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ASYMPTOTIC STABILITY FOR SECOND-ORDER DIFFERENTIAL EQUATIONS WITH COMPLEX COEFFICIENTS

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Abstract. We prove asymptotical stability and instability results for a general second-order differential equations with complex-valued functions as coefficients. To prove asymptotic stability of linear second-order differential equations, we use the technique of asymptotic representations of solutions and error estimates. For nonlinear second-order differential equations, we extend the asymptotic stability theorem of Pucci and Serrin to the case of complex-valued coefficients.

1. Main Results

Consider the linear second-order differential equation

$$L[x(t)] = x''(t) + 2f(t)x'(t) + g(t)x(t) = 0, \quad t > T > 0, \tag{1.1}$$

where the coefficients $2f(t)$ and $g(t)$ are complex-valued continuous functions of time t. The rest state $x(t) = x'(t) = 0$ of (1.1) is called asymptotically stable if

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = 0 \tag{1.2}$$

for every solution of (1.1).

The asymptotic stability for the classical equation 1.1 has been widely studied [1, 2, 3, 4, 6, 7, 9, 13]. However, most of the studies consider real-valued coefficients and are based on Lyapunov stability theorems. In this paper, we prove asymptotical stability and instability theorems for a general linear second-order equation (1.1) with complex-valued coefficients. For a linear case, we use the technique of asymptotic representations of solutions and error estimates [8, 5]. For a nonlinear second-order equations (1.25) with complex-valued variable coefficients, we generalize the asymptotic stability theorem of Pucci and Serrin (Theorem 1.8).

Denote

$$g_0(t) \equiv g(t) - f^2(t) - f'(t), \quad k_0(t) \equiv \frac{g_0'(t)}{4g_0^{3/2}(t)}, \tag{1.3}$$

$$G_0(t) \equiv -k_0'(t) - k_0^2(t)\sqrt{g_0(t)} = \frac{5g_0'(t)^2}{16g_0^{5/2}(t)} - \frac{g_0''(t)}{4g_0^{3/2}(t)}, \tag{1.4}$$

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$$\mu_{1,2}(t) = -f(t) - \frac{g_0'(t)}{4g_0(t)} \pm \sqrt{f^2(t) + f'(t) - g(t)}. \tag{1.5}$$

Denote by $L^1(T,\infty)$ the class of Lebesgue integrable in (T,∞) functions and by $C^1(T,\infty)$ the class of differentiable functions on (T,∞).

Theorem 1.1. *Let $f \in C^3(T,\infty), g \in C^2(T,\infty)$ be the complex-valued functions, and assume that there exists positive number N such that*

$$\int_T^\infty \Big|k_0^2(t)\sqrt{g_0(t)} + k_0'(t)\Big| e^{\pm 2\int_T^t \Re\sqrt{-g_0(s)}ds} dt \le N. \tag{1.6}$$

Then the rest state of (1.1) is asymptotically stable if and only if

$$\lim_{t\to\infty}\int_T^t \Re[\mu_j]dt = -\infty, \quad \lim_{t\to\infty}\int_T^t \Re\big[\mu_j + \frac{\mu_j'}{\mu_j}\big]dt = -\infty, \quad j = 1,2. \tag{1.7}$$

Remark. When f(t) and g(t) are constant, $k_0(t) \equiv 0$, conditions (1.6) are satisfied, and conditions (1.7) becomes the Routh-Hurwitz criterion of asymptotical stability:

$$\Re\big(-f \mp \sqrt{f^2 - g}\big) < 0.$$

Remark. Theorem 1.1 shows that asymptotic stability of (1.1) depends on the behavior of $\Re(f)$ and $g_0(t)$ as $t \to \infty$.

Example 1.1. Let $f(t) = t^\alpha(\ln t)^\beta, g(t) = 1$. From Theorem 1.1, equation (1.1) is asymptotically stable if $-1 < \alpha < 0$ or $\alpha = -1, \beta > -1$ (see section 3).

Example 1.2. *Let $f(t) = t^\alpha + it^\beta, g(t) = 1$. From Theorem 1.1, equation (1.1) is asymptotically stable if $-1 < \alpha < -\beta - 1, \alpha < 0$ (see section 3).*

Remark. For the small damping case: $g(t) = 1$, $\lim_{t\to\infty} f(t) = 0$, we have $\Re\sqrt{-g_0(t)} = 0$ and conditions (1.6) are not restrictive. For the large damping case: $\lim_{t\to\infty} f(t) = \infty$, we have $\lim_{t\to\infty}\Re\sqrt{-g_0(t)} = \infty$. and conditions (1.6) are very restrictive.

Remark. If $g_0(t) = g(t) - f^2(t) - f'(t) \ge 0$ then $\Re\sqrt{-g_0(t)} \equiv 0$ and (1.6) becomes

$$\int_T^\infty \big|k_0^2(t)\sqrt{g_0(t)} + k_0'(t)\big|dt \le N. \tag{1.8}$$

Remark. Condition (1.8) is close to the main assumption of asymptotic stability theorems in Pucci and Serrin, that $k_0(t)$ is the function of bounded variation ($\int_T^\infty |k_0'(t)|dt < \infty$). See [10, 11] or Theorem 1.8 in this paper.

Theorem 1.2. *Assume there exist the complex-valued functions $\varphi_{1,2} \in C^2(T,\infty)$ that satisfy the conditions*

$$\lim_{t\to\infty} \exp\int_T^t \Re\big(\frac{\varphi_j'}{\varphi_j}\big)ds = 0, \quad j = 1,2, \tag{1.9}$$

$$\varphi_2(t)L\varphi_1(t) = \varphi_1(t)L\varphi_2(t), \tag{1.10}$$

$$\Re\big(\frac{\varphi_j'(t)}{\varphi_j(t)}\big) \le 0, \quad j = 1,2,\ t \ge b \ \textit{for some}\ b \ge T, \tag{1.11}$$

$$\int_T^\infty |B_{21}(s)|ds < \infty, \tag{1.12}$$

$$B_{21}(t) \equiv \frac{\varphi_2(t)L\varphi_1(t)}{W[\varphi_1(t),\varphi_2(t)]}, \quad W[\varphi_1(t),\varphi_2(t)] \equiv \varphi_1(t)\varphi_2'(t) - \varphi_1'(t)\varphi_2(t) \tag{1.13}$$

then every solution of (1.1) satisfies $\lim_{t\to\infty} x(t) = 0$.

Theorem 1.3. *Assume there exist complex-valued functions* $\varphi_{1,2} \in C^2(T,\infty)$ *that satisfy conditions (1.9)-(1.12) and*

$$\lim_{t\to\infty} \int_T^t \Re\big[\frac{\varphi_j''(s)}{\varphi_j'(s)}\big] ds = -\infty, \quad j = 1, 2, \tag{1.14}$$

$$\big|\frac{\varphi_j'(t)}{\varphi_j(t)}\big| \le c\Big(\int_b^t |B_{21}(s)| ds\Big)^{-\delta}, \quad j = 1, 2,\ t \ge b,\ 0 < \delta < 1, \tag{1.15}$$

for some positive constants c *and* δ. *Then the rest state of (1.1) is asymptotically stable and for* $k = 0, 1$ *we have*

$$|x^{(k)}(t)| \le \sum_{j=1}^{2} \big|C_j \varphi_j^{(k)}(t)\big| + C\Big(\int_b^t |B_{21}(s)| ds\Big)^{-k\delta}\Big(-1 + \exp\big(\int_b^t |2B(s)| ds\big)\Big). \tag{1.16}$$

Theorem 1.4. *Let conditions (1.8) and*

$$\lim_{t\to\infty} \int_T^t \Re\big[f(t) + \frac{g_0'(t)}{4g_0(t)} \pm \sqrt{f^2(t) + f'(t) - g(t)}\big] dt = \infty, \tag{1.17}$$

$$\Re\big[f(t) + \frac{g_0'(t)}{4g_0(t)} \pm \sqrt{f^2(t) + f'(t) - g(t)}\big] \ge 0, \quad t \ge b \tag{1.18}$$

be satisfied for some number $b \ge T$. *Then every solution* $x(t)$ *of (1.1) satisfies* $\lim_{t\to\infty} x(t) = 0$.

Example 1.3. $f(t) = t^\alpha$, $g(t) = 1$. From Theorem 1.4 follows that if $-1 \le \alpha \le 1$ then all solutions of (1.1) approach to zero as $t \to \infty$. It is known that condition $-1 \le \alpha \le 1$ is necessary and sufficient condition of asymptotic stability in this case.

Remark. Example 1.3 shows that Theorem 1.4 covers small and large damping cases although example 1.1 shows that Theorem 1.1 covers only the small damping case ($\alpha < 0$).

Example 1.4. $f(t) = t^\alpha + it^\beta$, $g(t) = 1$. It can be checked that (1.8) is satisfied (see section 3). From Theorem 1.4 follows that if $-1 \le \alpha < 1$, $\beta \le (\alpha + 1)/2$ then all solutions of (1.1) approach to zero as $t \to \infty$.

Theorem 1.5. *Let conditions (1.8),(1.17), (1.18) and*

$$\lim_{t\to\infty} \int_T^t \Re\big[\mu_j + \frac{\mu_j'}{\mu_j}\big] dt = -\infty, \quad j = 1, 2. \tag{1.19}$$

$$|\mu_j(t)| \le c\Big(\int_T^t \big|k_0^2(t)\sqrt{g_0(t)} + k_0'(t)\big| dt\Big)^{-\delta}, \quad 0 < \delta < 1, \quad j = 1, 2 \tag{1.20}$$

be satisfied for some positive numbers c, δ. *Then the rest state of (1.1) is asymptotically stable.*

Theorem 1.6. *Let the complex-valued functions* $\varphi_{1,2} \in C^2(T,\infty)$ *satisfy conditions (1.10), (1.12) and*

$$|\varphi_{1,2}(t)| \quad \textit{be decreasing}, \tag{1.21}$$

$$|\varphi_1(\infty)| = \gamma > 0 \tag{1.22}$$

then the rest state of (1.1) is not asymptotically stable.

Theorem 1.7. *Let $f \in C^3(T,\infty)$, $g \in C^2(T,\infty)$ satisfy conditions (1.8) and*

$$\Re[f(t) + \frac{g_0'(t)}{4g_0(t)} \pm \sqrt{f^2(t) + f'(t) - g(t)}]dt \geq 0, \quad t \geq T, \tag{1.23}$$

$$\int_T^\infty \Re[f(t) + \frac{g_0'(t)}{4g_0(t)} - \sqrt{f^2(t) + f'(t) - g(t)}]dt < \infty\,. \tag{1.24}$$

Then the rest state of equation (1.1) is not asymptotically stable.

Consider a nonlinear second order differential equation

$$x''(t) + h(t, x(t), x'(t))x'(t) + j(t, x(t)) = 0, \quad t \in J = [T, \infty). \tag{1.25}$$

The following theorem is a generalization of the asymptotic stability theorem of Pucci and Serrin [10, Theorem 3.1], to the case of complex-valued coefficients.

Theorem 1.8. *If there exist a non-negative continuous function $k(t)$ of bounded variation on (T,∞), non-negative measurable functions $\sigma(t), \delta(t), \psi \in L^1(J)$ and positive numbers β, χ, M, c, m such that*

$$0 \leq \sigma \leq Re[h(t,x,x')], \quad t \in J, \tag{1.26}$$

$$|h(t,x,x')| \leq \delta(t), \quad t \in J, \tag{1.27}$$

$$|h(t,x,x')| \leq \gamma Re[h(t,x,x')], \quad t \in J,\ \gamma \geq 1, \tag{1.28}$$

$$0 \leq k(t) \leq \beta\sigma(t), \quad t \in J, \tag{1.29}$$

$$\lim_{t\to\infty} \int_T^t k(s)ds = \infty, \tag{1.30}$$

$$\int_T^t \delta(s)k(s)e^{\int_t^s k(z)dz}ds \leq M, \quad t \in J, \tag{1.31}$$

$$\bar{x}j(t,x,x') + x\bar{j}(t,x,x') \geq \chi > 0, \quad \text{for } |x| > 0,\ t \in J, \tag{1.32}$$

$$F(t,x) = \int j(t,x,x')d\bar{x} = \int \bar{j}(t,x,x')dx > 0, \quad \text{for } |x| > 0, \tag{1.33}$$

$$F(t,0) = 0, \quad F(t,x) \geq c|x|^m, \quad \partial_t F(t,x) \leq \psi(t), \quad t \in J. \tag{1.34}$$

Then the rest state of (1.25) is asymptotically stable.

Example 1.5. Let $j(t,x) = l(t)x|x|^{2q}$, $h(t,x,x') = t^\alpha + it^\beta$, $q > 0$ then from Theorem 1.8 it follows that the rest state of (1.1) is asymptotically stable if

$$0 \leq l_0 \leq l(t) \leq l_1 < \infty, \quad \int_T^\infty |l_1(t)|dt < \infty, \quad -1 \leq \alpha < 0, \quad \beta \leq \alpha. \tag{1.35}$$

2. Auxiliary theorems

Consider the system of ordinary differential equations

$$a'(t) = A(t)a(t), \quad t > T, \tag{2.1}$$

where $a(t)$ is a n-vector function and $A(t)$ is a continuous on (T,∞) $n \times n$ matrix-function. Suppose we can find the exact solutions of the system

$$\psi'(t) = A_1(t)\psi(t), \quad t > T, \tag{2.2}$$

with the matrix-function A_1 close to the matrix-function A. Let $\Psi(t)$ is the $n \times n$ fundamental matrix of the auxiliary system (2.2). Then the solutions of (2.1) can be represented in the form

$$a(t) = \Psi(t)(C + \varepsilon(t)), \tag{2.3}$$

where $a(t), \varepsilon(t), C$ are the n-vector columns: $a(t) = \text{colomn}(a_1(t), \dots, a_n(t))$, $\varepsilon(t) = \text{colomn}(\varepsilon_1(t), \dots, \varepsilon_n(t))$, $C = \text{colomn}(C_1, \dots, C_n), C_k$ are an arbitrary constants. We can consider (2.3) as definition of the error vector-function $\varepsilon(t)$.

Theorem 2.1 ([5])**.** *Assume there exist an invertible matrix function $\Psi(t) \in C^1[T, \infty)$ such that*

$$H(t) \equiv \Psi^{-1}(t)(A(t)\Psi(t) - \Psi'(t)) = \Psi^{-1}(t)(A(t) - A_1(t))\Psi(t) \in L^1(T, \infty). \tag{2.4}$$

Then every solution of (2.1) can be represented in form (2.3) and the error vector-function $\varepsilon(t)$ can be estimated as

$$\|\varepsilon(t)\| \le \|C\|\Big(-1 + \exp\Big[\int_T^t \|\Psi^{-1}(s)(A\Psi(s) - \Psi'(s))\| ds \Big]\Big), \tag{2.5}$$

where $\| \cdot \|$ is the Euclidean vector (or matrix) norm: $\|C\| = \sqrt{C_1^2 + \dots + C_n^2}$.

Remark. From (2.5) the error $\varepsilon(t)$ is small when $\int_T^t \|\Psi^{-1}(A - A_1)\Psi\| ds$ is small.

Proof of Theorem 2.1. Let $a(t)$ be a solution of (2.1). The substitution $a(t) = \Psi(t)u(t)$ transforms (2.1) into

$$u'(t) = H(t)u(t), \quad t > T, \tag{2.6}$$

where H is defined by (2.4). By integration we obtain

$$u(t) = C + \int_T^t H(s)u(s)ds, \quad t > T, \tag{2.7}$$

where the constant vector C is chosen as in (2.3). Estimating $u(t)$,

$$\|u(t)\| \le \|C\| + \int_T^t \|H(s)\| \cdot \|u(s)\| ds, \tag{2.8}$$

and by Gronwall's lemma we have

$$\|u(t)\| \le \|C\| \exp\Big(\int_T^t \|H(s)\| ds \Big). \tag{2.9}$$

From representation (2.3) and expression (2.7), we have

$$\varepsilon(t) = \Psi^{-1}a - C = u - C = \int_T^t H(s)u(s)ds.$$

Then using (2.9) we obtain the estimate (2.5):

$$\begin{aligned}
\|\varepsilon(t)\| &\le \int_T^t \|Hu\| ds \\
&\le \|C\| \int_T^t \|H(s)\| \exp\Big(\int_T^s \|H\| dy \Big) ds \\
&= \|C\| \Big(-1 + \exp\big(\int_T^t \|H\| ds \big) \Big).
\end{aligned}$$

$\square$

Theorem 2.2. *Let $\varphi_{1,2}(t) \in C^2(T,\infty)$ be complex-valued functions such that*

$$\int_T^\infty |B_{kj}(t)|dt < \infty, \quad k,j = 1,2, \tag{2.10}$$

where

$$B_{kj}(t) \equiv \frac{\varphi_k(t)L\varphi_j(t)}{W(\varphi_1,\varphi_2)}, \quad L \equiv \frac{d^2}{dt^2} + 2f(t)\frac{d}{dt} + g(t), \quad j = 1,2. \tag{2.11}$$

Then for arbitrary constants C_1,C_2 there exist solution of (1.1) that can be written in the form

$$x(t) = [C_1 + \varepsilon_1(t)]\,\varphi_1(t) + [C_2 + \varepsilon_2(t)]\,\varphi_2(t), \tag{2.12}$$

$$x'(t) = [C_1 + \varepsilon_1(t)]\,\varphi_1'(t) + [C_2 + \varepsilon_2(t)]\,\varphi_2'(t), \tag{2.13}$$

where the error function is estimated as

$$\|\varepsilon(t)\| \le \|C\|\big(-1 + \exp\int_T^t \|B(s)\|ds\big), \tag{2.14}$$

the matrix B has entries B_{kj} and has norm $\|B\|$.

Proof. Equation (1.1) we can rewrite in the form

$$v'(t) = A(t)v(t), \tag{2.15}$$

where

$$v(t) = \begin{pmatrix} x \\ x'(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} 0 & 1 \\ -g(t) & -2f(t) \end{pmatrix}.$$

By substitution

$$v(t) = \Psi w(t), \quad \Psi = \begin{pmatrix} \varphi_1(t) & \varphi_2(t) \\ \varphi_1'(t) & \varphi_2'(t) \end{pmatrix}. \tag{2.16}$$

in (2.15), we get

$$w'(t) = H(t)w(t), \quad H(t) = \begin{pmatrix} B_{21}(t) & B_{22}(t) \\ -B_{11}(t) & -B_{12}(t) \end{pmatrix}. \tag{2.17}$$

To apply Theorem 2.1 to the system (2.17), we choose $A(t) = H(t)$ in (2.1) and $A_1(t) \equiv 0$ in (2.2). Then the identity matrix $\Psi = I$ is the fundamental solution of (2.2) with $A_1(t) \equiv 0$. By direct calculations we get $\|H\| = \|\Psi^{-1}(A\Psi - \Psi')\| = \|B\|$, so condition (2.4) of Theorem 2.1 follows from (2.10). From Theorem 2.1 we have

$$w(t) = (C + \varepsilon(t)), \quad \text{or} \quad v(t) = \Psi(t)w(t) = \Psi(t)(C + \varepsilon(t)). \tag{2.18}$$

Representations (2.12), (2.13) and estimates (2.14) follow from Theorem 2.1. $\square$

Denote

$$x_j(t) = \exp\big(\int_T^t \mu_j(s)ds\big), \quad j = 1,2, \quad \mu_{1,2} = -f(t) - \frac{g_0'(t)}{4g_0(t)} \pm i\sqrt{g_0(t)}. \tag{2.19}$$

Theorem 2.3. *Let $g \in C^2(T,\infty)$, $f \in C^3(T,\infty)$ and*

$$\int_T^\infty |G_0(t)|\, e^{\pm 2\int_T^t \Im\sqrt{g_0(s)}ds}dt = \int_T^\infty \big|G_0(t)e^{\pm 2\int_T^t \Re[\sqrt{-g_0(s)}]ds}\big|dt < \infty, \tag{2.20}$$

where $G_0(t)$ is defined by (1.4). Then for any constants C_1, C_2 there exist solution of (1.1) that can be written in the form

$$x(t) = [C_1 + \varepsilon_1(t)]\,x_1(t) + [C_2 + \varepsilon_2(t)]\,x_2(t), \tag{2.21}$$

$$x'(t) = [C_1 + \varepsilon_1(t)]\, x_1'(t) + [C_2 + \varepsilon_2(t)]\, x_2'(t), \tag{2.22}$$

and for the error vector-function $\varepsilon(t) = \begin{pmatrix} \varepsilon_1(t) \\ \varepsilon_2(t) \end{pmatrix}$ *we have the estimate*

$$\|\varepsilon(t)\| \le \|C\|\big(-1 + \exp \int_T^t |G(t)| dt\big), \tag{2.23}$$

$$G(t) \equiv \max\Big(\big|G_0(s) e^{2\int_T^t \Im\sqrt{g_0} dz}\big|, \big|G_0(s) e^{-2\int_T^t \Im\sqrt{g_0} dz}\big| \Big). \tag{2.24}$$

Proof. We apply Theorem 2.2 with $\varphi_j(t) = x_j(t)$. By direct calculations, we have

$$\begin{gathered} \frac{x_1(t) L x_1(t)}{W[x_1, x_2]} = \frac{i G_0(t)}{2} e^{2i\int_T^t \sqrt{g_0(s)} ds}, \\ \frac{x_2(t) L x_2(t)}{W[x_1, x_2]} = \frac{G_0(t)}{2i} e^{-2i\int_T^t \sqrt{g_0(s)} ds}, \\ \frac{x_1(t) L x_2(t)}{W[x_1, x_2]} = \frac{x_2(t) L x_1(t)}{W[x_1, x_2]} = \frac{G_0}{2i}. \end{gathered} \tag{2.25}$$

From (2.20) and Cauchy-Schwarz inequality follows $\int_T^\infty |G_0| dt < \infty$. So conditions (2.10) of Theorem 2.2 follow from (2.20). Theorem 2.3 follows from Theorem 2.2. □

Theorem 2.4. *Let $\varphi_{1,2} \in C^2(T, \infty)$ satisfied (1.10)-(1.12). Then for any constants C_1, C_2 there exist solution $x(t)$ of (1.1) that can be written in the form (2.12),(2.13) and the error functions $\varepsilon_j(t)$ are estimated as*

$$|\varepsilon_j(t)| \le \frac{C\big(-1 + \exp\int_b^t |B_{21}| ds\big)}{|\varphi_j(t)|}, \quad j = 1, 2 \tag{2.26}$$

with some positive constant C not depending on b.

Remark. For the given functions $\varphi_1(t)$, $W(t)$ we can construct

$$\varphi_2(t) = \varphi_1(t) \int_T^t \frac{W(s) ds}{\varphi_1^2(s)}$$

such that (1.10) and (1.13) are satisfied.

Proof of Theorem 2.4. From (1.11) we have

$$\frac{d}{dt}|\varphi_j(t)| = |\varphi_j(b)| \frac{d}{dt}\Big| \exp\int_b^t \frac{\varphi_j'}{\varphi_j} ds\Big| = |\varphi_j(t)| Re\big(\frac{\varphi_j'(t)}{\varphi_j(t)}\big) \le 0, \quad j = 1, 2,\ t \ge b,$$

which means that the functions $|\varphi_j(t)|$ are decreasing. When (1.10) is satisfied then the functions $\varphi_{1,2}(t)$ are solutions of the homogeneous equation

$$u''(t) + 2f(t) u'(t) + \big(g(t) - \frac{L\varphi_1}{\varphi_1}\big) u(t) = 0$$

and any solution of (also of (1.1))

$$x''(t) + 2f(t) x'(t) + \big(g(t) - \frac{L\varphi_1}{\varphi_1}\big) x(t) = -\frac{L\varphi_1}{\varphi_1} x(t)$$

can be written in the form:

$$x(t) = \varphi_1(t) C_1 + \varphi_2(t) C_2 + \varphi_1(t) \int_b^t \frac{x(s) L\varphi_2 ds}{W[\varphi_1, \varphi_2]} - \varphi_2(t) \int_b^t \frac{x(s) L\varphi_1 ds}{W[\varphi_1, \varphi_2]}, \tag{2.27}$$

$$x(t) = \varphi_1(t)C_1 + \varphi_2(t)C_2 + \int_b^t \Big(\frac{\varphi_1(t)}{\varphi_1(s)} - \frac{\varphi_2(t)}{\varphi_2(s)}\Big)\frac{\varphi_1(s)L\varphi_2(s)}{W[\varphi_1(s),\varphi_2(s)]}x(s)ds, \quad (2.28)$$

$$x'(t) = \varphi_1'(t)C_1 + \varphi_2'(t)C_2 + \int_b^t \Big(\frac{\varphi_1'(t)}{\varphi_1(s)} - \frac{\varphi_2'(t)}{\varphi_2(s)}\Big)\frac{\varphi_1(s)L\varphi_2(s)}{W[\varphi_1(s),\varphi_2(s)]}x(s)ds, \quad (2.29)$$

or (2.12), (2.13) where

$$\varepsilon_1(t) = \int_b^t \frac{x(s)L\varphi_2(s)ds}{W[\varphi_1(s),\varphi_2(s)]}, \quad \varepsilon_2(t) = -\int_b^t \frac{x(s)L\varphi_1(s)ds}{W[\varphi_1(s),\varphi_2(s)]}. \quad (2.30)$$

Here C_1, C_2 and b are arbitrary constants and C_1, C_2 do not depend on b. Because the functions $|\varphi_j(t)|$ are decreasing they are bounded:

$$|\varphi_j(t)| \le N_j(T), \quad j = 1,2, \quad t \ge T. \quad (2.31)$$

From representation (2.28) we have the estimates:

$$|x(t)| \le |\varphi_1(t)C_1| + |\varphi_2(t)C_2| + 2\int_b^t \Big|\frac{x(s)\varphi_1(s)L\varphi_2(s)}{W(s)}\Big|ds,$$

$$|x(t)| \le |N_1C_1| + |N_2C_2| + 2\int_b^t |B_{21}x(s)|ds.$$

Applying Gronwall's lemma we have

$$|x(t)| \le C\exp\Big(\int_b^t 2|B_{21}(s)|ds\Big), \quad C = |N_1C_1| + |N_2C_2|. \quad (2.32)$$

From (2.30) and (2.32), because $\varphi_{1,2}(t)$ are decreasing, we obtain estimates (2.26):

$$|\varphi_j\varepsilon_j(t)| \le C\int_b^t |B_{21}(s)|e^{\int_b^s |B_{21}dz}ds = C\Big(-1 + \exp\int_b^t |2B_{21}|dz\Big), \quad j = 1,2.$$

□

3. Proofs of the main statements

Proof of Theorem 1.1. Let us choose $x_j(t)$ as in (2.19) and apply Theorem 2.3. From conditions (2.20) (which coincide with conditions (1.6)) of Theorem 1.1, by Theorem 2.3 we have representations (2.21), (2.22) and estimates (2.23). From

$$|x_j(t)| = \exp\int_T^t Re(\mu_j)ds,$$
$$|x_j'(t)| = |\mu_j(T)|exp\int_T^t \Re\Big(\mu_j(s) + \frac{\mu_j'(s)}{\mu_j(s)}\Big)ds, \quad j = 1,2, \quad (3.1)$$

and (1.7) we have

$$\lim_{t\to\infty}|x_j(t)| = \lim_{t\to\infty}|x_j'(t)| = 0, \quad j = 1,2.$$

From (2.21)-(2.23) and (1.6) we have $|\varepsilon_j(t)| \le \text{const}$, $t > T$, $j = 1,2$ and the asymptotic stability.

Now prove that if one of the conditions in (1.7) is not satisfied, then there exist an asymptotically unstable solution $x(t)$. By contradiction assume that (1.2) is satisfied, and, for example, the first condition of (1.7) is not satisfied. Then there exist the sequence $t_n \to \infty$ such that

$$\lim_{t_n\to\infty}|x_1(t_n)| = \lambda_1 \ne 0. \quad (3.2)$$

There exist the subsequence $t_{n_j} \equiv t_m$ of the sequence t_n such that

$$\lim_{t_m \to \infty} |x_2(t_m)| = \lambda_2. \tag{3.3}$$

From Theorem 2.3 for any constants C_1, C_2 there exists the solution $x(t)$ of (1.1) that can be represented in form (2.21), or

$$x(t_m) = [C_1 + \varepsilon_1(t_m)]\, x_1(t_m) + [C_2 + \varepsilon_2(t_m)]\, x_2(t_m), \tag{3.4}$$

where $|\varepsilon_j(t)| \le const, \quad t > T, \quad j = 1, 2$. From representation (3.2),(3.4) $\lambda_{1,2}$ must be finite numbers, otherwise left side of (3.4) vanished and right side approaches to infinity when $t_m \to \infty$ by appropriate choice of C_j. Let choose $C_1 = 1, \quad C_2 = 0$ and denote $N = \exp\big(\int_T^\infty |G| ds\big)$ then from (2.23) we get

$$|\varepsilon_j(t)| \le \|\varepsilon\| \le e^N - 1. \tag{3.5}$$

There exist the subsequence t_k of sequence t_m such that exist $\lim_{t_k \to \infty} \varepsilon_j(t_k)$. So from (3.4) we obtain

$$\begin{gathered} 0 = \lambda_1 + \lambda_1 \lim_{t_k \to \infty} \varepsilon_1(t_k) + \lambda_2 \lim_{t_k \to \infty} \varepsilon_2(t_k), \\ -1 = \lim_{t_k \to \infty} \varepsilon_1(t_k) + \frac{\lambda_2}{\lambda_1} \lim_{t_k \to \infty} \varepsilon_2(t_k), \end{gathered}$$

which is impossible because the right side can be made small in view of estimate (3.5) by choosing T big, which makes N and ε_j small. □

To prove the statement of Example 1.1 let us show that if $-1 < \alpha < 0$, or $\alpha = -1$, $\beta > -1$, then conditions (1.6), (1.7) of Theorem 1.1 are satisfied. From the estimates

$$\begin{gathered} f(t) = o(1), \quad g_0 \equiv 1 - f^2(t) - f'(t) = 1 + o(1), \quad t \to \infty, \\ g_0 \ge 0.5, \quad \Im\sqrt{g_0} = 0, \quad t > T, \\ |g_0'(t)| \le C|f'(t)|, \quad |g_0''(t)| \le C|f'(t)| \in L_1(T, \infty), \end{gathered}$$

conditions (1.6) follows:

$$\begin{aligned} \int_T^\infty \big|G_0(s) e^{\pm 2\int_T^s \Im\sqrt{g_0(y)}dy} ds\big| &\le \int_T^\infty \big(|g_0'(s)|^2 + |g_0''(s)|\big)\, ds \\ &\le C \int_T^\infty |f'(s)| ds < \infty. \end{aligned}$$

Further from the estimates

$$\begin{gathered} \mu_{1,2} = -f - \frac{g_0'}{4g_0} \pm i\sqrt{g_0} = -f + O(f'f) \pm i\sqrt{g_0} = \pm i + o(1), \\ |\mu_j'| = \Big| - f'(t) \pm \frac{i g_0'(t)}{2g_0^{3/2}} - \big(\frac{g_0'}{4g_0}\big)'\Big| \le |f'| + c_1|g_0'| + c_2|g_0'|^2 + c_3|g_0''| \le C|f'|, \\ Re\big(\frac{\mu_j'}{\mu_j}\big) \le \big|\frac{\mu_j'}{\mu_j}\big| \le c_4|\mu_j'(t)| \le C|f'|, \quad f' \in L_1(T, \infty), \\ \int_T^\infty f dt = \int_T^\infty t^\alpha \ln^\beta t dt = \infty, \quad \alpha > -1, \text{ or } \alpha = -1, \beta > -1, \\ \int_T^\infty \Re(\mu_j) dt = \int_T^\infty (-f + O(f'f) dt) = -\infty \end{gathered}$$

conditions (1.7) follows.

To prove the statement of example 1.2 let us show that if $-1 < \alpha < -1-\beta$, $\alpha < 0$ then conditions (1.6), (1.7) of Theorem 1.1 are satisfied. From the estimates

$$f(t) = o(1), \quad g_0 = 1 - f^2(t) - f'(t) = 1 + o(1), \quad t \to \infty,$$
$$|f'(t)| \le \frac{\sqrt{\alpha^2 t^{2\alpha} + \beta^2 t^{2\beta}}}{t} \in L_1(0,\infty),$$
$$|g_0| \ge 0.5, \quad |g_0'(t)| \le C|f'(t)|, \quad |g_0''(t)| \le C|f'(t)|,$$
$$P \equiv \Re(-g_0) = -1 + t^{2\alpha} - t^{2\beta} + \alpha t^{\alpha-1}, \quad Q \equiv \Im(-g_0) = 2t^{\alpha+\beta} + \beta t^{\beta-1},$$
$$P = -1 + o(1), \quad Q = 2t^{\alpha+\beta}(1+o(1)), \quad R \equiv \sqrt{P^2+Q^2} = 1 + o(1), \quad t \to \infty,$$
$$\Re\sqrt{-g_0} = \sqrt{\frac{P+R}{2}} = \sqrt{\frac{R^2-P^2}{2(R-P)}} = \frac{|Q|}{2(1+o(1))},$$
$$\int_T^t \Re\sqrt{-g_0}dt = \int_T^t \frac{|Q|}{2(1+o(1))} \le C\int_T^t s^{\alpha+\beta}ds < \text{const}, \quad t \to \infty$$

conditions (1.6) follow:

$$\begin{aligned}\int_T^\infty \Big|G_0(s)e^{\pm\int_T^t \Re\sqrt{-g_0}dy}ds\Big| &\le C\int_T^\infty |G_0(s)|ds \le C\int_T^\infty (|g_0'|^2 + |g_0''|)ds \\ &\le C\int_T^\infty |f'(t)|dt < \infty.\end{aligned}$$

Further from the estimates

$$\Re(\mu_j) = Re\big(-f - \frac{g_0'}{4g_0} \pm i\sqrt{g_0}\big) = -Re(f) + O(g_0') \pm O(|Q|/2), \;\; j = 1,2, \; t \to \infty,$$
$$\Re(\mu_j) = -t^\alpha + O(g_0') \pm O(t^{\alpha+\beta}), \quad j = 1,2, \; t \to \infty,$$
$$\int_T^t \Re(\mu_j)dt \to -\infty, \quad \alpha > -1, \; t \to \infty,$$
$$\mu_j = -f - \frac{g_0'}{4g_0} \pm i\sqrt{g_0} = \pm i + o(1), \quad j = 1,2, \quad t \to \infty,$$
$$|\mu_j'| = \big|f' + (\frac{g_0'}{4g_0})' \pm \frac{ig_0'}{g_0^{3/2}}\big| \le C|f'(t)| \quad f' \in L_1(T,\infty),$$
$$\frac{\mu_j'}{\mu_j} \in L_1(T,\infty), \quad j = 1,2$$

conditions (1.7) follow.

Proof of Theorem 1.2. From representation (2.28) of the solutions of (1.1) we have the estimates:

$$|x(t)| \le \sum_{j=1}^2 |C_j\varphi_j(t)| + \int_b^t \big|\frac{\varphi_1(t)}{\varphi_1(s)} - \frac{\varphi_2(t)}{\varphi_2(s)}\big|\big|\frac{x(s)\varphi_2(s)L\varphi_1(s)}{W[\varphi_1(s),\varphi_2(s)]}\big|ds \tag{3.6}$$

or because the functions $|\varphi_j(t)|$ are decreasing:

$$|x(t)| \le \sum_{j=1}^2 |C_j\varphi_j(t)| + 2\int_b^t \big|\frac{x(s)\varphi_2(s)L\varphi_1(s)}{W[\varphi_1(s),\varphi_2(s)]}\big|ds. \tag{3.7}$$

From (2.31),

$$|x(t)| \le C + 2\int_b^t |x(s)B_{21}(s)|ds, \tag{3.8}$$

where $C = |N_1C_1| + |N_2C_2|$ depends on T and does not depend on b. From (1.10) we have $B_{21} = B_{12}$. Applying Gronwall's lemma we have

$$|x(t)| \le C \exp \int_b^t 2|B_{21}(s)|ds. \tag{3.9}$$

From (3.7), (3.9) we have

$$|x(t)| \le \sum_{j=1}^{2} |C_j\varphi_j(t)| + C\int_b^t |B_{21}| \exp\big(2\int_b^s |B_{21}(y)|dy\big)ds. \tag{3.10}$$

From (1.9),

$$\lim_{t\to\infty} |\varphi_j(t)| = |\varphi_j(T)| \lim_{t\to\infty} \exp\int_T^t \Re(\frac{\varphi_j'}{\varphi_j})ds = 0, \quad j = 1, 2.$$

In view of (1.12)

$$|x(t)| \le \sum_{j=1}^{2} |C_j\varphi_j(t)| + C\Big(-1 + \exp\int_b^t |2B_{21}(s)|ds\Big) \to 0,$$

when $t \to \infty$ and $b \to \infty$, because C_j, C do not depend on b. $\square$

Proof of Theorem 1.3. From

$$|\varphi_j^{(k-1)}(t)| = |\varphi_j^{(k-1)}(T)| \exp\int_T^t Re\Big(\frac{\varphi_j^{(k)}(s)}{\varphi_j^{(k-1)}(s)}\Big)ds, \quad k = 1, 2 \tag{3.11}$$

and conditions (1.9), (1.14) we have

$$\lim_{t\to\infty} |\varphi_j^{(k-1)}(t)| = 0, \quad j, k = 1, 2. \tag{3.12}$$

From (3.12) we have

$$|\varphi_j^{(k-1)}(t) \le S < \infty, \quad j, k = 1, 2,\ t \ge T, \tag{3.13}$$

where S depend on T and does not depend on b. From representations (2.28),(2.29) we have the estimates:

$$|x^{(k)}(t)| \le \sum_{j=1}^{2} |\varphi_j^{(k)}(t)C_j| + \int_b^t \Big|\frac{\varphi_1^{(k)}(t)}{\varphi_1(s)} - \frac{\varphi_2^{(k)}(t)}{\varphi_2(s)}\Big| \cdot |B_{21}(s)x(s)|ds, \quad k = 0, 1 \tag{3.14}$$

or because the functions $|\varphi_j(t)|$ are decreasing we get for $k = 0, 1$

$$|x^{(k)}(t)| \le \sum_{j=1}^{2} |\varphi_j^{(k)}(t)C_j| + \Big(\Big|\frac{\varphi_1^{(k)}(t)}{\varphi_1(t)}\Big| + \Big|\frac{\varphi_2^{(k)}(t)}{\varphi_2(t)}\Big|\Big)\int_b^t |B_{21}(s)x(s)|ds. \tag{3.15}$$

In view of (1.15) and (3.9) we obtain for $k = 0, 1$ the estimates

$$|x^{(k)}(t)| \le \sum_{j=1}^{2} |\varphi_j^{(k)}(t)C_j| + cC\Big(\int_b^t |B_{21}(s)|ds\Big)^{-k\delta} \int_b^t |B_{21}(s)|e^{\int_b^s |B_{21}|dz}ds,$$

from which follow estimates (1.16):

$$|x^{(k)}(t)| \le \sum_{j=1}^{2} \big|C_j\varphi_j^{(k)}(t)\big| + cC\Big(\int_b^t |B(s)|ds\Big)^{-k\delta}\Big(-1+e^{\int_b^t |2B(s)|ds}\Big) \to 0,$$

when $t\to\infty$ and $b\to\infty$, and C_1, C_2, c, C do not depend on b. □

Proof of Theorem 1.4. Let us choose $\varphi_j(t) = x_j(t)$ as in (2.19). From (1.17), (1.18), (2.25) and (1.8), conditions (1.9)-(1.12) of Theorem 1.2 follow. Theorem 1.4 follows from Theorem 1.2. □

Note that Theorem 1.5 follows from Theorem 1.3 by choosing $\varphi_j(t) = x_j(t)$ as in (2.19).

The statement in the example 1.3. follows from example 1.4 when $\beta\to-\infty$. To prove the statement in example 1.4 let us show that if

$$-1 \le \alpha < 1, \quad \beta \le \frac{\alpha+1}{2}$$

then conditions (1.8),(1.17), (1.18) of Theorem 1.4 are satisfied. From conditions (1.17) or

$$\exp\Big(\int_T^t \Re\big(f+\frac{g_0'}{4g_0} \pm\sqrt{-g_0}\big)dt\Big) = C|g_0|^{1/4}\exp\Big(\int_T^t \Re\left(f\pm\sqrt{-g_0}\right)dt\Big)\to\infty \tag{3.16}$$

it follows that for the product of these expressions we have

$$|g_0|^{1/2}\exp\Big(\int_T^t Re(f)dt\Big) = |g_0|^{1/2}\exp\Big(\int_T^t t^\alpha dt\Big)\to\infty$$

from which, because g_0 has polynomial growth or decay, we get the necessary condition $\alpha\ge-1$. Denote

$$U(t)\equiv Re(f-\sqrt{-g_0}) = t^\alpha - \sqrt{(R+P)/2},$$
$$P\equiv\Re(-g_0),\quad Q\equiv\Im(-g_0),\quad R\equiv\sqrt{P^2+Q^2}.$$

Then one of conditions (3.16) turns

$$R^{1/4}\exp(\int_T^t U(s)ds)\to\infty.$$

Because $R(t)$ has polynomial growth or decay in most of the cases to prove (1.17) it is sufficient to prove that

$$\int_T^t U(s)ds = O(t^\lambda)\to\infty,\quad t\to\infty,\quad \lambda>0.$$

By direct calculations

$$\begin{aligned} U &= \frac{t^{2\alpha}-(P+R)/2}{t^\alpha+\sqrt{(P+R)/2}} = \frac{2t^{2\alpha}-P-R}{2t^\alpha+\sqrt{2P+2R}} \\ &= \frac{K}{(2t^\alpha+\sqrt{2P+2R})(2t^{2\alpha}+R-P)}, \end{aligned}$$

$K = 4t^{2\alpha}-4\alpha t^{3\alpha-1}-4\beta t^{\alpha+2\beta-1}-\beta^2t^{2\beta-2}$. Dividing the plane (α,β) on 6 regions $\{\alpha\ge\beta,\alpha>0\}$, $\{\alpha\le\beta,\beta>0\}$, $\{\alpha<0,\beta<0\}$, $\{\alpha<0,\beta=0\}$, $\{\alpha=0,\beta<0\}$ and $\{\alpha=0,\beta=0\}$ we check conditions of Theorem 1.4 in each region separately.

Case 1: $\alpha \geq \beta,\ \alpha > 0$. From

$$g_0 = O(t^{2\alpha}), \quad g_0'(t) = O(t^{2\alpha-1}), \quad t \to \infty,$$
$$|G_0(t)| \leq C\left(|g_0'(t)|^2/|g_0(t)|^{5/2} + |g_0''(t)|/|g_0(t)|^{3/2}\right) \leq Ct^{-\alpha-2}$$

condition (1.8) follows.

To prove (1.17) note that

$$P = -1 + t^{2\alpha} - t^{2\beta} + \alpha t^{\alpha-1} = t^{2\alpha}(1+o(1)), \quad t \to \infty,$$
$$Q = 2t^{\alpha+\beta}(1+o(1)), \quad R = \sqrt{P^2+Q^2} = t^{2\alpha}(1+o(1)), \quad t \to \infty,$$
$$(2t^\alpha + \sqrt{2P+2R})(2t^{2\alpha} + 2R - P) = 12t^{3\alpha}(1+o(1)), \quad t \to \infty,$$
$$U(t) = \frac{4t^{2\alpha} - 4\alpha t^{3\alpha-1} - 4\beta t^{\alpha+2\beta-1} - \beta^2 t^{2\beta-2}}{12t^{3\alpha}(1+o(1))}$$

When $\alpha < 1$,

$$\int_T^t U(s)ds = \frac{t^{1-\alpha}(1+o(1))}{3} \to \infty, \quad t \to \infty.$$

Conditions (1.17),(1.18) are satisfied.

Case 2: $-1 < \alpha \leq \beta,\ \beta > 0$. From

$$g_0 = O(t^{2\beta}), \quad g_0'(t) = O(t^{2\beta-1}), \quad t \to \infty,$$
$$|G_0(t)| \leq C\left(|g_0'(t)|^2/|g_0(t)|^{5/2} + |g_0''(t)|/|g_0(t)|^{3/2}\right) \leq Ct^{-\beta-2}$$

condition (1.8) follows. Further

$$P = t^{2\beta}(-1+o(1)), \quad Q = O(t^{\alpha+\beta}), \quad t \to \infty,$$
$$R = \sqrt{P^2+Q^2} = t^{2\beta}(1+o(1)) \quad R - P = 2t^{2\beta}(1+o(1)), \quad t \to \infty,$$
$$\begin{aligned}(2t^\alpha + \sqrt{2P+2R})(2t^{2\alpha} + 2R - P) &= (2t^\alpha + \frac{|Q|\sqrt{2}}{\sqrt{R-P}})(2t^{2\alpha} + 2R - P)\\ &= 6t^{\alpha+2\beta}(1+o(1)), \quad t \to \infty,\end{aligned}$$
$$U(t) = \frac{4t^{2\alpha} - 4\alpha t^{3\alpha-1} - 4\beta t^{\alpha+2\beta-1} - \beta^2 t^{2\beta-2}}{6t^{\alpha+2\beta}(1+o(1))}.$$

If $\beta < \frac{\alpha+1}{2}$ then (1.17) is satisfied:

$$\int_T^t U(s)ds = \frac{4t^{1+\alpha-2\beta}}{\alpha-2\beta+1}(1+o(1)) \to \infty, \quad t \to \infty.$$

If $\beta = \frac{\alpha+1}{2} < 1$, we have

$$U(t) = \frac{2}{3}(1-\beta)t^{-1} - \frac{2}{3}(2\beta-1)t^{2\beta-3} - \frac{\beta^2}{6t^{2\beta+1}},$$
$$\int_T^t U(s)ds = \frac{2}{3}(1-\beta)\ln t - \frac{4\beta-2}{3t^{2(1-\beta)}} - \frac{t^{-2\beta}}{6} \to \infty$$
$$R^{1/4}\exp(\int_T^t U(s)ds) \to \infty, \quad t \to \infty.$$

Then conditions (1.17),(1.18) are satisfied.

Case 3: $\beta < 0,\ -1 < \alpha < 0$. From

$$g_0 = O(1), \quad |g_0'(t)| \le \frac{C}{t}, \quad |g_0''(t)| \le \frac{C}{t^2}, \quad t \to \infty,$$
$$|G_0(t)| \le C|g_0'(t)|^2/|g_0(t)|^{5/2} + |g_0''(t)|/|g_0(t)|^{3/2} \le Ct^{-2}$$

condition (1.8) follows.

Then

$$P = -1 + o(1), \quad t \to \infty,$$
$$Q = 2t^{\alpha+\beta}(1 + o(1)), \quad R = \sqrt{P^2 + Q^2} = 1 + o(1), \quad t \to \infty,$$
$$(2t^\alpha + \sqrt{2P + 2R})(2t^{2\alpha} + 2R - P) = 6t^\alpha(1 + o(1)), \quad t \to \infty,$$
$$U(t) = \frac{4t^{2\alpha} - 4\alpha t^{3\alpha-1} - 4\beta t^{\alpha+2\beta-1} - \beta^2 t^{2\beta-2}}{6t^\alpha(1 + o(1))},$$
$$\int_T^t U(s)ds = \frac{2t^{1+\alpha}}{3(1+\alpha)}(1 + o(1)) \to \infty, \quad t \to \infty.$$

Conditions (1.17), (1.18) are satisfied.

Case 4: $\beta = 0,\ -1 < \alpha < 0$. From

$$g_0 = O(2), \quad |g_0'(t)| \le \frac{C}{t}, \quad |g_0''(t)| \le \frac{C}{t^2}, \quad t \to \infty,$$

condition (1.8) follows. Then

$$P = -2 + o(1), \quad t \to \infty,$$
$$Q = 2t^\alpha(1 + o(1)), \quad R = 2 + o(1), \quad t \to \infty,$$
$$(2t^\alpha + \sqrt{2P + 2R})(2t^{2\alpha} + 2R - P) = \Big(2t^\alpha + \frac{|Q|\sqrt{2}}{\sqrt{R - P}}\Big)(2t^{2\alpha} + 2R - P) = O(t^\alpha),$$
$$U(t) = O\Big(\frac{4t^{2\alpha} - 4\alpha t^{3\alpha-1}}{t^\alpha}\Big),$$
$$\int_T^t U(s)ds = O\left(t^{1+\alpha}\right) \to \infty, \quad t \to \infty.$$

Conditions (1.17), (1.18) are satisfied.

Case 5: $\beta < 0,\ \alpha = 0$. From

$$g_0 = O(t^{2\beta} - 2it^\beta) = O(-2it^\beta), \quad |g_0'(t)| \le \frac{C}{t}, \quad |g_0''(t)| \le \frac{C}{t^2}, \quad t \to \infty,$$

condition (1.8) follows. Then

$$P = -t^{2\beta}, \quad Q = O(2t^\beta), \quad R = O(2t^\beta), \quad t \to \infty$$
$$(2t^\alpha + \sqrt{2P + 2R})(2t^{2\alpha} + 2R - P) = (2 + \sqrt{4t^\beta - 2t^{2\beta}})(2 + 4t^\beta + t^{2\beta}) = O(4),$$
$$U(t) = O(1 - \beta t^{2\beta-1} - \frac{\beta^2}{4}t^{2\beta-2}), \quad t \to \infty,$$
$$R^{1/4}\int_T^t U(s)ds = O\Big(t^{\beta/4}\exp(4t - 2t^{2\beta} - \frac{\beta^2}{2\beta - 1}t^{2\beta-1})\Big) \to \infty, \quad t \to \infty.$$

Conditions (1.17), (1.18) are satisfied.

Case 6: $\beta = 0,\ \alpha = 0$. From

$$P = -1, \quad Q = 2, \quad R = \sqrt{5}, \quad U = O(1)$$

conditions (1.8), (1.17), (1.18) follow.

Proof of Theorem 1.6. From representation (2.27) we have the estimate:

$$|x(t)| \le \sum_{j=1}^{2} |\varphi_j(t)C_j| + |\varphi_1(t)| \int_b^t \Big|\frac{x(s)L\varphi_2(s)}{W[\varphi_1,\varphi_2]}\Big| ds + |\varphi_2(t)| \int_b^t \Big|\frac{x(s)L\varphi_1(s)}{W[\varphi_1,\varphi_2]}\Big| ds \tag{3.17}$$

or because the functions $|\varphi_j|$ are decreasing and bounded:

$$|x(t)| \le \sum_{j=1}^{2} |\varphi_j(t)C_j| + 2\int_b^t \Big|\frac{x(s)\varphi_1 L\varphi_2(s)}{W[\varphi_1,\varphi_2]}\Big| ds \le C + 2\int_b^t |x(s)B_{21}(s)| ds. \tag{3.18}$$

Applying Gronwall's lemma we have

$$|x(t)| \le C \exp \int_b^t |2B_{21}(s)| ds \le C \exp \int_T^\infty |2B_{21}| ds \equiv C_0. \tag{3.19}$$

By choosing $C_2 = 0$ from representation (2.27) we have the estimates:

$$|x(t)| \ge |\varphi_1(t)C_1| - |\varphi_1(t)| \int_b^t \Big|\frac{x(s)L\varphi_2(s)}{W[\varphi_1,\varphi_2]}\Big| ds - |\varphi_2(t)| \int_b^t \Big|\frac{x(s)L\varphi_1(s)}{W[\varphi_1,\varphi_2]}\Big| ds$$

or

$$|x(t)| \ge |\varphi_1(t)C_1| - 2\int_b^t \Big|\frac{x\varphi_2 L\varphi_1}{W[\varphi_1,\varphi_2]}\Big| ds. \tag{3.20}$$

From (1.12)

$$\alpha(b) \equiv \int_b^\infty \Big|\frac{\varphi_2 L\varphi_1}{W[\varphi_1,\varphi_2]}\Big| ds \to 0 \tag{3.21}$$

when $b \to \infty$. Because positive constants $|C_1|, C_0, \gamma$ do not depend on b by choosing b big enough we can make

$$\alpha(b) < \frac{|C_1|\gamma}{2C_0}.$$

Thus from (3.20) and $|\varphi_1(t)| \ge |\varphi_1(\infty)| = \gamma > 0$ for $t > b$ we have

$$|x(t)| \ge |C_1|\gamma - 2\alpha(b)C_0 > 0$$

and Theorem 1.6 is proved. □

Proof of Theorem 1.7. By choosing $\varphi_j(t) = x_j(t)$ as in (2.19) from (2.25), (1.23), (1.8) it follows (1.10), (1.12) and that the functions $|\varphi_j|$ are decreasing. From (1.24) follows $|\varphi_1(\infty)| = \gamma > 0$. So Theorem 1.7 follows from Theorem 1.6. □

Proof of Theorem 1.8. We prove this theorem by the method of Pucci and Serrin. First we prove the theorem in the case the function of bounded variation $k(t)$ is of class $C^1(J)$. Multiplying equation (1.25) by $\bar{x}'(t)$ we get

$$\bar{x}'(t)x''(t) + h(t,x,x')|x'(t)|^2 + \bar{x}'(t)j(t,x) = 0. \tag{3.22}$$

Adding the conjugate equation

$$\bar{x}''(t)x'(t) + \bar{h}(t,x,x')|x'(t)|^2 + x'(t)\bar{j}(t,x) = 0$$

we get

$$\frac{d}{dt}(|x'(t)|^2 + F(t,x)) + 2Re[h(t,x,x')]|x'(t)|^2 = F_t(t,x), \tag{3.23}$$

$$|x'(t)|^2 + F(t,x) + 2\int_T^t Re[h(s,x,x')]|x'(s)|^2 ds = C + \int_T^t F_s(s,x) ds. \tag{3.24}$$

From $F(t,x) \geq 0$ and $\int_T^\infty F_t(t,x)dt < \infty$ we have

$$\int_T^\infty |x'(t)|^2 Re[h(t,x,x')]dt < \infty. \tag{3.25}$$

Indeed otherwise the right side of (3.24) is finite when $t \to \infty$ and the left side approaches to positive infinity and we get contradiction. So when $t \to \infty$ from (3.24) we get

$$|x'(t)|^2 + F(t,x) = l + \varepsilon(t), \quad l \geq 0, \quad \lim_{t\to\infty} \varepsilon(t) = 0. \tag{3.26}$$

From this expression and (1.34) we see that $x(t)$ and $x'(t)$ are bounded:

$$|x(t)| \leq L, \quad |x'(t)| \leq C, \quad \text{for } t \in J. \tag{3.27}$$

To prove that $l = 0$ assume for contradiction $l > 0$. Multiplying (3.23) by the positive non decreasing function

$$\omega(t) = \exp \int_T^t k(s)ds \tag{3.28}$$

we get

$$\begin{aligned}
&\frac{d}{dt}(\omega|x'(t)|^2 + \omega F(t,x)) \\
&= \omega F_t(t,x) + \omega'(t)(|x'(t)|^2 + F(t,x)) - 2\omega Re[h(t,x,x')]|x'(t)|^2,
\end{aligned}$$

or

$$\begin{aligned}
&\frac{d}{dt}\left(\omega|x'|^2 + \omega F + \alpha\omega'\bar{x}'x + \alpha\omega'\bar{x}x'\right) \\
&= \alpha\omega''(\bar{x}'x + x'\bar{x}) - \alpha\omega'\bar{x}(x'h + j) - \alpha\omega' x(\bar{x}'\bar{h} + \bar{j}) \\
&\quad + 2\alpha\omega'|x'|^2 + \omega F_t + \omega'(|x'|^2 + F) - 2\omega Re[h)]|x'|^2,
\end{aligned} \tag{3.29}$$

where α is a positive number. Denote

$$R(t) \equiv \frac{d}{dt}\left(\omega|x'|^2 + \omega F + \alpha k\omega(\bar{x}'x + x'\bar{x})\right), \tag{3.30}$$

then from $\omega' = k\omega$, $\omega'' = (k' + k^2)\omega$ and

$$\begin{aligned}
\frac{R}{\omega} &= F_t + k\left(|x'|^2 + F - \alpha\bar{x}j - \alpha x\bar{j}\right) - 2Re(h)|x'|^2 \\
&\quad + \alpha(k' + k^2)(x\bar{x})' + 2k\alpha|x'|^2 - \alpha k(x'\bar{x}h + x\bar{x}'\bar{h}).
\end{aligned} \tag{3.31}$$

We take T_1 large so that

$$|x'|^2 + F \geq \frac{3l}{4} \text{ on } J_1 = (T_1, \infty) \quad \text{and} \quad \int_{T_1}^\infty |\psi(t)|dt \leq \frac{l}{4}. \tag{3.32}$$

Let us estimate R when $t \in J_1$ and α is suitably small. Suppose that $k = k' \equiv 0$ on $t \in J \setminus I$, $\quad J = [T, \infty)$. Then from (1.26),(1.34), (3.31):

$$\frac{R}{\omega} \leq \psi(t), \quad t \in J_1 \setminus I. \tag{3.33}$$

On the remaining set $I' = I \bigcap J_1$, we partition R in the form

$$\frac{R}{\omega} = F_t + k(|x'|^2 + F) + \sum_{k=1}^5 R_k, \tag{3.34}$$

where

$$
\begin{gathered}
R_1 = -\frac{2Re(h)}{5}|x'|^2 - k\alpha(\bar{x}j + x\bar{j}), \\
R_2 = -\frac{2Re(h)}{5}|x'|^2 + 2k\alpha|x'|^2, \\
R_3 = -\frac{2Re(h)}{5}|x'|^2 + k^2\alpha(x\bar{x}' + \bar{x}x'), \\
R_4 = -\frac{2Re(h)}{5}|x'|^2 + k'\alpha(x\bar{x}' + \bar{x}x'), \\
R_5 = -\frac{2Re(h)}{5}|x'|^2 - k\alpha(\bar{x}x'h + \bar{x}'x\bar{h}).
\end{gathered}
$$

To prove the estimate

$$
R_1 \le -k\alpha\chi, \quad \text{for } t \in I' \text{ and small } \alpha \tag{3.35}
$$

let us fix $p_1 = \sqrt{l/4}$ so that $|x'|^2 = |p|^2 \le \frac{l}{4}$when $|x| \le L$ and $|p| \le |p_1|$. From (3.32)

$$
F(t,x) \ge \frac{l}{2} \quad \text{on } I_1 = \{t \in I' : |x'(t)| \le p_1\}.
$$

On other hand

$$
F(t,x) = F(T_1,x) + \int_{T_1}^{t} F_s(s,x)ds \le F(T_1,x) + \int_{T_1}^{t} \psi(s)ds \le F(T_1,x) + \frac{l}{4}
$$

for $t \in J_1$. Thus $F(t,x) \ge l/4$ in I_1. Since $F(T,0) = 0$ it follows that there exist a number $u_0 > 0$ such that $|x(t)| > u_0$ for $t \in I_1$. From (1.26), (1.32) we get (3.35) for $t \in I_1$. In the remaining set $I' \setminus I_1$ we have $|x'(t) > p_1$ and if

$$
\alpha \le \frac{2p_1^2}{5\beta\chi}. \tag{3.36}
$$

then

$$
\begin{gathered}
2Re(h)|x'(t)|^2 \ge 2\sigma p_1^2 \ge \frac{2kp_1^2}{\beta} \ge 5\alpha k\chi. \\
R_1 = -\frac{2}{5}Re(h)|x'(t)|^2 - \alpha k(\bar{x}j + x\bar{j}) \le -\alpha k\chi
\end{gathered}
$$

and (3.35) is valid for all $t \in I'$. We claim that

$$
R_2 \le \frac{\alpha k\chi}{8}, \quad \text{for } t \in I' \text{ and small } \alpha. \tag{3.37}
$$

Indeed

$$
\begin{gathered}
|x'(t)|^2 \le \frac{\chi}{16} \quad \text{if} \quad |x'(t)| \le p_2 \equiv \frac{\chi}{16}, \\
R_2 = \big(2\alpha k - \frac{2Re(h)}{5}\big)|x'(t)|^2 \le 2\alpha k|x'(t)|^2 \le \frac{\alpha k\chi}{8}.
\end{gathered}
$$

Otherwise, if $|x'(t)| > p_2$, then from

$$
\alpha \le \frac{1}{5\beta} \tag{3.38}
$$

we have

$$
2\alpha k \le 2\alpha\beta\sigma \le \frac{2\sigma}{5} \le \frac{2Re(h)}{5} \quad \text{and } R_2 \le 0.
$$

Let us prove that
$$R_3 \le \frac{\alpha k\chi}{8}, \quad \text{for } t \in I' \text{ and small } \alpha. \tag{3.39}$$
Indeed for $p_3 \equiv \frac{\chi}{16L\sup(k)}$, we have
$$2\alpha k^2|x'(t)\bar{x}| \le 2\alpha k^2 L p_3 \le \frac{\alpha k\chi}{8} \quad \text{if} \quad |x'(t)| \le p_3.$$
Otherwise if $|x'(t)| > p_3$ then from $|x'(t)| \le C$ and
$$\alpha \le \frac{p_3}{5\beta L\sup(k)} \tag{3.40}$$
we have
$$2\alpha k^2|x'\bar{x}| \le \frac{2\alpha k^2 L}{p_3}|x'|^2 \le \frac{2k}{5\beta}|x'|^2 \le \frac{2\sigma}{5}|x'|^2 \le \frac{2Re(h)}{5}|x'|^2, \quad R_3 \le 0.$$
So (3.39) is proved. From (3.27) we have
$$R_4 = -\frac{2}{5}Re(h)|x'|^2 + \alpha k'(\bar{x}'x + \bar{x}(t)x') \le 2\alpha|k'|CL, \quad \text{for} \quad t \in I'. \tag{3.41}$$
To prove the estimate
$$R_5 \le 10\alpha^2 L^2\gamma\delta k\sup(k) \tag{3.42}$$
define the set
$$I_5 = \{t \in I' : |x'(t)| \ge \alpha\Lambda, \quad \Lambda = 5L\gamma\sup(k)\}.$$
In this set
$$-\alpha k(\bar{x}x'h + \bar{x}'x\bar{h}) \le 2\alpha k|x'xh| \le 2\alpha kL\frac{|x'|^2\gamma\Re(h)}{5\alpha L\gamma\sup(k)} \le \frac{2Re(h)}{5}|x'|^2$$
and we have $R_5 \le 0$.

In $I' \setminus I_5$ we have $|x'(t)| \le \alpha L$ and estimate (3.42):
$$R_5 \le 2\alpha k|\bar{x}x'(t)h| \le 10\alpha^2 L^2\gamma\delta k\sup(k) = 2\alpha^2\delta kL\Lambda.$$
Thus we have the estimates
$$\frac{R}{\omega} \le \psi + k\big(|x'(t)|^2 + F - \alpha\chi + \frac{2\alpha\chi}{8}\big) + 2\alpha CL|k'(t)| + 10\alpha^2 L^2\gamma\delta k\sup(k),$$
$$R \le \omega\left(\psi + 2\alpha CL|k'| + 10\alpha^2 L^2\gamma\delta k\sup(k)\right) + \omega'\big(l + \varepsilon - \alpha\chi + \frac{2\alpha\chi}{8}\big), \tag{3.43}$$
where $\delta k \equiv 0$ on $J \setminus I$. Let us fix α so small that (3.36), (3.38), (3.40) and
$$\alpha \le \frac{\chi}{80ML^2\gamma\sup(k)} \tag{3.44}$$
are satisfied. Moreover in view of (3.26) and $k \in BV(J), k' \in L_1(J)$ we can take $T_2 > T_1$ such that
$$|\varepsilon(t)| \le \frac{\alpha\chi}{8} \quad \text{for } t > T_2, \tag{3.45}$$
$$\int_{T_2}^{\infty}\psi(s)ds \le \frac{\alpha\chi}{8}, \quad \int_{T_2}^{\infty}|k'(s)|ds \le \frac{\chi}{16CL}. \tag{3.46}$$
Then from (3.30) and (3.43),
$$\begin{aligned}&\frac{d}{dt}\left(\omega|x'|^2 + \omega F + \alpha k\omega(\bar{x}'x + x'\bar{x})\right)\\ &\equiv R \le \omega\left(\psi + 2\alpha CL|k'| + 10\alpha^2 L^2\gamma\delta k\sup(k)\right) + \omega'(l + \alpha\chi/8 - 3\alpha\chi/4)\end{aligned}$$

which integrating yields

$$\omega(|x'(t)|^2 + F + \alpha k(\bar{x}'x + x'\bar{x})) \\ \leq \int_{T_2}^{\infty} \omega\psi ds + 2\alpha CL \int_{T_2}^{t} \omega|k'|ds + 10\alpha^2 L^2\gamma \sup(k) \int_{T_2}^{t} \omega\delta k\, ds + \omega(l - 5\alpha\chi/8) + c.$$

So the function

$$\begin{aligned} \Psi &= \omega\left(|x'(t)|^2 + F + \alpha k(\bar{x}'x + x'\bar{x}) - l + 5\alpha\chi/8\right) \\ &\quad - \int_{T_2}^{t} \omega\psi ds - 2\alpha CL \int_{T_2}^{t} \omega|k'|ds - 10\alpha^2 L^2\gamma \sup(k) \int_{T_2}^{t} \omega\delta k\, ds \end{aligned} \tag{3.47}$$

is decreasing.

Now we claim that there exist a sequence t_n such that $t_n \uparrow \infty$ and

$$k(t_n)|x'(t_n)|^2 \to 0, \quad n \to \infty. \tag{3.48}$$

Otherwise, because of boundedness of $k(t), |x'(t)|$ there exist numbers $k_0 > 0$, $p_0 > 0$, $\bar{t}$ such that

$$k(t) \geq k_0 > 0 \quad \text{and} \quad |x'(t)| \geq p_0 > 0 \quad \text{for } t > \bar{t}.$$

In turn, since $k(t) \equiv 0$ on $J \setminus I$, we must have $I \supset [\bar{t}, \infty)$,

$$\sigma(t) \geq \frac{k_0}{\beta} \quad \text{and} \quad |x'(t)| \geq p_0 > 0 \quad \text{for } t > \bar{t}.$$

So $\mathrm{Re}(h)|x'(t)|^2 > 0$ for $t > \bar{t}$, which contradicts (3.25).

From (3.44)-(3.47) we have

$$\begin{aligned} \frac{\Psi(t_n)}{\omega(t_n)} &\geq \varepsilon - 2\alpha Lk|x'| + 5\alpha\chi/8 - 3\alpha\chi/8 \\ &\geq -\alpha\chi/8 - 2\alpha Lk|x'| + 2\alpha\chi/8 \\ &\geq \alpha\chi/8 - 2\alpha Lk|x'|. \end{aligned}$$

From (3.48), $Lk(t_n)|x'(t_n)| \leq \chi/32$ for $n > n_0$ and

$$\frac{\Psi(t_n)}{\omega(t_n)} \geq \alpha\chi/16.$$

Hence $\Psi(t_n) \to \infty$ as $\to \infty$. This contradicts the fact that $\Psi(t)$ is decreasing. So $l = 0$ or $\lim_{t\to\infty}(|x'|^2 + F) = l = 0$ from which follows (1.2).

The proof of the general case $k \in BV(J)$ follows from the lemma below. □

Lemma 3.1 ([10]). *Let $k(t)$ be a non-negative continuous function of bounded variation on J ($k \in BV(J)$). Then for every constant $\theta > 1$ there exists a function $\bar{k} \in C^1(J)$ and an open set $E \subset J$ such that*

(i) $\theta k \geq \bar{k} \geq \begin{cases} k, & \text{in } J \setminus E \\ 0, & \text{in } E \end{cases}$

(ii) $\mathrm{Var}(\bar{k}) \leq \theta\, \mathrm{Var}(k)$

(iii) $\int_E k dt \leq 1$.

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