L^q(0,\tau; W^{-1,q}(\Omega))$ and $W_0 = \{u \in V_0 : u_t \in V_0^*\}$. Let $\|\cdot\|_V$ and $\|\cdot\|_{V_0}$ be the usual norms defined on $V$ and $V_0$ (and similarly on $V^*$ and $V_0^*$):

$$\|u\|_V = \Big( \int_0^\tau \|u(t)\|^p_{W^{1,p}(\Omega)}\, dt\Big)^{1/p}, \quad \|u\|_{V_0} = \Big( \int_0^\tau \|u(t)\|^p_{W_0^{1,p}(\Omega)}\, dt\Big)^{1/p}.$$

We use the notation $\langle \cdot, \cdot \rangle$ for any of the dual pairings between $V$ and $V^*$, $V_0$ and $V_0^*$, $W^{1,p}(\Omega)$ and $[W^{1,p}(\Omega)]^*$, and $W_0^{1,p}(\Omega)$ and $W^{-1,q}(\Omega)$. For example, with $f \in V^*, u \in V$,

$$\langle f, u\rangle = \int_0^\tau \langle f(t), u(t)\rangle\, dt.$$

Let $L := \partial/\partial t$ and its domain of definition $D(L)$ given by

$$D(L) = \{u \in V_0 : u_t \in V_0^* \text{ and } u(0) = 0\}.$$

The linear operator $L : D(L) \subset V_0 \to V_0^*$ can be shown to be closed, densely defined and maximal monotone, e.g., cf. [9, Chap. 32].

We assume $f \in V_0^*$ and impose the following hypotheses of Leray-Lions type on the coefficient functions $a_i$, $i = 1, \dots, N$, of the operator $A$:

(A1) $a_i : Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions, i.e. $a_i(\cdot,\cdot,s,\xi) : Q \to \mathbb{R}$ is measurable for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and $a_i(x,t,\cdot,\cdot) : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is continuous for a.e. $(x,t) \in Q$. In addition, one has

$$|a_i(x,t,s,\xi)| \le k_0(x,t) + c_0 \left(|s|^{p-1} + |\xi|^{p-1}\right)$$

for a.e. $(x,t) \in Q$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, for some constant $c_0 > 0$ and some function $k_0 \in L^q(Q)$.

(A2) $\sum_{i=1}^N (a_i(x,t,s,\xi) - a_i(x,t,s,\xi'))(\xi_i - \xi_i') > 0$ for a.e. $(x,t) \in Q$, for all $s \in \mathbb{R}$ and all $\xi, \xi' \in \mathbb{R}^N$ with $\xi \neq \xi'$.

(A3) $\sum_{i=1}^N a_i(x,t,s,\xi)\xi_i \ge \nu|\xi|^p - k_1(x,t)$ for a.e. $(x,t) \in Q$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, for some constant $\nu > 0$ and some function $k_1 \in L^1(Q)$.

(A4) $|a_i(x,t,s,\xi) - a_i(x,t,s',\xi)| \le [k_2(x,t) + |s|^{p-1} + |s'|^{p-1} + |\xi|^{p-1}]\omega(|s-s'|)$ for a.e. $(x,t) \in Q$, for all $s, s' \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$, for some function $k_2 \in L^q(Q)$ and a continuous function $\omega : [0,+\infty) \to [0,+\infty)$ satisfying

$$\int_{0^+} \frac{1}{\omega(r)}\, dr = +\infty.$$

For example, we can take $\omega(r) = cr$, with $c > 0$, in (A4).

The operator $A : V \to V^* \subset V_0^*$ related with the quasilinear elliptic operator is defined as follows:

$$\langle A(u), v\rangle = \sum_{i=1}^{N} \int_Q a_i(\cdot,\cdot,u,\nabla u) v_{x_i}\, dxdt, \tag{2.1}$$

for all $v, u \in V$. Due to (A1) the operator $A : V \to V^* \subset V_0^*$ is continuous and bounded, and due to (A2) and (A3) the operator $A : D(L) \subset V_0 \to V_0^*$ is pseudomonotone with respect to the graph norm topology of $D(L)$ (with respect to $D(L)$ for short), and coercive, see, e.g., [1, Theorem E.3.2]. Thus the evolution hemivariational inequality (1.1) may be rewritten as:

$$u \in D(L) : \langle Lu + A(u) - f, v - u\rangle + \int_Q j^o(u; v-u)\, dxdt \geq 0, \quad \forall v \in V_0. \tag{2.2}$$

A partial ordering in $L^p(Q)$ is defined by $u \leq w$ if and only if $w - u$ belongs to the positive cone $L^p_+(Q)$ of all nonnegative elements of $L^p(Q)$. This induces a corresponding partial ordering also in the subspace $W$ of $L^p(Q)$, and if $u, w \in W$ with $u \leq w$ then

$$[u, w] = \{v \in W : u \leq v \leq w\}$$

denotes the order interval formed by $u$ and $w$. Further, for $u, v \in V$, and $U_1, U_2 \subset V$, we use the notation $u \wedge v = \min\{u, v\}$, $u \vee v = \max\{u, v\}$, $U_1 * U_2 = \{u * v : u \in U_1, v \in U_2\}$, $u * U_1 = \{u\} * U_1$ with $* \in \{\wedge, \vee\}$.

Our basic notion of sub-and supersolution of (1.1) is defined as follows:

**Definition 2.1.** A function $\underline{u} \in W$ is called a *subsolution* of (1.1) if the following holds:

(i) $\underline{u}(\cdot, 0) \leq 0$ in $\Omega$, $\underline{u} \leq 0$ on $\Gamma$,
(ii) $\langle \underline{u}_t + A\underline{u} - f, v - \underline{u}\rangle + \int_Q j^o(\underline{u}; v - \underline{u})\, dx\, dt \geq 0, \quad \forall\, v \in \underline{u} \wedge V_0$.

**Definition 2.2.** $\bar{u} \in W$ is a *supersolution* of (1.1) if the following holds:

(i) $\bar{u}(\cdot, 0) \geq 0$ in $\Omega$, $\bar{u} \geq 0$ on $\Gamma$,
(ii) $\langle \bar{u}_t + A\bar{u} - f, v - \bar{u}\rangle + \int_Q j^o(\bar{u}; v - \bar{u})\, dx\, dt \geq 0, \quad \forall\, v \in \bar{u} \vee V_0$.

We assume the following hypothesis for $j$:

(H) The function $j : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and its Clarke's generalized gradient $\partial j$ satisfies the following growth conditions:
  (i) there exists a constant $c_1 \geq 0$ such that
  $$\xi_1 \leq \xi_2 + c_1(s_2 - s_1)^{p-1}$$
  for all $\xi_i \in \partial j(s_i),\ i = 1, 2$, and for all $s_1,\ s_2$ with $s_1 < s_2$.
  (ii) there is a constant $c_2 \geq 0$ such that
  $$\xi \in \partial j(s) : \quad |\xi| \leq c_2\,(1 + |s|^{p-1}), \quad \forall\, s \in \mathbb{R}.$$

**Remark 2.3.** The notion of sub-supersolution introduced here extends that for inclusions of hemivariational type introduced in [2]. To see this let, for example, $\bar{u}$ be a supersolution of the inclusion (1.5), i.e., $\bar{u} \in W$ and there is a function $\eta \in L^q(Q)$ such that $\bar{u}(\cdot, 0) \geq 0$ in $\Omega$, $\bar{u} \geq 0$ on $\Gamma$, $\eta(x,t) \in \partial j(\bar{u}(x,t))$ and the following inequality holds:

$$\langle \bar{u}_t + A\bar{u} - f, \varphi\rangle + \int_Q \eta(x,t)\varphi(x,t)\, dx\, dt \geq 0, \quad \forall\, \varphi \in V_0 \cap L^p_+(Q). \tag{2.3}$$

Thus (2.3), in particular, holds for $\varphi$ in the form $\varphi = (w - \bar{u})^+$, for any $w \in V_0$, which yields by applying the definition of Clarke's generalized gradient the following inequality

$$\langle \bar{u}_t + A\bar{u} - f, (w-\bar{u})^+ \rangle + \int_Q j^o(\bar{u}(x,t); (w-\bar{u})^+(x,t))\,dx\,dt \ge 0, \quad \forall\, w \in V_0. \tag{2.4}$$

Since $\bar{u} \vee w = \bar{u} + (w - \bar{u})^+$, we see that (2.4) is equivalent with Definition 2.2. In the case that $j$ is regular in the sense of Clarke (see [4, Chap. 2.3]) one can prove that the reverse is true, i.e., in this case any supersolution of (1.1) according to Definition 2.2 is also a supersolution of the associated inclusion (1.5). Analogous results hold for subsolutions. Moreover, any solution of (1.1) is both a subsolution and supersolution according to Definition 2.1 and Definition 2.2, respectively.

In the next section we provide some preliminaries used in the proofs of our main results in Sections 4 and 5.

## 3. Preliminaries

First let us recall a general surjectivity result for multivalued operators $\mathcal{A} : X \to 2^X$ in a real reflexive Banach space $X$. To this end we introduce the notion of multivalued pseudomonotone and generalized pseudomonotone operators and their relation to each other, cf., e.g., [8, Chapter 2]. Let $X$ be a real reflexive Banach space.

**Definition 3.1.** The operator $\mathcal{A} : X \to 2^{X^*}$ is called *pseudomonotone* if the following conditions hold:

(i) The set $\mathcal{A}(u)$ is nonempty, bounded, closed and convex for all $u \in X$.
(ii) $\mathcal{A}$ is upper semicontinuous from each finite dimensional subspace of $X$ to the weak topology on $X^*$.
(iii) If $(u_n) \subset X$ with $u_n \rightharpoonup u$, and if $u_n^* \in \mathcal{A}(u_n)$ is such that $\limsup\langle u_n^*, u_n - u\rangle \le 0$, then to each element $v \in X$ there exists $u^*(v) \in \mathcal{A}(u)$ with

$$\liminf\langle u_n^*, u_n - v\rangle \ge \langle u^*(v), u - v\rangle.$$

**Definition 3.2.** The operator $\mathcal{A} : X \to 2^{X^*}$ is called *generalized pseudomonotone* if the following holds:

Let $(u_n) \subset X$ and $(u_n^*) \subset X^*$ with $u_n^* \in \mathcal{A}(u_n)$. If $u_n \rightharpoonup u$ in $X$ and $u_n^* \rightharpoonup u^*$ in $X^*$ and if $\limsup\langle u_n^*, u_n - u\rangle \le 0$, then the element $u^*$ lies in $\mathcal{A}(u)$ and

$$\langle u_n^*, u_n\rangle \to \langle u^*, u\rangle.$$

**Proposition 3.3.** *If the operator $\mathcal{A} : X \to 2^{X^*}$ is pseudomonotone then $\mathcal{A}$ is generalized pseudomonotone.*

Under an additional boundedness condition the following reverse statement is true.

**Proposition 3.4.** *Let $\mathcal{A} : X \to 2^{X^*}$ be a bounded generalized pseudomonotone operator. If for each $u \in X$ we have that $\mathcal{A}(u)$ is a nonempty, closed and convex subset of $X^*$, then $\mathcal{A}$ is pseudomonotone.*

**Definition 3.5.** *The operator $\mathcal{A} : X \to 2^{X^*}$ is called coercive if either the domain of $\mathcal{A}$ denoted by $D(\mathcal{A})$ is bounded or $D(\mathcal{A})$ is unbounded and*

$$\frac{\inf\{\langle v^*, v\rangle : v^* \in \mathcal{A}(v)\}}{\|v\|_X} \to +\infty \quad \text{as} \quad \|v\|_X \to \infty,\ v \in D(\mathcal{A}).$$

Let $L : D(L) \subset X \to X^*$ be a linear, closed, densely defined and maximal monotone operator. We finally introduce the notion of multivalued pseudomonotone operators with respect to the graph norm topology of $D(L)$ (with respect to $D(L)$ for short).

**Definition 3.6.** *The operator $\mathcal{A} : X \to 2^{X^*}$ is called pseudomonotone with respect to D(L) if (i) and (ii) of Definition 3.1 and the following one hold:*

(iv) *If $(u_n) \subset D(L)$ with $u_n \rightharpoonup u$ in $X$, $Lu_n \rightharpoonup Lu$ in $X^*$, $u_n^* \in \mathcal{A}(u_n)$ with $u_n^* \rightharpoonup u^*$ in $X^*$ and $\limsup\langle u_n^*, u_n - u\rangle \le 0$, then $u^* \in \mathcal{A}(u)$ and $\langle u_n^*, u_n\rangle \to \langle u^*, u\rangle$.*

The following surjectivity result which will be used later can be found, e.g., in [5, Theorem 1.3.73, p. 62].

**Theorem 3.7.** *Let $X$ be a real reflexive, strictly convex Banach space with dual space $X^*$, and let $L : D(L) \subset X \to X^*$ be a closed, densely defined and maximal monotone operator. If the multivalued operator $\mathcal{A} : X \to 2^{X^*}$ is pseudomonotone with respect to $D(L)$, bounded and coercive, then $L+\mathcal{A}$ is surjective, i.e., range$(L+\mathcal{A}) = X^*$.*

As already mentioned in Section 2 the operator $L = \partial/\partial t : D(L) \subset V_0 \to V_0^*$ is closed, densely defined and maximal monotone, and under hypotheses (A1)–(A3) the operator $A : V_0 \to V_0^*$ is pseudomonotone with respect to $D(L)$.

Consider the function $J : L^p(Q) \to \mathbb{R}$ defined by

$$J(v) = \int_Q j(v(x,t))\,dx\,dt, \quad \forall\, v \in L^p(Q). \tag{3.1}$$

Using the growth condition (H) (ii) and Lebourg's mean value theorem, we note that the function $J$ is well-defined and Lipschitz continuous on bounded sets in $L^p(Q)$, thus locally Lipschitz so that Clarke's generalized gradient $\partial J : L^p(Q) \to 2^{L^q(Q)}$ is well-defined. Moreover, the Aubin-Clarke theorem (see [4, p. 83]) ensures that, for each $u \in L^p(Q)$ we have

$$\xi \in \partial J(u) \Longrightarrow \xi \in L^q(Q) \text{ with } \xi(x,t) \in \partial j(u(x,t)) \text{ for a.e. } (x,t) \in Q. \tag{3.2}$$

Denote the restriction of $J$ to $V_0$ by $J|_{V_0}$, then the following result holds.

**Lemma 3.8.** *Under hypothesis (H)(ii) Clarke's generalized gradient $\partial(J|_{V_0}) : V_0 \to 2^{V_0^*}$ is pseudomonotone with respect to $D(L)$.*

*Proof.* The growth condition (H) (ii) implies that $\partial(J|_{V_0}) : V_0 \to 2^{V_0^*}$ is bounded. From the calculus of Clarke's generalized gradient (see [4, Chap. 2]) we know that $\partial(J|_{V_0})(u)$ is nonempty, closed and convex. Condition (ii) in Definition 3.1 is also satisfied (see [4, p.29]). Therefore, in view of Proposition 3.4 we only need to show that $\partial(J|_{V_0})$ satisfies property (iv) of Definition 3.6. To this end let $(u_n) \subset D(L)$ with $u_n \rightharpoonup u$ in $V_0$, $Lu_n \rightharpoonup Lu$ in $V_0^*$, $u_n^* \in \partial(J|_{V_0})(u_n)$ with $u_n^* \rightharpoonup u^*$ in $V_0^*$. We are going to show that already under these assumptions we get $u^* \in \partial(J|_{V_0})(u)$ and $\langle u_n^*, u_n\rangle \to \langle u^*, u\rangle$, which is (iv). By the assumptions on $(u_n)$ we have $u_n \rightharpoonup u$ in $W_0$, which implies $u_n \to u$ in $L^p(Q)$ due to the compact embedding $W_0 \subset L^p(Q)$. Since $V_0$ is dense in $L^p(Q)$ we know that $u_n^* \in \partial J(u_n)$, see [4, p. 47], and thus $u_n^* \in L^q(Q)$ with $u_n^* \rightharpoonup u^*$ in $L^q(Q)$. Because the mapping $\partial J : L^p(Q) \to 2^{L^q(Q)}$ is weak-closed (cf. [4, p. 29] and note $L^q(Q)$ is reflexive), we deduce that $u^* \in \partial J(u)$, and, moreover, the following holds:

$$\langle u_n^*, u_n\rangle_{V_0^*,V_0} = \langle u_n^*, u_n\rangle_{L^q(Q),L^p(Q)} \to \langle u^*, u\rangle_{L^q(Q),L^p(Q)} = \langle u^*, u\rangle_{V_0^*,V_0},$$

which completes the proof. □

**Corollary 3.9.** *Assume hypotheses (A1)–(A3) and (H)(ii), and let $A : V_0 \to V_0^*$ be the operator as defined in (2.1). Then $A + \partial(J|_{V_0}) : V_0 \to 2^{V_0^*}$ is pseudomonotone with respect to $D(L)$ and bounded.*

*Proof.* The Leray-Lions conditions (A1)–(A3) imply that the (singlevalued) operator $A$ is pseudomonotone with respect to $D(L)$, and by Lemma 3.8 the multivalued operator $\partial(J|_{V_0}) : V_0 \to 2^{V_0^*}$ is pseudomonotone with respect to $D(L)$ as well. To prove that $A + \partial(J|_{V_0}) : V_0 \to 2^{V_0^*}$ is pseudomonotone with respect to $D(L)$ note first that $A + \partial(J|_{V_0}) : V_0 \to 2^{V_0^*}$ is bounded. Thus we only need to verify property (iv) of Definition 3.6. To this end assume $(u_n) \subset D(L)$ with $u_n \rightharpoonup u$ in $V_0$, $Lu_n \rightharpoonup Lu$ in $V_0^*$, $u_n^* \in (A + \partial(J|_{V_0}))(u_n)$ with $u_n^* \rightharpoonup u^*$ in $V_0^*$, and

$$\limsup_n \langle u_n^*, u_n - u\rangle \le 0. \tag{3.3}$$

We need to show that $u^* \in (A + \partial(J|_{V_0}))(u)$ and $\langle u_n^*, u_n\rangle \to \langle u^*, u\rangle$. Due to $u_n^* \in (A + \partial(J|_{V_0}))(u_n)$ we have $u_n^* = Au_n + \eta_n$ with $\eta_n \in \partial(J|_{V_0}))(u_n)$, and (3.3) reads

$$\limsup_n \langle Au_n + \eta_n, u_n - u\rangle \le 0. \tag{3.4}$$

Because the sequence $(\eta_n) \subset L^q(Q)$ is bounded and $u_n \to u$ in $L^p(Q)$ we obtain

$$\langle \eta_n, u_n - u\rangle = \int_Q \eta_n\,(u_n - u)\,dx\,dt \to 0 \quad \text{as} \quad n \to \infty. \tag{3.5}$$

From (3.4) and (3.5) we deduce

$$\limsup_n \langle Au_n, u_n - u\rangle \le 0. \tag{3.6}$$

The sequence $(Au_n) \subset V_0^*$ is bounded, so that there is some subsequence $(Au_k)$ with $Au_k \rightharpoonup v$. Since $A$ is pseudomonotone with respect to $D(L)$, it follows that $v = Au$ and $\langle Au_k, u_k\rangle \to \langle Au, u\rangle$. This shows that each weakly convergent subsequence of $(Au_n)$ has the same limit $Au$, and thus the entire sequence $(Au_n)$ satisfies

$$Au_n \rightharpoonup Au \quad \text{and} \quad \langle Au_n, u_n\rangle \to \langle Au, u\rangle. \tag{3.7}$$

From (3.7) and $u_n^* = Au_n + \eta_n \rightharpoonup u^*$ we obtain $\eta_n = u_n^* - Au_n \rightharpoonup u^* - Au$, which in view of (3.5) and the pseudomonotonicity of $\partial(J|_{V_0})$ implies $u^* - Au \in \partial(J|_{V_0})(u)$, and thus $u^* \in (A + \partial(J|_{V_0}))(u)$, and, moreover

$$\langle u_n^* - Au_n, u_n\rangle \to \langle u^* - Au, u\rangle,$$

which yields $\langle u_n^*, u_n\rangle \to \langle u^*, u\rangle$. □

## 4. Existence and comparison result

The main result of this paper is the following theorem.

**Theorem 4.1.** *Let hypotheses (A1)–(A4) and (H) be satisfied. Given subsolutions $\underline{u}_i$ and supersolutions $\bar{u}_i$, $i = 1, 2$, of (1.1) such that $\max\{\underline{u}_1, \underline{u}_2\} =: \underline{u} \le \bar{u} := \min\{\bar{u}_1, \bar{u}_2\}$. Then there exist solutions of (1.1) within the order interval $[\underline{u}, \bar{u}]$.*

*Proof.* The proof will be carried out in three steps: (a), (b), and (c).
(a) Auxiliary hemivariational inequality.
Let us first introduce the cut-off function $b : Q \times \mathbb{R} \to \mathbb{R}$ related with the ordered pair of functions $\underline{u}$, $\bar{u}$, and given by

$$b(x,t,s) = \begin{cases} (s - \bar{u}(x,t))^{p-1} & \text{if } s > \bar{u}(x,t), \\ 0 & \text{if } \underline{u}(x,t) \le s \le \bar{u}(x,t), \\ -(\underline{u}(x,t) - s)^{p-1} & \text{if } s < \underline{u}(x,t). \end{cases} \tag{4.1}$$

One readily verifies that $b$ is a Carathéodory function satisfying the growth condition

$$|b(x,t,s)| \le k_2(x,t) + c_3\,|s|^{p-1} \tag{4.2}$$

for a.e. $(x,t) \in Q$, for all $s \in \mathbb{R}$, with some function $k_2 \in L^q_+(Q)$ and a constant $c_3 > 0$. Moreover, one has the following estimate

$$\int_Q b(x,t,u(x,t))\,u(x,t)\,dxdt \ge c_4\,\|u\|^p_{L^p(Q)} - c_5, \quad \forall u \in L^p(Q), \tag{4.3}$$

where $c_4$ and $c_5$ are some positive constants. In view of (4.2) the Nemytskij operator $B : L^p(Q) \to L^q(Q)$ defined by

$$Bu(x,t) = b(x,t,u(x,t))$$

is continuous and bounded, and thus due to the compact embedding $W_0 \subset L^p(Q)$ it follows that $B : W_0 \to L^q(Q) \subset V_0^*$ is completely continuous, which implies that $B : V_0 \to V_0^*$ is compact with respect to $D(L)$. Let us consider the following auxiliary evolution hemivariational inequality:

$$u \in D(L) : \langle Lu + A(u) + \lambda\,B(u) - f, v - u\rangle + \int_Q j^o(u; v-u)\,dx\,dt \ge 0, \quad \forall\, v \in V_0, \tag{4.4}$$

where $\lambda$ is some positive constant to be specified later. The existence of solutions of (4.4) will be proved by using Theorem 3.7. To this end consider the multivalued operator $A + \lambda\,B + \partial(J|_{V_0}) : V_0 \to 2^{V_0^*}$, where $J$ is the locally Lipschitz functional defined in (3.1) and $\partial(J|_{V_0})$ is the generalized Clarke's gradient of the restriction $J|_{V_0}$. By Corollary 3.9 and the property of $B$ we readily see that $A + \lambda\,B + \partial(J|_{V_0}) : V_0 \to 2^{V_0^*}$ is pseudomonotone with respect to $D(L)$ and bounded. In order to apply Theorem 3.7 we need to show the coercivity of $A + \lambda\,B + \partial(J|_{V_0}) : V_0 \to 2^{V_0^*}$. For any $v \in V_0 \setminus \{0\}$ and any $w \in \partial(J|_{V_0})(v)$ we obtain by applying (A3), (H) (ii) and (4.3) the estimate

$$\begin{aligned}
&\frac{1}{\|v\|_{V_0}}\langle Av + \lambda B(v) + w, v\rangle \\
&= \frac{1}{\|v\|_{V_0}}\Big[\int_Q \sum_{i=1}^N a_i(\cdot,\cdot,v,\nabla v)\frac{\partial v}{\partial x_i}\,dx\,dt + \lambda\langle B(v), v\rangle + \int_Q wv\,dx\,dt\Big] \\
&\ge \frac{1}{\|v\|_{V_0}}\Big[\nu\int_Q |\nabla v|^p\,dx\,dt - \int_Q k_1\,dx\,dt + c_4\lambda\|v\|^p_{L^p(Q)} \\
&\quad - c_5\lambda - c_2\int_Q (1 + |v|^{p-1})|v|\,dx\,dt\Big] \\
&\ge \frac{1}{\|v\|_{V_0}}\big[\nu\|v\|^p_{V_0} - C_0\big],
\end{aligned}$$

for some constant $C_0 > 0$, by choosing the constant $\lambda$ sufficiently large such that $c_4\lambda > c_2$, which implies the coercivity. Thus we may apply Theorem 3.7 to ensure that $range\,(L+A+\lambda\,B+\partial(J|_{V_0})) = V_0^*$, which yields the existence of an $u \in D(L)$ such that $f \in Lu + A(u) + \lambda\,B(u) + \partial(J|_{V_0})(u)$, i.e., there exists an $\xi \in \partial(J|_{V_0})(u)$ such that

$$u \in D(L): \quad Lu + A(u) - f + \lambda B(u) + \xi = 0 \quad \text{in } V_0^*. \tag{4.5}$$

Since $V_0$ is dense in $L^p(Q)$ we get $\xi \in \partial J(u)$ and thus by the characterization (3.2) of $\partial J(u)$ it follows that $\xi \in L^q(Q)$ and $\xi(x,t) \in \partial j(u(x,t))$, so that from (4.5) we get

$$\langle Lu + A(u) - f + \lambda B(u), \varphi\rangle + \int_Q \xi(x,t)\varphi(x,t)\,dx\,dt = 0, \quad \forall\,\varphi \in V_0. \tag{4.6}$$

By definition of Clarke's generalized gradient $\partial j$ it follows

$$\int_Q \xi(x,t)\,\varphi(x,t)\,dx\,dt \le \int_Q j^o(u(x,t);\varphi(x,t))\,dx\,dt, \quad \forall\,\varphi \in V_0. \tag{4.7}$$

In view of (4.6) and (4.7), (4.4) has a solution. Next we shall show that any solution $u$ of the auxiliary evolution hemivariational inequality (4.4) satisfies $\underline{u} \le u \le \bar u$.

(b) Comparison: $u \in [\underline{u}, \bar u]$.
Let $u$ be any solution of (4.4). We are going to show that $\underline{u}_k \le u \le \bar u_j$ holds, where $k,\, j = 1,2$, which implies the assertion. Let us first prove that $u \le \bar u_j$ is true. By Definition 2.2 $\bar u_j$ satisfies $\bar u_j(\cdot,0) \ge 0$ in $\Omega$, $\bar u_j \ge 0$ on $\Gamma$, and

$$\langle \frac{\partial \bar u_j}{\partial t} + A\bar u_j - f, v - \bar u_j\rangle + \int_Q j^o(\bar u_j; v - \bar u_j)\,dx\,dt \ge 0, \quad \forall v \in \bar u_j \vee V_0, \tag{4.8}$$

which implies due to $v = \bar u_j \vee \varphi = \bar u_j + (\varphi - \bar u_j)^+$ with $\varphi \in V_0$ and $w^+ = w \vee 0$ the following inequality

$$\langle \frac{\partial \bar u_j}{\partial t} + A\bar u_j - f, (\varphi - \bar u_j)^+\rangle + \int_Q j^o(\bar u_j; (\varphi - \bar u_j)^+)\,dx\,dt \ge 0, \quad \forall\,\varphi \in V_0. \tag{4.9}$$

Let $M := \{(\varphi - \bar u_j)^+ : \varphi \in V_0\}$, then one can show that the closure $\overline{M}^{V_0} = V_0 \cap L^p_+(Q)$. Since $s \mapsto j^o(r;s)$ is continuous, we get from (4.9) by using Fatou's lemma the inequality

$$\langle \frac{\partial \bar u_j}{\partial t} + A\bar u_j - f, \psi\rangle + \int_Q j^o(\bar u_j; \psi)\,dx\,dt \ge 0, \quad \forall\,\psi \in V_0 \cap L^p_+(Q). \tag{4.10}$$

Taking in (4.4) the special test function $v = u - \psi$ and adding (4.4) and (4.10) we obtain:

$$\langle \frac{\partial u}{\partial t} - \frac{\partial \bar u_j}{\partial t} + A(u) - A(\bar u_j) + \lambda\,B(u), \psi\rangle \le \int_Q \Big(j^o(\bar u_j;\psi) + j^o(u;-\psi)\Big)\,dx\,dt \tag{4.11}$$

for all $\psi \in V_0 \cap L^p_+(Q)$. Now we construct a special test function in (4.11). By (A4), for any fixed $\varepsilon > 0$ there exists $\delta(\varepsilon) \in (0,\varepsilon)$ such that

$$\int_{\delta(\varepsilon)}^{\varepsilon} \frac{1}{\omega(r)}\,dr = 1.$$

We define the function $\theta_\varepsilon : \mathbb{R} \to \mathbb{R}_+$ by

$$\theta_\varepsilon(s) = \begin{cases} 0 & \text{if } s < \delta(\varepsilon) \\ \displaystyle\int_{\delta(\varepsilon)}^s \frac{1}{\omega(r)}\,dr & \text{if } \delta(\varepsilon) \le s \le \varepsilon \\ 1 & \text{if } s > \varepsilon. \end{cases}$$

We readily verify that, for each $\varepsilon > 0$, the function $\theta_\varepsilon$ is continuous, piecewise differentiable and the derivative is nonnegative and bounded. Therefore the function $\theta_\varepsilon$ is Lipschitz continuous and nondecreasing. In addition, it satisfies

$$\theta_\varepsilon \to \chi_{\{s>0\}} \quad \text{as } \varepsilon \to 0, \tag{4.12}$$

where $\chi_{\{s>0\}}$ is the characteristic function of the set $\{s > 0\}$. Moreover, one has

$$\theta'_\varepsilon(s) = \begin{cases} 1/\omega(s) & \text{if } \delta(\varepsilon) < s < \varepsilon \\ 0 & \text{if } s \notin [\delta(\varepsilon), \varepsilon]. \end{cases}$$

Taking in (4.11) the test function $\theta_\varepsilon(u - \bar u_j) \in V_0 \cap L^p_+(Q)$ we get

$$\begin{aligned} &\langle \frac{\partial(u-\bar u_j)}{\partial t}, \theta_\varepsilon(u-\bar u_j)\rangle + \langle A(u) - A(\bar u_j), \theta_\varepsilon(u-\bar u_j)\rangle \\ &+ \lambda \int_Q B(u)\,\theta_\varepsilon(u-\bar u_j)\,dx\,dt \\ &\le \int_Q \Big( j^o(\bar u_j; \theta_\varepsilon(u-\bar u_j)) + j^o(u; -\theta_\varepsilon(u-\bar u_j))\Big)\,dx\,dt. \end{aligned} \tag{4.13}$$

Let $\Theta_\varepsilon$ be the primitive of the function $\theta_\varepsilon$ defined by

$$\Theta_\varepsilon(s) = \int_0^s \theta_\varepsilon(r)\,dr.$$

We obtain for the first term on the left-hand side of (4.13) (cf., e.g., [3]) that

$$\langle \frac{\partial(u-\bar u_j)}{\partial t}, \theta_\varepsilon(u-\bar u_j)\rangle = \int_\Omega \Theta_\varepsilon(u-\bar u_j)(x,\tau)\,dx \ge 0. \tag{4.14}$$

Using (A4) and (A2), the second term on the left-hand side of (4.13) can be estimated as follows

$$\begin{aligned} &\langle A(u) - A(\bar u_j), \theta_\varepsilon(u-\bar u_j)\rangle \\ &= \sum_{i=1}^N \int_Q (a_i(x,t,u,\nabla u) - a_i(x,t,\bar u_j,\nabla \bar u_j)) \frac{\partial}{\partial x_i}\theta_\varepsilon(u-\bar u_j)\,dx\,dt \\ &\ge \sum_{i=1}^N \int_Q (a_i(x,t,u,\nabla u) - a_i(x,t,u,\nabla \bar u_j)) \frac{\partial(u-\bar u_j)}{\partial x_i}\theta'_\varepsilon(u-\bar u_j)\,dx\,dt \\ &- N\int_Q (k_2 + |u|^{p-1} + |\bar u_j|^{p-1} + |\nabla \bar u_j|^{p-1})\,\omega(|u-\bar u_j|)\theta'_\varepsilon(u-\bar u_j)|\nabla(u-\bar u_j)|\,dx\,dt \\ &\ge -N \int_{\{\delta(\varepsilon) < u - \bar u_j < \varepsilon\}} \gamma\,|\nabla(u-\bar u_j)|\,dx\,dt, \end{aligned} \tag{4.15}$$

where $\gamma = k_2 + |u|^{p-1} + |\bar u_j|^{p-1} + |\nabla \bar u_j|^{p-1} \in L^q(Q)$. The term on the right-hand side of (4.15) tends to zero as $\varepsilon \to 0$.

Using (4.12) and applying Lebesgue's dominated convergence theorem it follows

$$\lim_{\varepsilon\to 0}\int_Q B(u)\,\theta_\varepsilon(u-\bar u_j)\,dx\,dt=\int_Q B(u)\,\chi_{\{u-\bar u_j>0\}}\,dx\,dt. \tag{4.16}$$

Again by applying Fatou's lemma and the continuity of $s\mapsto j^o(r;s)$ we obtain the following estimate for the right-hand side of (4.13)

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\Big(\int_Q\Big(j^o(\bar u_j;\theta_\varepsilon(u-\bar u_j))+j^o(u;-\theta_\varepsilon(u-\bar u_j))\Big)\,dx\,dt\Big)\\ &\le\int_Q\Big(j^o(\bar u_j;\chi_{\{u-\bar u_j>0\}})+j^o(u;-\chi_{\{u-\bar u_j>0\}})\Big)\,dx\,dt.\end{aligned} \tag{4.17}$$

Finally from (4.13)–(4.17) one gets the inequality:

$$\lambda\int_Q B(u)\chi_{\{u-\bar u_j>0\}}\,dx\,dt\le\int_Q\Big(j^o(\bar u_j;\chi_{\{u-\bar u_j>0\}})+j^o(u;-\chi_{\{u-\bar u_j>0\}})\Big)\,dx\,dt. \tag{4.18}$$

Note that $\bar u=\min\{\bar u_1,\bar u_2\}$, which by definition of the operator $B$ yields

$$\lambda\int_Q B(u)\chi_{\{u-\bar u_j>0\}}\,dx\,dt=\lambda\int_{\{u>\bar u_j\}}(u-\bar u)^{p-1}dx\,dt\ge\lambda\int_{\{u>\bar u_j\}}(u-\bar u_j)^{p-1}dx\,dt. \tag{4.19}$$

The function $r\mapsto j^o(s;r)$ is finite and positively homogeneous, $\partial j(s)$ is a nonempty, convex and compact subset of $\mathbb{R}$, and one has

$$j^o(s;r)=\max\{\xi\,r:\xi\in\partial j(s)\}. \tag{4.20}$$

By using (H)(i), (4.20) and the properties of $j^o$ and $\partial j$ we get for certain $\xi(x,t)\in\partial j(u(x,t))$ and $\bar\xi_j(x,t)\in\partial j(\bar u_j(x,t))$ with $\xi$, $\bar\xi_j\in L^q(Q)$ the following estimate:

$$\begin{aligned}&\int_Q\Big(j^o(\bar u_j;\chi_{\{u-\bar u_j>0\}})+j^o(u;-\chi_{\{u-\bar u_j>0\}})\Big)\,dx\,dt\\ &=\int_{\{u>\bar u_j\}}\Big(j^o(\bar u_j;1)+j^o(u;-1)\Big)\,dx\,dt\\ &=\int_{\{u>\bar u_j\}}(\bar\xi_j(x,t)-\xi(x,t))\,dx\,dt\le c_1\int_{\{u>\bar u_j\}}(u(x,t)-\bar u_j(x,t))^{p-1}\,dx\,dt.\end{aligned} \tag{4.21}$$

Thus (4.18), (4.19) and (4.21) result in

$$(\lambda-c_1)\int_{\{u>\bar u_j\}}(u-\bar u_j)^{p-1}\,dx\,dt\le 0. \tag{4.22}$$

Selecting $\lambda$ large enough such that $\lambda>c_1$, then (4.22) implies that meas$\{u>\bar u_j\}=0$, and thus $u\le\bar u_j$ in $Q$, where $j=1,2$, which shows that $u\le\bar u$. The proof of the inequality $\underline{u}\le u$ can be done analogously.

(c) Completion of the proof of the theorem.
From steps (a) and (b) it follows that any solution $u$ of the auxiliary evolution hemivariational inequality (4.4) with $\lambda>0$ sufficiently large satisfies $u\in[\underline{u},\bar u]$, which implies $B(u)=0$, and hence $u$ is a solution of the original evolution hemivariational inequality (1.1) within the interval $[\underline{u},\bar u]$. This completes the proof of Theorem 4.1. □

The following corollaries are immediate consequences of Theorem 4.1.

**Corollary 4.2.** *Let $\underline{w}$ and $\bar{w}$ be any subsolution and supersolution, respectively of (1.1) satisfying $\underline{w} \le \bar{w}$. Then there exist solutions of (1.1) within the order interval $[\underline{w}, \bar{w}]$.*

*Proof.* Set $\underline{w} = \underline{u}_1 = \underline{u}_2$ and $\bar{w} = \bar{u}_1 = \bar{u}_2$ and apply Theorem 4.1. □

Let $\mathcal{S}$ denote the set of all solutions of (1.1) within the interval $[\underline{w}, \bar{w}]$ of an ordered pair of sub- and supersolutions. We introduce the following notion from set theory.

**Definition 4.3.** Let $(\mathcal{P}, \le)$ be a partially ordered set. A subset $\mathcal{C}$ of $\mathcal{P}$ is said to be *upward directed* if for each pair $x, y \in \mathcal{C}$ there is a $z \in \mathcal{C}$ such that $x \le z$ and $y \le z$, and $\mathcal{C}$ is *downward directed* if for each pair $x, y \in \mathcal{C}$ there is a $w \in \mathcal{C}$ such that $w \le x$ and $w \le y$. If $\mathcal{C}$ is both upward and downward directed it is called *directed*.

**Corollary 4.4.** *The solution set $\mathcal{S}$ of (1.1) is a directed set.*

*Proof.* Let $u_1$, $u_2 \in \mathcal{S}$. Since any solution of (1.1) is a subsolution and a supersolution as well, by Theorem 4.1 there exist solutions of (1.1) within $[\max\{u_1, u_2\}, \bar{w}]$ and also within $[\underline{w}, \min\{u_1, u_2\}]$, which proves the directedness. □

## 5. Compactness and Extremality Results

In this section we show that the solution set $\mathcal{S}$ of (1.1) within the interval of an ordered pair of sub-and supersolutions $[\underline{w}, \bar{w}]$ possesses the smallest and greatest elements with respect to the given partial ordering. The smallest and greatest element of $\mathcal{S}$ are called the *extremal solutions* of (1.1) within $[\underline{w}, \bar{w}]$. We shall assume hypotheses (A1)–(A4) and (H) throughout this section.

**Theorem 5.1.** *The solution set $\mathcal{S}$ is weakly sequentially compact in $W_0$ and compact in $V_0$.*

*Proof.* The solution set $\mathcal{S} \subset [\underline{w}, \bar{w}]$ is bounded in $L^p(Q)$. We next show that $\mathcal{S}$ is bounded in $W_0$. Let $u \in \mathcal{S}$ be given, and take as a special test function in (1.1) $v = 0$. This leads to

$$\langle u_t + Au, u\rangle \le \langle f, u\rangle + \int_Q j^o(u; -u)\, dx\, dt. \tag{5.1}$$

Since

$$\langle u_t, u\rangle = \frac{1}{2}\|u(\cdot, \tau)\|^2_{L^2(\Omega)} \ge 0,$$

and

$$\int_Q j^o(u; -u)\, dx\, dt \le c_2 \int_Q (1 + |u|^{p-1})\, |u|\, dx\, dt,$$

we get from (5.1) by means of (A3) and taking the $L^p(Q)$-boundedness of $\mathcal{S}$ into account the following uniform estimate

$$\|u\|_{V_0} \le C, \quad \forall\, u \in \mathcal{S}. \tag{5.2}$$

Taking in (1.1) the special test function $v = u - \varphi$, where $\varphi \in B = \{v \in V_0 : \|v\|_{V_0} \le 1\}$ we obtain

$$|\langle u_t, \varphi\rangle| \le |\langle f, \varphi\rangle| + |\langle Au, \varphi\rangle| + \Big|\int_Q j^o(u; -\varphi)\, dx\, dt\Big|. \tag{5.3}$$

In view of (5.2), we obtain from (5.3)

$$|\langle u_t, \varphi\rangle| \le \text{const}, \quad \forall\, \varphi \in B, \tag{5.4}$$

where the constant on the right-hand side of (5.4) does not depend on $u$, and thus from (5.2) and (5.4) we get

$$\|u\|_{W_0} \le C, \quad \forall\, u \in \mathcal{S}. \tag{5.5}$$

Now let $(u_n) \subset \mathcal{S}$ be any sequence. Then by (5.5) there exists a weakly convergent subsequence $(u_k)$ with

$$u_k \rightharpoonup u \ \text{ in } \ W_0.$$

Since $u_k$ are solutions of (1.1), we have

$$\langle \frac{\partial u_k}{\partial t} + Au_k - f, v - u_k\rangle + \int_Q j^o(u_k; v - u_k)\, dx\, dt \ge 0, \quad \forall\ v \in V_0. \tag{5.6}$$

Taking as special test function the weak limit $u$ we get

$$\begin{aligned} \langle Au_k, u_k - u\rangle &\le \langle \frac{\partial u_k}{\partial t} - f, u - u_k\rangle + \int_Q j^o(u_k; u - u_k)\, dx\, dt \\ &\le \langle \frac{\partial u}{\partial t} - f, u - u_k\rangle + \int_Q j^o(u_k; u - u_k)\, dx\, dt. \end{aligned} \tag{5.7}$$

The weak convergence of $(u_k)$ in $W_0$ implies $u_k \to u$ in $L^p(Q)$ due to the compact embedding $W_0 \subset L^p(Q)$, and thus by applying (H) (ii) the right-hand side of (5.7) tends to zero as $k \to \infty$, which yields

$$\limsup_k \langle Au_k, u_k - u\rangle \le 0. \tag{5.8}$$

Since $A$ is pseudomonotone with respect to $D(L)$, from (5.8) we get

$$Au_k \rightharpoonup Au \quad \text{and} \quad \langle Au_k, u_k\rangle \to \langle Au, u\rangle, \tag{5.9}$$

and, moreover, because $A$ has the $(\mathrm{S}_+)-$property with respect to $D(L)$ the strong convergence $u_k \to u$ in $V_0$ holds, see, e.g., [1, Theorem E.3.2]. The convergence properties of the subsequence $(u_k)$ obtained so far and the upper semicontinuity of $j^o : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ finally allow the passage to the limit in (5.6), which completes the proof. □

**Theorem 5.2.** *The solution set $\mathcal{S}$ possesses extremal elements.*

*Proof.* We prove the existence of the greatest solution of (1.1) within $[\underline{w}, \bar{w}]$, i.e., the greatest element of $\mathcal{S}$. The proof of the smallest element can be done in a similar way. Since $W_0$ is separable, $\mathcal{S} \subset W_0$ is separable as well, and there exists a countable, dense subset $Z = \{z_n \,:\, n \in \mathbb{N}\}$ of $\mathcal{S}$. By Corollary 4.4 $\mathcal{S}$ is a directed set. This allows the construction of an increasing sequence $(u_n) \subset \mathcal{S}$ as follows. Let $u_1 = z_1$. Select $u_{n+1} \in \mathcal{S}$ such that

$$\max\{z_n, u_n\} \le u_{n+1} \le \overline{w}.$$

The existence of $u_{n+1}$ is due to Corollary 4.4. Since $(u_n)$ is increasing and both bounded and order-bounded, we deduce by applying Lebesgue's dominated convergence theorem that $u_n \to w := \sup_n u_n$ strongly in $L^p(Q)$. By Theorem 5.1 we find a subsequence $(u_k)$ of $(u_n)$, and an element $u \in \mathcal{S}$ such that $u_k \rightharpoonup u$ in $W_0$, and $u_k \to u$ in $L^p(Q)$ and in $V_0$. Thus $u = w$ and each weakly convergent subsequence must have the same limit $w$, which implies that the entire increasing sequence $(u_n)$ satisfies:

$$u_n,\, w \in \mathcal{S}: \quad u_n \rightharpoonup w \;\text{ in } W_0, \;\; u_n \to w \;\text{ in }\; V_0. \tag{5.10}$$

By construction, we see that $\max\{z_1, z_2, \dots, z_n\} \le u_{n+1} \le w$, for all $n$; thus $Z \subset [\underline{w}, w]$. Since the interval $[\underline{w}, w]$ is closed in $W_0$, we infer

$$\mathcal{S} \subset \overline{Z} \subset \overline{[\underline{w}, w]} = [\underline{w}, w],$$

which in conjunction with $w \in \mathcal{S}$ ensures that $w$ is the greatest element of $\mathcal{S}$. $\square$

**Remark 5.3.** It should be noted that our main results of Section 4 and Section 5 remain valid also in case that the operator $A$ involves quasilinear first order terms, i.e., operators $A$ in the form

$$Au(x,t) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x,t,u(x,t),\nabla u(x,t)) + a_0(x,t,u(x,t),\nabla u(x,t)), \tag{5.11}$$

where $a_0 : Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the same regularity and growth condition as $a_i$, $i = 1, \dots, N$.

Next we provide examples to demonstrate the applicability of the theory developed in this paper.

**Example 5.4.** Let $c_P$ denote the best constant in Poincaré's inequality, i.e.,

$$\int_Q |\nabla v|^p \,dx\,dt \ge c_P \int_Q |v|^p \,dx\,dt, \quad \forall v \in V_0.$$

Assume that (A1)–(A4) and (H) hold, and suppose in addition

(a) $a_i(x,t,0,0) = 0$ for a.e. $(x,t) \in Q$, $i = 1, \dots, N$.
(b) $f \in L^q(Q)$ satisfying $f(x,t) \ge \max\{0, \min_{\zeta \in \partial j(0)} \zeta\}$ for a.e. $(x,t) \in Q$.
(c) $k_1 = 0$ in assumption (A3).
(d) $c_P\,\nu > c_2$, where $\nu$ and $c_2$ are the constants in (A3) and (H) (ii), respectively.

Under these assumptions, problem (1.1) admits an extremal nonnegative solution.

First, we check that $\underline{u} = 0$ is a subsolution of problem (1.1). Indeed, using Definition 2.1 we have to check the inequality

$$\langle A0 - f, v\rangle + \int_Q j^o(0; v)\,dx\,dt \ge 0,$$

for all $v \in 0 \wedge V_0 = \{\min\{0, w\} : w \in V_0\} = \{-w^- : w \in V_0\}$ (where $w^- = \max\{0, -w\}$). Taking into account assumption (a), this reduces to

$$\int_Q (j^o(0; -1) + f) w^- \,dx\,dt \ge 0, \quad \forall\, w \in V_0.$$

This is true due to assumption (b) because

$$f(x,t) \ge \min_{\zeta \in \partial j(0)} \zeta = -\max_{\zeta \in \partial j(0)} \zeta(-1) = -j^o(0; -1) \text{ for a.e. } (x,t) \in Q.$$

The claim that $\underline{u} = 0$ is a subsolution of (1.1) is verified.

Consider now the initial boundary value problem

$$\begin{gathered}
\frac{\partial u}{\partial t}-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i(x,t,u,\nabla u)-c_2(1+|u|^{p-1})=f \quad \text{in } Q,\\
u(\cdot,0)=0 \quad \text{in } \Omega,\\
u=0 \quad \text{on } \Gamma,
\end{gathered} \tag{5.12}$$

which may be rewritten as the following abstract problem:

$$u\in D(L): Lu+A(u)+G(u)=f \quad \text{in } \ V_0^*, \tag{5.13}$$

where $G: V_0\to V_0^*$ is defined by

$$\langle G(u),v\rangle=-c_2\int_Q(1+|u|^{p-1})v\,dx\,dt.$$

One easily verifies that $A+G: V_0\to V_0^*$ is bounded, continuous and pseudomonotone with respect to $D(L)$, and due to condition (d) given above $A+G: V_0\to V_0^*$ is also coercive. Thus $L+A+G: D(L)\subset V_0\to V_0^*$ is surjective, which implies that (5.13) and hence (5.12) possesses solutions.

We are going to show that any solution of (5.12) is nonnegative and a supersolution of (1.1). Let $\bar u\in W_0$ be any solution of (5.12).

Testing the equation by $-\bar u^-$ we find

$$\begin{aligned}
&\int_Q\frac{\partial \bar u}{\partial t}(-\bar u^-)\,dx\,dt+\sum_{i=1}^{N}\int_Q a_i(x,t,\bar u,\nabla\bar u)\frac{\partial}{\partial x_i}(-\bar u^-)\,dx\,dt\\
&=\int_Q(c_2(1+|\bar u|^{p-1})+f)(-\bar u^-)\,dx\,dt.
\end{aligned}$$

Since

$$\int_Q\frac{\partial \bar u}{\partial t}(-\bar u^-)\,dx\,dt=\frac12\int_\Omega(\bar u^-)^2(x,\tau)\,dx\ge 0$$

and using assumption (A3), it follows that

$$\begin{aligned}
&\nu\int_{\{\bar u\le 0\}}|\nabla\bar u|^p\,dx\,dt+c_2\int_{\{\bar u\le 0\}}|\bar u|^p\,dx\,dt\\
&\le c_2\int_{\{\bar u\le 0\}}\bar u\,dx\,dt+\int_{\{\bar u\le 0\}}f\bar u\,dx\,dt\le 0.
\end{aligned}$$

Here we used also the assumptions (b) and (c). Taking into account that $\nu>0$ we conclude that $\bar u\ge 0$.

To obtain the desired conclusion concerning the existence of extremal nonnegative solutions of (1.1), it is sufficient to show that $\bar u$ is a supersolution of problem (1.1). Towards this, we see that every $v\in\bar u\vee V_0$ can be written as $v=\bar u+(w-\bar u)^+$ with $w\in V_0$. Then we have

$$\begin{aligned}
&\langle\frac{\partial \bar u}{\partial t}+A\bar u-f,(w-\bar u)^+\rangle+\int_Q j^o(\bar u;(w-\bar u)^+)\,dx\,dt\\
&\ge\langle\frac{\partial \bar u}{\partial t}+A\bar u-f,(w-\bar u)^+\rangle-c_2\int_Q(1+|\bar u|^{p-1})(w-\bar u)^+\,dx\,dt=0,\quad\forall\, w\in V_0,
\end{aligned}$$

where hypothesis (H) (ii) has been used as well as the fact that $\bar u$ solves the initial boundary value problem (5.12). Therefore, $\bar u\ge 0$ is a supersolution of problem

(1.1). Consequently, Theorem 5.2 yields extremal solutions in the order interval $[0, \bar{u}]$.

**Remark 5.5.** In case we have $p = 2$ in Example 1 then condition (d) is not needed.

**Example 5.6.** Here we provide sufficient conditions for sub-supersolutions as constants. Let us assume that $a_i(x, t, u, 0) = 0$ for a.e. $(x, t) \in Q$, all $u \in \mathbb{R}$, $i = 1, \dots, N$. Then we have the following proposition.

**Proposition 5.7.** *Let $D \in \mathbb{R}$.*

(a) *If $D \le 0$ and $f(x, t) \ge -j^o(D; -1)$ for a.e. $(x, t) \in Q$, then $\underline{u} = D$ is a subsolution of (1.1).*
(b) *If $D \ge 0$ and $f(x, t) \le j^o(D; 1)$ for a.e. $(x, t) \in Q$, then $\bar{u} = D$ is a supersolution of (1.1).*

*Proof.* (a) We only need to check (ii) in Definition 2.2. Note that $\underline{u}_t = 0$ and $A\underline{u} = 0$. Let $v \in D \wedge V_0$. Since $v - \underline{u} \le 0$ in $Q$, we have

$$\begin{aligned}
&\langle \underline{u}_t + A\underline{u} - f, v - \underline{u} \rangle + \int_Q j^o(\underline{u}; v - \underline{u}) dx\, dt \\
&= \int_Q [j^o(D; v - \underline{u}) - f(v - \underline{u})] dx\, dt \\
&= \int_Q [j^o(D; -1) + f] |v - \underline{u}| dx\, dt \ge 0.
\end{aligned}$$

(b) Similarly, in the second case, we have $v - D \ge 0$ for $v \in D \vee V_0$ and

$$\begin{aligned}
&\langle \bar{u}_t + A\bar{u} - f, v - \bar{u} \rangle + \int_Q j^o(\bar{u}; v - \bar{u}) dx\, dt \\
&= \int_Q [j^o(D; v - \bar{u}) - f(v - \bar{u})] dx\, dt \\
&= \int_Q [j^o(D; 1) - f](v - \bar{u}) dx\, dt \ge 0.
\end{aligned}$$

□

As consequence, for example, if there exists $D > 0$ such that

$$-j^o(0; -1) \le f(x, t) \le j^0(D; 1) \quad \text{for a.e. } (x, t) \in Q, \tag{5.14}$$

then (1.1) has a nonnegative bounded solution (in the interval $[0, D]$). Similarly, if there is $D < 0$ such that

$$-j^o(D; -1) \le f(x, t) \le j^0(0; 1) \quad \text{for a.e. } (x, t) \in Q, \tag{5.15}$$

then (1.1) has a nonpositive bounded solution (in $[D, 0]$).

It should be noted that, e.g., condition (5.14) may also formulated in terms of the generalized gradient as follows:

$$\min_{\zeta \in \partial j(0)} \zeta \le f(x, t) \le \max_{\zeta \in \partial j(D)} \zeta \quad \text{for a.e. } (x, t) \in Q. \tag{5.16}$$

**Example 5.8.** Finally, here we characterize a class of locally Lipschitz functions $j$ satisfying the hypothesis (H).

Let $j_1 : (-\infty, 0) \to \mathbb{R}$ be a convex function and let $j_2 : [0, +\infty) \to \mathbb{R}$ be a continuously differentiable function such that

(1) $\lim_{s\to 0} j_1(s) = j_2(0)$;
(2) For all $t < 0$ and all $s \geq 0$,
$$-c_2(1+|t|^{p-1}) \leq \min_{\xi\in\partial j_1(t)} \xi \leq \max_{\xi\in\partial j_1(t)} \xi \leq j_2'(s) \leq c_2(1+|s|^{p-1})$$
(3)
$$\sup_{0\leq s_1<s_2} \frac{j_2'(s_1)-j_2'(s_2)}{(s_2-s_1)^{p-1}} \leq c_1.$$
Here $c_1$ and $c_2$ are positive constants.

Then $j : \mathbb{R} \to \mathbb{R}$ defined as $j(s) = j_1(s)$ for $s < 0$ and $j(s) = j_2(s)$ for $s \geq 0$ satisfies (H).

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Siegfried Carl

Fachbereich Mathematik und Informatik, Institut für Analysis, Martin-Luther - Universität Halle-Wittenberg, 06099 Halle, Germany

*E-mail address*: carl@mathematik.uni-halle.de

Vy K. Le

Department of Mathematics and Statistics, University of Missouri - Rolla, Rolla, MO 65401, USA

*E-mail address*: vy@umr.edu

Dumitru Motreanu

Département de Mathématiques, Université de Perpignan, 52 Avenue Paul Alduy, 66860 Perpignan, France

*E-mail address*: motreanu@univ-perp.fr