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ON GLOBAL SOLUTIONS FOR THE VLASOV-POISSON SYSTEM

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ABSTRACT. In this article we show that the Vlasov-Poisson system has a unique weak solution in the space $L_1 \cap L_\infty$. For this purpose, we use the method of characteristics, unlike the approach in [12].

1. Introduction

Consider the classical Vlasov-Poisson system

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \nabla_v f \cdot E(x,t) = 0, \quad (t,x,v) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3,\ f = f(t,x,v), \tag{1.1}$$

$$E(x,t) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \nabla U(x-y) f(t,y,v) dy\, dv, \quad U(x) = \kappa |x|^{-1}, \tag{1.2}$$

$$f(0,x,v) = f_0(x,v), \tag{1.3}$$

where all quantities are real, $a \cdot b$ means the usual scalar product of $a, b \in \mathbb{R}^3$, $\kappa = \pm 1$ is a constant, and f is an unknown function that has the sense of a distribution function of particles in the (x,v)-space. In view of the sense of f, we require

$$f \geq 0 \quad \text{and} \quad \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(t,x,v) dx\, dv \equiv 1. \tag{1.4}$$

Everywhere L_p denotes the standard Lebesgue space $L_p(\mathbb{R}^3 \times \mathbb{R}^3)$ with the standard norm (here $1 \leq p \leq \infty$). In what follows, we look for weak solutions of (1.1)–(1.4) that belong to $L_1 \cap L_\infty$ for each fixed $t \in \mathbb{R}$.

The Vlasov-Poisson system has applications in particular in plasma physics and stellar dynamics. There is a numerous literature devoted to studies of Vlasov equations. Here we mention the following papers. In [1, 4, 6, 14], the Vlasov equation with a smooth bounded potential U is considered; the existence and uniqueness of a weak solution with values in the space of normalized nonnegative measures is proved. In [2, 3, 7, 9], the Vlasov-Poisson system is investigated (see also [1]). In [3], the existence and uniqueness of radial solutions is proved. In [1, 2, 7, 9], weak solutions of this system are studied (we note that in these papers the question about

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the uniqueness of weak solutions similar to ours is left open). We also mention paper [5] where weak solutions of the Vlasov-Maxwell system are considered. In [13], the existence of a global smooth solution to (1.1)–(1.4) is demonstrated. In [8], the Vlasov equation with potentials of higher singularities is considered and in [15], a two-time problem for the equation with a smooth bounded potential is treated. In addition, when the present article was already prepared, the author learned about paper [12] where a result quite similar to ours is obtained by a completely different method.

Here we prove in particular the uniqueness of a weak solution of (1.1)–(1.4). In this connection, we mention paper [10] where for the Vlasov-Poisson system in the one-dimensional case the non-uniqueness of solutions is shown. This does not contradict our result because in that article the solution is weaker than ours: measure-valued solutions are considered there.

Definition. Let $f(t,\cdot,\cdot) \in C(I; L_p)$ for all $1 \le p < \infty$ where $I \subset \mathbb{R}$ is an interval containing 0 and $\|f(t,\cdot,\cdot)\|_{L_\infty} \le C$ for all $t \in I$. Then, we call f a weak solution of (1.1)–(1.4) if (1.2)–(1.4) are satisfied and if for any function $\eta = \eta(t,x,v)$ in $I \times \mathbb{R}^3 \times \mathbb{R}^3$ continuously differentiable and equal to zero from outside of a compact set one has for all $t \in I$:

$$\begin{aligned} &\int_{\mathbb{R}^3\times\mathbb{R}^3} dx\, dv\, [\eta(t,x,v) f(t,x,v) - \eta(0,x,v) f_0(x,v)] \\ &- \int_0^t ds \int_{\mathbb{R}^3\times\mathbb{R}^3} dx\, dv\, f(s,x,v) \\ &\times \big\{\eta_s(s,x,v) + v\cdot\nabla_x \eta(s,x,v) + \nabla_v \eta(s,x,v)\cdot E(x,s)\big\} = 0. \end{aligned} \tag{1.5}$$

The main result in the present paper reads as follows.

Theorem 1.1. *For any $f_0 \in L_1 \cap L_\infty$ with a compact support problem* (1.1)–(1.4) *has a unique weak solution $f(t,x,v)$ global in t such that its (x,v)-support is bounded uniformly in t from an arbitrary finite interval. The energy of the system,*

$$E(f) = \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 f(t,x,v)\, dx\, dv - \int_{\mathbb{R}^{12}} dx\, dx'\, dv\, dv'\; f(t,x,v) U(x-x') f(t,x',v'),$$

does not depend on t.

2. Proof of Main Theorem

To (1.1)–(1.4), we associate the system

$$\dot{x}(t,x_0,v_0) = v(t,x_0,v_0), \tag{2.1}$$

$$\dot{v}(t,x_0,v_0) = w(x(t,x_0,v_0),t) := \int_{\mathbb{R}^3\times\mathbb{R}^3} \nabla U(x(t,x_0,v_0)-y) f(t,y,v)\, dy\, dv, \tag{2.2}$$

$$(x(0,x_0,v_0), v(0,x_0,v_0)) = (x_0,v_0), \tag{2.3}$$

$$\begin{gathered} f(t,x(t,x_0,v_0),v(t,x_0,v_0)) = f_0(x_0,v_0) \\ \text{for almost all } (x_0,v_0) \in \mathbb{R}^3\times\mathbb{R}^3 \text{ for a fixed } t, \end{gathered} \tag{2.4}$$

where (x_0,v_0) runs over the entire $\mathbb{R}^3\times\mathbb{R}^3$. Formally, if $(x(t,x_0,v_0), v(t,x_0,v_0))$ is a solution of system (2.1)–(2.4) and if $f(t,x,v)$ is given by (2.4), then f satisfies (1.1)–(1.4). Here, we are aimed in particular to justify this fact.

For $g \in L_1 \cap L_\infty$, set

$$(Tg)(x) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \nabla U(x-y) g(y,v) dy\, dv.$$

Let $\omega(\cdot)$ be a nonnegative even C^∞-function with a compact support in $\mathbb{R}^3$ satisfying $\int_{\mathbb{R}^3} \omega(x)\ dx = 1$ and let $U_n(x) = (U(\cdot) \star n^3\omega(n\cdot))(x)$ where the star means the convolution and $n = 1, 2, 3, \dots$. Consider the sequence of approximations of system (2.1)–(2.4) that occur by substitutions of U_n in place of U in (2.1)–(2.4). We denote these approximations by (2.1n)–(2.4n). Let also T_n be the integral operator which is defined by analogy with T with the change of U by U_n.

It is the well known result proved in fact in [1, 4, 6, 14] that for each n system (2.1n)–(2.4n) possesses a unique global solution $(x_n(t,x_0,v_0), v_n(t,x_0,v_0))$; also, for any fixed t the map S_n^t transforming (x_0, v_0) into $(x_n(t,x_0,v_0), v_n(t,x_0,v_0))$ is a diffeomorphism of $\mathbb{R}^3 \times \mathbb{R}^3$ onto itself (i. e. it is a one-to-one map continuously differentiable with its inverse), and the corresponding function defined by $f_n(t, x_n(t,x_0,v_0), v_n(t,x_0,v_0)) \equiv f_0(x_0,v_0)$ is finite for each fixed t and it is a weak solution of the problem arising from (1.1)–(1.4) by replacing U by U_n; in addition, $\operatorname{diam}(\operatorname{supp} f_n(t,\cdot,\cdot))$ is continuous in t. Also, for each n the corresponding energy $E_n(f_n)$, which occurs by replacing U by U_n in the representation for E, and the norms $\|f_n(t,\cdot,\cdot)\|_{L_p}$ with $1 \le p \le \infty$ do not depend on t. In addition, according to [16] $\det J_n(t,x_0,v_0) \equiv 1$ where

$$J_n = \frac{\partial(x_n(t,x_0,v_0), v_n(t,x_0,v_0))}{\partial(x_0,v_0)}$$

is the Jacobi matrix. Denote $D_n^x(t) = \sup\{p \in [0,\infty) : \operatorname{ess\,sup}_{|x|>p} f_n(t,x,v) > 0\}$ and $D_n^v(t) = \sup\{q \in [0,\infty) : \operatorname{ess\,sup}_{|v|>q} f_n(t,x,v) > 0\}$. Also, for any $(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3, n$ and $s,t \in \mathbb{R}$ there exists a unique point $(x_0,v_0) \in \mathbb{R}^3 \times \mathbb{R}^3$ such that $(x_n(s,x_0,v_0), v_n(s,x_0,v_0)) = (x,v)$. By $(x_n,v_n)(t,s,x,v)$ we denote the point of the corresponding trajectory $(x_n(\tau,x_0,v_0), v_n(\tau,x_0,v_0))$, where $\tau \in \mathbb{R}$, taken at the time $\tau = t$. Clearly $(x_n,v_n)(t,s,x_n(s,\tau,x,v), v_n(s,\tau,x,v)) \equiv (x_n,v_n)(t,\tau,x,v)$ and $f_n(t,x,v) \equiv f_0((x_n,v_n)(0,t,x,v))$.

Lemma 2.1. *For any finite $f_0 \in L_1 \cap L_\infty$ there exist $D_0 > 0$ and $T = T(D_0) > 0$, where $T(s)$ is a nonincreasing function of $s > 0$, such that $D_n^x(t) + D_n^v(t) \le D_0$ for all n and all $t \in [-T,T]$.*

Proof. We consider only the case $t > 0$ because for $t < 0$ all our estimates can be made analogously. First of all, we have the estimate

$$|(T_n f_n)(x,t)| \le C([D_n^v(t)]^3 \|f_n(t,\cdot,\cdot)\|_{L_\infty} + \|f_n(t,\cdot,\cdot)\|_{L_1}). \tag{2.5}$$

' Further, it can be easily derived from (2.1n),(2.2n) and (2.5) that for any $(x_0,v_0) \in \operatorname{supp}(f_0)$,

$$|x_n(t,x_0,v_0)| \le |x_0| + \int_0^t |v_n(s,x_0,v_0)|\, ds \le |x_0| + \int_0^t D_n^v(s)\, ds$$

and

$$|v_n(t,x_0,v_0)| \le C_1 + |v_0| + C_2 \int_0^t [D_n^v(s)]^3\, ds\,.$$

Hence

$$D_n^x(t) \le D_n^x(0) + \int_0^t D_n^v(s)\,ds$$

$$D_n^v(t) \le C_3 + D_n^v(0) + C_4 \int_0^t [D_n^v(s)]^3\,ds$$

with constants $C_3, C_4 > 0$ independent of $t \in [0,1]$ and n, which easily implies our claim. $\square$

Corollary 2.2. *There exists $C > 0$ such that $|(T_n f_n)(x,t)| \le C$ for all x and $t \in [-T, T]$.*

The proof of this corollary follows from (2.5) and Lemma 2.1.

Lemma 2.3. *There exists a positive constant C such that*

$$|(T_n g)(x_1) - (T_n g)(x_2)| \le -C|x_1 - x_2| \ln |x_1 - x_2|$$

for all $g \in L_1 \cap L_\infty$ satisfying $\|g\|_{L_1} + \|g\|_{L_\infty} \le 1$ and $g(x,v) = 0$ if $|x| + |v| > D_0$, all $n = 1,2,3,\dots$ and for all $x_1, x_2 \in \mathbb{R}^3$ such that $|x_1 - x_2| \le 1/2$.

Proof. Take arbitrary $x, h \in \mathbb{R}^3$, such that $|h| \le 1/2$, and g. Then, we have

$$\begin{aligned}(T_n g)(x+h) - (T_n g)(x) = \Big(\int_{B_{2|h|}(x)} + \int_{B_{D_0}(0)\setminus B_{2|h|}(x)} \Big) dy \big(\nabla U(x+h-y) \\ - \nabla U(x-y)\big) \int_{B_{D_0}(0)} g(y,v)\,dv = I_1 + I_2.\end{aligned}$$

Since $\|g\|_{L_\infty} \le 1$, for I_1 we have

$$|I_1| \le C_1 D_0^3(T) \int_{B_{2|h|}(0)} |y|^{-2}\,dy \le C'|h|.$$

For I_2, we deduce

$$|I_2| \le C_2 |h| D_0^3(T) \int_{B_{D_0}(0)\setminus B_{2|h|}(0)} |y|^{-3}\,dy \le -C_3 |h| \ln |h|.$$

$\square$

Corollary 2.4. *One has*

$$\big|(Tg)(x+h) - (Tg)(x)\big| \le -C|h| \ln |h|$$

for all x and $h:\ |h| \le 1/2$ and for all $g \in L_1 \cap L_\infty$ satisfying $\|g\|_{L_1} + \|g\|_{L_\infty} \le 1$ and $g(x,v) = 0$ if $|x| + |v| > D_0$.

Lemma 2.5. *For any $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\big|(x_n(t,x_0,v_0), v_n(t,x_0,v_0)) - (x_n(t,x_1,v_1), v_n(t,x_1,v_1))\big| < \epsilon$$

for all n and all $t \in [-T,T]$ if $|(x_0,v_0) - (x_1,v_1)| < \delta$.

Proof. We consider only the case $t > 0$ because for $t < 0$ the proof can be made by analogy. We have by Lemma 2.3:

$$|x_n(t,x_0,v_0) - x_n(t,x_1,v_1)| \le |x_0 - x_1| + \int_0^t |v_n(s,x_0,v_0) - v_n(s,x_1,v_1)|\,ds$$

and

$$
\begin{aligned}
&|v_n(t,x_0,v_0) - v_n(t,x_1,v_1)| \\
&\le |v_0 - v_1| - C\int_0^t |x_n(s,x_0,v_0) - x_n(s,x_1,v_1)| \ln |x_n(s,x_0,v_0) - x_n(s,x_1,v_1)|\, ds
\end{aligned}
$$

until $|x_n(t,x_0,v_0) - x_n(t,x_1,v_1)| \le 1/2$. Now our claim follows by standard arguments similar to those used when proving the Gronwell's lemma (see also [11]). $\square$

Now, applying the Arzéla-Ascoli theorem, in view of Lemmas 2.1–2.5 and estimate (2.5) we deduce that the sequence $\{(x_n(t,x_0,v_0), v_n(t,x_0,v_0))\}_{n=1,2,3,\dots}$ of functions from $[-T,T]\times\mathbb{R}^3\times\mathbb{R}^3$ into $\mathbb{R}^3\times\mathbb{R}^3$ contains a subsequence still denoted $\{(x_n(t,x_0,v_0), v_n(t,x_0,v_0))\}_{n=1,2,3,\dots}$ which converges to a pair of continuous functions $(x(t,x_0,v_0), v(t,x_0,v_0))$ uniformly in (t,x_0,v_0) from an arbitrary compact subset of $[-T,T]\times\mathbb{R}^3\times\mathbb{R}^3$.

Lemma 2.6. *For any $t\in[-T,T]$ there exists $f(t,\cdot,\cdot)\in L_1\cap L_\infty$ such that for any $p\in[1,\infty)$ the sequence $\{f_n(t,\cdot,\cdot)\}_{n=1,2,3,\dots}$ converges to $f(t,\cdot,\cdot)$ strongly in L_p.*

Proof. Take a sequence $h^k(\cdot,\cdot)$ of continuous functions converging to f_0 strongly in L_p and almost everywhere, bounded in L_∞ and such that $h^k(x,v)=0$ if $|x|+|v|> D_0+1$ for all k. Then, we have

$$
\begin{aligned}
&\|f_n(t,\cdot,\cdot) - f_m(t,\cdot,\cdot)\|_{L_p} \\
&\le \|h^k(x_n(0,t,\cdot,\cdot), v_n(0,t,\cdot,\cdot)) - h^k(x_m(0,t,\cdot,\cdot), v_m(0,t,\cdot,\cdot))\|_{L_p} \\
&\quad + \|h^k(x_n(0,t,\cdot,\cdot), v_n(0,t,\cdot,\cdot)) - f_0(x_n(0,t,\cdot,\cdot), v_n(0,t,\cdot,\cdot))\|_{L_p} \\
&\quad + \|h^k(x_m(0,t,\cdot,\cdot), v_m(0,t,\cdot,\cdot)) - f_0(x_m(0,t,\cdot,\cdot), v_m(0,t,\cdot,\cdot))\|_{L_p}.
\end{aligned}
$$

Then, obviously, for any $\epsilon>0$ the second and third terms in the right-hand side of this inequality are smaller than $\epsilon/3$ for all sufficiently large k and for all n and m, and the first term is smaller than $\epsilon/3$ for the same (fixed) values of k and for all sufficiently large n and m. $\square$

Corollary 2.7. *One has $\|f(t,\cdot,\cdot)\|_{L_p}\equiv\|f_0\|_{L_p}$ for all $p\ge 1$ and all t.*

The proof of this corollary follows from Lemma 2.6 and $\frac{d}{dt}\|f_n(t,\cdot,\cdot)\|_{L_p}\equiv 0$ which holds for all $t\in\mathbb{R}$ and for all $p\in[1,\infty)$, which are well known.

Lemma 2.8. *Let $\{g^n\}_{n=1,2,3,\dots}\subset L_1\cap L_\infty$, each $g^n=0$ if $|x|+|v|>D_0$, this sequence is bounded in L_∞, and let for any $p\in[1,\infty)$ $g^n\to g$ strongly in L_p. Then, $(T_n g^n)(x)\to(Tg)(x)$ uniformly in $x\in\mathbb{R}^3$.*

Proof. First, we have

$$
|(T_n g^n)(x) - (Tg)(x)| \le |((T_n - T)g^n)(x)| + |(T(g^n - g))(x)|.
$$

The first term in the right-hand side of this inequality tends to 0 as $n\to\infty$ because

$$
\begin{aligned}
|((T_n - T)g^n)(x)| &\le \int_{B_{D_0}(0)} dy\ |\nabla U_n(x-y) - \nabla U(x-y)| \int_{B_{D_0}(0)} g^n(y,v)\, dv \\
&\le C\int_{B_{D_0}(0)} |\nabla U_n(x-y) - \nabla U(x-y)|\, dy \to 0
\end{aligned}
$$

uniformly in $x \in \mathbb{R}^3$. As for the second term, we have

$$|T(g^n - g)(x)| \leq \int_{B_{D_0}(0)} dy\, |\nabla U(x-y)| \int_{B_{D_0}(0)} |g^n(y,v) - g(y,v)|\, dv \to 0.$$

□

Taking the limit $n \to \infty$ in (2.1n),(2.2n), we obtain by Lemmas 2.6 and 2.8,

$$x(t, x_0, v_0) = x_0 + \int_0^t v(s, x_0, v_0)\, ds, \tag{2.6}$$

$$v(t, x_0, v_0) = v_0 + \int_0^t w(x(s, x_0, v_0), s)\, ds, \tag{2.7}$$

where the function w is given by (2.2), and (2.2) and (2.7) hold for all (x_0, v_0) and $t \in [-T, T]$. Now, it follows by the known uniqueness theorem for ODEs (see, for example, [11]) and by Corollary 2.4 that system (2.6)–(2.7) may have at most one solution. It is also easy to see from (2.6)–(2.7) that for any fixed t the transformation $(x_0, v_0) \to (x(t, x_0, v_0), v(t, x_0, v_0))$ is a one-to-one map of $\mathbb{R}^3 \times \mathbb{R}^3$ onto itself continuous with the inverse (to see this, it suffices to consider the initial value problem for $(x(t, x_0, v_0), v(t, x_0, v_0))$ with initial data given at an arbitrary time $t_0 \in [-T, T]$).

Theorem 2.9. *Denote $S^t(x_0, v_0) = (x(t, x_0, v_0), v(t, x_0, v_0))$. Then, for any fixed $t \in [-T, T]$ S^t is a one-to-one map continuous with its inverse of $\mathbb{R}^3 \times \mathbb{R}^3$ onto itself so that in particular it transforms Borel subsets of $\mathbb{R}^3 \times \mathbb{R}^3$ into Borel ones. For any Borel set $A \subset \mathbb{R}^3 \times \mathbb{R}^3$ one has $m(A) = m(S^t(A))$ where $m(\cdot)$ is the Lebesgue measure in $\mathbb{R}^3 \times \mathbb{R}^3$.*

Proof. Take an arbitrary open bounded set $A \subset \mathbb{R}^3 \times \mathbb{R}^3$. As well known, for any $\epsilon > 0$ there exists compact $K_\epsilon \subset A$ such that $m(A \setminus K_\epsilon) < \epsilon$. Let $\alpha = \operatorname{dist}(K_\epsilon, \partial A) > 0$ and $A_\beta = \{z \in S^t(A) : \operatorname{dist}(z, \partial S^t(A)) \geq \beta\}$. Let also $\beta = \operatorname{dist}(S^t(K_\epsilon), \partial S^t(A)) > 0$. For any $z \in K_\epsilon$ take a ball $B_r(z) \subset A$ such that $S^t(B_r(z)) \in A_{\frac{\beta}{2}}$. Let $B_{r_1}(z_1), \dots, B_{r_l}(z_l)$ be a finite covering of K_ϵ by these balls. Then, by construction, there exists a number N such that $S^t_n(K_\epsilon) \subset S^t(A_{\frac{\beta}{4}})$ for all $n \geq N$. Now, we have

$$\begin{aligned} m(A) - \epsilon &\leq m(K_\epsilon) \leq m\Big(\cup_{k=1}^l B_{r_k}(z_k)\Big) = m\Big(S^t_n\Big(\cup_{k=1}^l B_{r_k}(z_k)\Big)\Big) \\ &\leq m(A_{\frac{\beta}{4}}) \leq m(S^t(A)) \end{aligned}$$

so that $m(A) - \epsilon \leq m(S^t(A))$. The inequality $m(S^t(A)) - \epsilon \leq m(A)$ can be obtained by the complete analogy by considering the map $S^{0,t}$ inverse to S^t. So, $m(A) = m(S^t(A))$.

For an unbounded open set A the same equality follows in view of representations

$$A = \cup_{k=1}^\infty A \cap B_k(0) \quad \text{and} \quad S^t(A) = \cup_{k=1}^\infty S^t(A \cap B_k(0)).$$

This also implies the same equality for closed sets. For an arbitrary Borel set $A \subset \mathbb{R}^3 \times \mathbb{R}^3$ the equality $m(A) = m(S^t(A))$ now can be obtained by approximations of A by open sets from outside. □

Since the map $(x_0, v_0) \to (x(t, x_0, v_0), v(t, x_0, v_0))$ is a homeomorphism of $\mathbb{R}^3 \times \mathbb{R}^3$ onto itself, for any s and $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ there exists a unique point (x_0, v_0) such that $(x(s, x_0, v_0), v(s, x_0, v_0)) = (x, v)$. Using this fact, for any $s, t \in \mathbb{R}$

and $(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$ we denote by $(x,v)(t,s,x,v)$ the point of the trajectory $(x(\tau,x_0,v_0),v(\tau,x_0,v_0))$ at the time $\tau = t$ with those initial value (x_0,v_0) for which $(x(s,x_0,v_0),v(s,x_0,v_0)) = (x,v)$.

Lemma 2.10. *For any fixed $t \in [-T,T]$ one has $f(t,x(t,x_0,v_0),v(t,x_0,v_0)) = f_0(x_0,v_0)$ for almost all $(x_0,v_0) \in \mathbb{R}^3 \times \mathbb{R}^3$.*

Proof. We have that, on a subsequence, $f_n(t,\cdot,\cdot) \to f(t,\cdot,\cdot)$ almost everywhere and $f_n(t,x_1,v_1) \equiv f_0((x_n,v_n)(0,t,x_1,v_1))$. So, in view of Theorem 2.9 to prove Lemma, it suffices to show that $f_0((x_n,v_n)(0,t,\cdot,\cdot)) \to f_0((x,v)(0,t,\cdot,\cdot))$ almost everywhere. Set

$$\begin{aligned}&(x_n(0,t,x_1,v_1),v_n(0,t,x_1,v_1))\\&= (x(0,t,x_1,v_1)+\delta_n(x_1,v_1),v(0,t,x_1,v_1)+\gamma_n(x_1,v_1)),\end{aligned}$$

where $(\delta_n,\gamma_n) \to 0$ as $n \to \infty$ uniformly in an arbitrary compact set, and show that $f_0(x(0,t,x_1,v_1)+\delta_n, v(0,t,x_1,v_1)+\gamma_n) \to f_0(x(0,t,x_1,v_1),v(0,t,x_1,v_1))$ almost everywhere over a subsequence. Let φ_k be a sequence of continuous functions, uniformly bounded in L_∞ and supports of which are uniformly bounded, converging to f_0 almost everywhere. Then, for $p \in [1,\infty)$,

$$\begin{aligned}&\|f_0(x(0,t,\cdot,\cdot)+\delta_n,v(0,t,\cdot,\cdot)+\gamma_n) - f_0(x(0,t,\cdot,\cdot),v(0,t,\cdot,\cdot))\|_{L_p}\\&\le \|\varphi_k(x(0,t,\cdot,\cdot)+\delta_n,v(0,t,\cdot,\cdot)+\gamma_n) - \varphi_k(x(0,t,\cdot,\cdot),v(0,t,\cdot,\cdot))\|_{L_p}\\&\quad+ \|f_0(x(0,t,\cdot,\cdot)+\delta_n,v(0,t,\cdot,\cdot)+\gamma_n) - \varphi_k(x(0,t,\cdot,\cdot)+\delta_n,v(0,t,\cdot,\cdot)+\gamma_n)\|_{L_p}\\&\quad+ \|f_0(x(0,t,\cdot,\cdot),v(0,t,\cdot,\cdot)) - \varphi_k(x(0,t,\cdot,\cdot),v(0,t,\cdot,\cdot))\|_{L_p}.\end{aligned}$$

The third term in the right-hand side tends to 0 as $k \to \infty$ in view of Theorem 2.9. As for the second one, since the map $(x_1,v_1) \to (x(0,t,x_1,v_1) + \delta_n(x_1,v_1), v(0,t,x_1,v_1)+\gamma_n(x_1,v_1))$ is one-to-one continuous with the inverse and preserving the Lebesgue measure, we have that it is equal to $\|f_0 - \varphi_k\|_{L_p} \to 0$ as $k \to \infty$. So, for a given $\epsilon > 0$ the second and third terms can be made smaller than $\epsilon/3$ by taking a sufficiently large k, uniformly in n. As for the first term, it can be made smaller than $\epsilon/3$ by taking the same sufficiently large fixed k and sufficiently large n. □

Proposition 2.11. *In the time interval $[-T,T]$, System (2.1)–(2.4) has a unique solution $(x(t,x_0,v_0),v(t,x_0,v_0))$ such that the (x,v)-support of the corresponding function $f(t,x,v)$ is bounded uniformly for $t \in [-T,T]$.*

Proof. We have to prove only the uniqueness of a solution. Suppose the opposite. Without the loss of generality we can accept that there exists $t_0 \in [0,T)$ such that two different solutions $(x_i,v_i)(t,x_0,v_0)$, $i=1,2$, coincide for $t \in [0,t_0]$ and for all (x_0,v_0) and that in an arbitrary small right half-neighborhood of t_0 there are points t where $(x_1,v_1)(t,x_0,v_0) \neq (x_2,v_2)(t,x_0,v_0)$ for some (x_0,v_0). Let also $f = f_i(t,x,v)$, $i=1,2$, be the corresponding functions given by (2.4). Without the loss of generality we accept that $f_i(t,x_i(t,x_0,v_0),v_i(t,x_0,v_0)) = f_0(x_0,v_0)$ for all (x_0,v_0) and t. Set also $(x,v)(t,x_0,v_0) = [(x_1,v_1)-(x_2,v_2)](t,x_0,v_0)$, $h(t) = \max_{(x_0,v_0)\in\operatorname{supp}(f_0)} |x(t,x_0,v_0)|$ and $r(t) = \max_{(x_0,v_0)\in\operatorname{supp}(f_0)} |v(t,x_0,v_0)|$. Then, applying Lemma 2.3, we obtain for $t > t_0$ sufficiently close to t_0,

$$h(t) \le \int_{t_0}^{t} r(s)\,ds, \tag{2.8}$$

$$\begin{aligned} |v(t,x_0,v_0)| &\le -C\int_{t_0}^{t} h(s)\ln h(s)\,ds \\ &+ \int_{t_0}^{t} ds\Big|\int_{\mathbb{R}^3\times\mathbb{R}^3} \nabla U(x_1(s,x_0,v_0)-y)[f_1(s,y,v)-f_2(s,y,v)]\,dy\,dv\Big| = I + II. \end{aligned} \tag{2.9}$$

Let us estimate II. Since the maps $S_i^t : (x_0,v_0) \to (x_i,v_i)(t,x_0,v_0)$ are homeomorphisms of $\mathbb{R}^3\times\mathbb{R}^3$ onto itself, for any i,s and $(x,v)\in\mathbb{R}^3\times\mathbb{R}^3$ there exists a unique point (x_0,v_0) such that $(x_i,v_i)(s,x_0,v_0) = (x,v)$. Using this fact, for any $i = 1,2$, $s,t \in \mathbb{R}$ and $(x,v)\in\mathbb{R}^3\times\mathbb{R}^3$ we denote by $(x_i,v_i)(t,s,x,v)$ the point of the trajectory $(x_i,v_i)(\tau,x_0,v_0)$ at the time t with those initial value (x_0,v_0) for which $(x_i,v_i)(s,x_0,v_0) = (x,v)$. Respectively, $S_i(t,s)$ denote homeomorphisms from $\mathbb{R}^3\times\mathbb{R}^3$ onto itself mapping (x,v) into $(x_i,v_i)(t,s,x,v)$. We have that $f_i(t,x,v)$ are solutions of the linear transport equations with the exterior forces

$$E_i(x,t) = \int_{\mathbb{R}^3\times\mathbb{R}^3} \nabla U(x-y) f_i(t,y,v)\,dy\,dv$$

and so, again the mappings $S_i(s,t_0)$ preserve the Lebesgue measure in $\mathbb{R}^3\times\mathbb{R}^3$. Denote also $P_i(k,s) = \operatorname{supp}(f_i(s,\cdot,\cdot)) \setminus (B_{kh(s)}(x_1(s,x_0,v_0))\times\mathbb{R}^3)$ and $D_2^x(t) = \sup\{p\in[0,\infty) : \operatorname{ess\,sup}_{|x|>p} f_2(t,x,v) > 0\}$ and apply the obvious fact that

$$|y'-y| \le h(s) \tag{2.10}$$

for any $(y,v)\in\operatorname{supp} f_1(s,\cdot,\cdot))$ and $(y',v') = S_2(s,t_0)(S_1(t_0,s)(y,v))$. Then, we have for $t > t_0$ sufficiently close to t_0,

$$\begin{aligned} II = \int_{t_0}^{t} ds\Big|\Big(\int_{B_{4h(s)}(x_1(s,x_0,v_0))} + \int_{\mathbb{R}^3\setminus B_{4h(s)}(x_1(s,x_0,v_0))}\Big)dy \\ \int_{\mathbb{R}^3} dv\,\nabla U(x_1(s,x_0,v_0)-y)[f_1(s,y,v)-f_2(s,y,v)]\Big| \end{aligned} \tag{2.11}$$

where clearly the first term can be estimated as follows

$$\Big|\int_{B_{4h(s)}(x_1(s,x_0,v_0))} dy \int_{\mathbb{R}^3} dv\ \nabla U(x_1(s,x_0,v_0)-y)[f_1(s,y,v)-f_2(s,y,v)]\Big| \le C_1 h(s). \tag{2.12}$$

Further, making the change of variables $(y,v)\to(x_1,v_1)(t_0,s,y,v)$ and using the facts that the operators $S_i(t,s)$ preserve the Lebesgue measure and that $f_1(s,y,v) = f_2(t_0,x_1(t_0,s,y,v),v_1(t_0,s,y,v))$ almost everywhere, one can easily derive

$$\begin{aligned} &\int_{(\mathbb{R}^3\setminus B_{4h(s)}(x_1(s,x_0,v_0)))\times\mathbb{R}^3} \nabla U(x_1(s,x_0,v_0)-y) f_1(s,y,v)\,dy\,dv \\ &= \int_{S_1(t_0,s)(P_1(4,s))} \nabla U(x_1(s,x_0,v_0) - x(s,t_0,y,v) - x_2(s,t_0,y,v)) f_2(t_0,y,v)\,dy\,dv. \end{aligned} \tag{2.13}$$

Now, applying (2.10) and making in one of the integrals in the right-hand side the change of variables $(y,v) \to (x_2,v_2)(s,t_0,y,v)$, we obtain from (2.11)–(2.13),

$$
\begin{aligned}
&II \\
&\le \int_{t_0}^t ds\Big\{C_1h(s) + C_2\int_{B_{5h(s)}(x_1(s,x_0,v_0))\setminus B_{3h(s)}(x_1(s,x_0,v_0))} |\nabla U(x_1(s,x_0,v_0)-y)|\,dy \\
&\quad + \int_{P_2(3,s)} |\nabla U(x_1(s,x_0,v_0) - x(s,t_0,x_2(t_0,s,y,v),v_2(t_0,s,y,v)) - y) \\
&\quad - \nabla U(x_1(s,x_0,v_0)-y)|f_2(s,y,v)\,dy\,dv\Big\} \\
&\le \int_{t_0}^t ds\Big\{C_3h(s) + C_4h(s)\int_{D_2^x(s)\setminus B_{3h(s)}(x_1(s,x_0,v_0))} |x_1(s,x_0,v_0)-y|^{-3}\,dy\Big\} \\
&\le -C_5\int_{t_0}^t h(s)\ln h(s)\,ds.
\end{aligned}
\tag{2.14}
$$

Estimates (2.9) and (2.14) yield

$$
h(t) \le \int_{t_0}^t r(s)\,ds \quad \text{and} \quad r(t) \le -C_6\int_{t_0}^t h(s)\ln h(s)\,ds,
$$

which easily imply that $h(t) \equiv r(t) \equiv 0$ in a right half-neighborhood of t_0. □

Proposition 2.12. *The function $f(t,x,v)$ is a weak solution of* (1.1)–(1.4).

Proof. Obviously, $f(t,\cdot,\cdot) \in C([-T,T];L_p)$ for each $p \in [1,\infty)$. Let $\eta(t,x,v)$ be an admissible function for (1.5). Then, equality (1.5) for f can be obtained by writing it for $f_n(t,x,v)$ with the further passing to the limit $n \to \infty$. □

Proposition 2.13. *Let $f(t,\cdot,\cdot) \in C(I;L_p)$ for each $p \in [1,\infty)$, where $0 \in I \subset [-T,T]$ and I is an interval, and let it be a weak solution of* (1.1)–(1.4). *Then, $f(t,x(t,x_0,v_0),v(t,x_0,v_0)) = f_0(x_0,v_0)$ for almost all (x_0,v_0), where the functions $x(t,x_0,v_0)$, $v(t,x_0,v_0)$ are the solution of* (2.1)–(2.3) *corresponding to this f.*

Proof. Consider the linear transport equation

$$
\frac{\partial}{\partial t}g + v\cdot\nabla_x g + \nabla_v g\cdot w(x,t) = 0, \quad g = g(t,x,v), \tag{2.15}
$$

$$
g(0,x,v) = g_0(x,v), \tag{2.16}
$$

where now the function w does not depend on the unknown g, is continuous and bounded in (x,t) and satisfies the estimate

$$
|w(x+h,t) - w(x,t)| \le -C|h|\ln|h|, \quad 0 < |h| \le 1/2,
$$

$g_0 \in L_1 \cap L_\infty$ and has a compact support. Again, for any point (x_0,v_0) the characteristic system for (2.15)–(2.16),

$$
\dot{x} = v, \quad \dot{v} = w(x(t),t), \quad x(0) = x_0, \quad v(0) = v_0,
$$

has a unique solution $(x(t,x_0,v_0),v(t,x_0,v_0))$ and the mapping $G_t:\ (x_0,v_0) \mapsto (x(t,x_0,v_0),v(t,x_0,v_0))$ is a one-to-one function from $\mathbb{R}^3\times\mathbb{R}^3$ onto itself continuous with the inverse. Again, the function $g(t,x,v)$ defined by the relation $g(t,x(t,x,v),v(t,x,v)) \equiv g_0(x,v)$ is a weak solution of (2.15)–(2.16) whose (x,v)-support is bounded uniformly with respect to t in a bounded interval. To prove Proposition 2.13, it suffices to show the uniqueness of this solution. Let us do this.

Suppose the existence of another compactly supported solution $g_1(t,x,v)$ and let $q = g - g_1$. Then, $q(t,x,v)$ is a solution of equation (2.15) and $q(0,x,v) \equiv 0$. Let $\omega(x)$ be a nonnegative even C_0^∞-function in $\mathbb{R}^3$ satisfying $\int_{\mathbb{R}^3} \omega(x)dx = 1$ and let $\omega_h(x) = h^{-3}(hx)$, $h > 0$. Let us take a $T > 0$ and let $R > 0$ be so large that $\operatorname{supp}(q(t,\cdot,\cdot)) \in B_{R-1}(0)$ for all $t \in [-T,T]$. Observe that for any $h \in H_0^1(B_R(0))$ the expressions $h\nabla_x q \cdot v$ and $h\nabla_v q \cdot w(x,t)$ are correctly determined by

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} h\nabla_x q \cdot v \, dx\, dv = \int_{\mathbb{R}^3\times\mathbb{R}^3} q\nabla_x h \cdot v \, dx\, dv \,,$$

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} h\nabla_v q \cdot w(x,t) \, dx\, dv = \int_{\mathbb{R}^3\times\mathbb{R}^3} q\nabla_v h \cdot w(x,t) \, dx\, dv$$

so that $\nabla_x q \cdot v,\ \nabla_v q \cdot w \in H^{-1}$ where H^{-1} is the Sobolev space dual to $H_0^1(B_R(0))$ with respect to the scalar product in L_2. Moreover, it is easy to see that $v \cdot \nabla_x q$, $\nabla_v q \cdot w \in C([-T,T];H^{-1})$. Now, it follows from the corresponding identity for solutions of (2.15) that can be obtained from (1.5) by replacing E by w that

$$\frac{\partial}{\partial t} q = -v \cdot \nabla_x q - \nabla_v q \cdot w \in C([-T,T];H^{-1}).$$

Hence, $\omega_h \star_x (\omega_h \star_v q_t)$, $\omega_h \star_x (\omega_h \star_v (\nabla_x q \cdot v))$, and $\omega_h \star_x (\omega_h \star_v (\nabla_v q \cdot w))$ are in $C([-T,T];C^1(\mathbb{R}^3\times\mathbb{R}^3))$, and the supports of these functions are bounded uniformly in $t \in [-T,T]$ and $0 < h < 1$. So, taking $\eta(t,x,v) = \omega_h \star_x (\omega_h \star_v q(t,\cdot,\cdot))(t,x,v)$ for the identity similar to (1.5), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3\times\mathbb{R}^3} \eta(t,x,v) q(t,x,v) \, dx\, dv \\
&= \int_0^t ds \int_{\mathbb{R}^3\times\mathbb{R}^3} q(s,x,v)\big\{ -\omega_h \star_x (\omega_h \star_v (\nabla_x q \cdot v))(s,x,v) \\
&\quad - \omega_h \star_x (\omega_h \star_v (\nabla_v q \cdot w))(s,x,v) + v \cdot \nabla_x \eta(s,x,v) + \nabla_v \eta(s,x,v) \cdot w(x,s)\big\} dx\, dv.
\end{aligned}$$

Now, using the evenness of ω_h and integrating by parts in the right-hand side of this equality, one can easily verify that the sums of the first and third and of the second and fourth terms are equal to 0. So,

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} \eta(t,x,v) q(t,x,v) dx\, dv \equiv 0$$

and letting here $h \to +0$, we deduce $\int_{\mathbb{R}^3\times\mathbb{R}^3} q^2(t,x,v) dx\, dv \equiv 0$. $\square$

So, we have proved the existence and uniqueness of a local solution to (1.1)–(1.4) finite for any fixed t. The relation $\frac{d}{dt}E(f) \equiv 0$ is also obvious and the mapping S^t preserves the measure. According to the result proved in [13], for any $p > \frac{33}{17}$ there exists $C > 0$ such that $D^x(t) + D^v(t) \le C(1+|t|)^p$ for all t from an arbitrary interval of the existence of our solution, where $D^x(t) = \sup\{p \in [0,\infty) : \operatorname{ess\,sup}_{|x|>p} f(t,x,v) > 0\}$ and $D^v(t) = \sup\{q \in [0,\infty) : \operatorname{ess\,sup}_{|v|>q} f(t,x,v) > 0\}$ (in fact, in [13], smooth compactly supported solutions of problem (1.1)–(1.4) are considered, but the proof in this paper is based only on general properties of solutions as the conservation of the energy and the preservation of the Lebesgue measure so that it holds in our case). This immediately yields that our solution $f(t,x,v)$ can be uniquely continued onto the entire real line $t \in \mathbb{R}$ and that it is finite for any fixed t. So, our proof of Theorem 1.1 is complete.

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