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STABILITY OF STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS AND THE W-TRANSFORM

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Abstract. The paper contains a systematic presentation of how the so-called "W-transform" can be used to study stability of stochastic functional differential equations. The W-transform is an integral transform which typically is generated by a simpler differential equation ("reference equation") via the Cauchy representation of its solutions ("variation-of-constant formula"). This other equation is supposed to have prescribed asymptotic properties (in this paper: Various kinds of stability). Applying the W-transform to the given equation produces an operator equation in a suitable space of stochastic processes, which depends on the asymptotic property we are interested in. In the paper we justify this method, describe some of its general properties, and illustrate the results by a number of examples.

1. Introduction

Stability analysis for stochastic delay and functional differential equations is usually based on the classical Lyapunov-Krasovskii-Razumikhin method (see e.g. [12] and [16]), where one tries to find a suitable Lyapunov function (a Lyapunov-Krasovskii functional) which ensures the prescribed stability property. Another way is more straightforward: it uses direct estimates on solutions [12].

On the other hand, a recent progress in stability theory for deterministic functional differential equations shows (see e.g. [2]) that at least for linear delay equations it seems to be more convenient to use special integral transforms to study various asymptotic properties.

The idea of using the W-transform in stability theory was originally proposed by Berezansky in his pioneer work [3]. Later on, this idea was developed by him and his collaborators in a series of papers (see [1]). The method can be briefly outlined as follows. Instead of studying stability of a given linear delay equation with respect to the initial function, one first moves the initial function over to the right hand side of the equation. By this, one arrives at another property called in the literature "admissibility of pairs of spaces" (see e.g. [13]). One proves then that any kind of stability with respect to the initial function is implied by admissibility of certain pairs of functional spaces. To check admissibility one choose a simpler equation (called "a reference equation"), which already has the required property

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of admissibility. This new equation gives then rise to an integral transform (traditionally called "the W-transform") which, when applied to the original equation, produces an integral equation of the form $x - \Theta x = f$. If the latter equation is solvable (for instance, if $\|\Theta\| < 1$), then admissibility, and hence stability, is proved.

In this paper, we try to extend this method to the case of stochastic functional differential equations. We exploit the scheme that (in the deterministic case) was developed in [1] replacing functional spaces by certain spaces of stochastic processes. This enables us to put the study Lyapunov stability (e.g. asymptotic stability and exponential stability) of stochastic delay equations into a unified framework in a natural way. As we show, this method covers more general stochastic functional differential equations and produces sufficient stability results in an efficient way.

Let us remark that the main purpose of the present paper is to give a theoretical justification of the W-method in connection with stochastic stability. That is why all the examples we present are illustrative and are not compared with the stability results which can be found in the literature. A more specific analysis of some stochastic delay equations, including a comparison with the existing stability criteria, will be a subject of a forthcoming paper.

2. Notation and main assumptions

Basic notation. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ be a complete filtered probability space (see e. g. [11, p. 9]), $Z := (z^1, \dots, z^m)^T$ be a m-dimensional semimartingale [11, p. 73] on it (we distinguish column vectors $(a^1, \dots a^n)^T$ and row vectors $(a^1, \dots a^n)$).

In the sequel, we let $|\cdot|$ denote the norm in $\mathbb{R}^n$; $\mathbb{R}^{k\times n}$ will be a linear space consisting of all real $k \times n$-matrices with the norm $\|.\|$ that agrees with the chosen vector norm in $\mathbb{R}^n$. We write $\bar{0}$ for the zero column vector in $\mathbb{R}^n$, the symbol $\bar{E}$ denotes the unit matrix, while $\mathbf{E}$ stands for the expectation.

For convenience, we denote by λ^* the complete measure on an interval I, generated by a nondecreasing function $\lambda(t)$ $(t \in I)$.

The following linear spaces of stochastic processes will be used in the sequel:

$L^n(Z)$ consists of predictable $n \times m$-matrix functions defined on $[0, \infty)$ with the rows that are locally integrable with respect to the semimartingale Z, see e. g. [5];

k^n consists of n-dimensional $\mathcal{F}_0$-measurable random variables (we set also $k := k^1$);

D^n consists of n-dimensional stochastic processes on $[0, \infty)$, which can be represented in the following form:

$$x(t) = x(0) + \int_0^t H(s)dZ(s) \quad (t \geq 0),$$

where $x(0) \in k^n$, $H \in L^n(Z)$;

L^λ_q consists of scalar functions defined on $[0, \infty)$, which are q-integrable $(1 \leq q < \infty)$ with respect to the measure λ^*, generated by a nondecreasing function $\lambda(t)$ $(t \in [0, \infty))$;

L^λ_∞ consists of scalar functions defined on $[0, \infty)$, which are measurable and a. s. bounded with respect to the measure λ^*, generated by a nondecreasing function $\lambda(t)$ $(t \in [0, \infty))$;

L_q stands for L^λ_q in the case when $\lambda(t) = t$ $(1 \leq q \leq \infty)$.

In addition, we will implicitly assume that the real numbers p, q satisfy the inequalities $1 \leq p < \infty$, $1 \leq q \leq \infty$.

The following notational agreement will be used in the sequel: $\int_a^b = \int_{[a,b]}$ (otherwise we will write $\int_{(a,b]}$, or $\int_{(a,b)}$ etc.

The variation of a function over a closed interval $[a,b]$ will be denoted by $\bigvee_{[a,b]}$, and we will also write $\bigvee_{(a,b]}$ for $\lim_{\delta\to 0+}\bigvee_{[a+\delta,b]}$ and $\bigvee_{[a,b)}$ for $\lim_{\delta\to 0+}\bigvee_{[a,b-\delta]}$, respectively.

Now we are able to formulate the main assumption on the semimartingale $Z(t)$. In what follows we always assume that the semimartingale $Z(t)$ $(t\in[0,\infty))$ can be represented as a sum

$$Z(t)=b(t)+c(t)\quad (t\ge 0), \tag{2.1}$$

where $b(t)$ is a predictable stochastic process of locally finite variation and $c(t)$ is a local squire-integrable martingale [11, p. 28] such that all the components of the process $b(t)$ as well as the predictable characteristics $< c^i, c^j > (t)$, $1\le i,j\le m$ of the process $c(t)$ [11, p. 48] are absolutely continuous with respect to to a nondecreasing function $\lambda:[0,\infty)\to\mathbb{R}_+$. In this case, we can write

$$b^i=\int_0^{\cdot} a^i d\lambda,\quad \langle c^i,c^j\rangle=\int_0^{\cdot} A^{ij}d\lambda,\quad i,j=1,\dots,m. \tag{2.2}$$

For example, $\lambda(t)=t$ for Itô equations. Without loss of generality, it will be convenient in the sequel to assume that the first component of the semimartingale $Z(t)$ coincides with $\lambda(t)$, i.e. $z^1(t)=\lambda(t)$. Clearly, we can always do it adding, if necessary, a new, $(m+1)$-th component to the m-dimensional semimartingale $Z(t)$.

It is known [5] that under the assumption (2.1) the space $L^n(Z)$ can be described as a set of all predictable $n\times m$-matrices $H(t)=[H^{ij}(t)]$, for which

$$\int_0^t (|Ha|+\|HAH^\top\|)d\lambda<\infty \ \text{ a. s.} \tag{2.3}$$

for any $t\ge 0$. Here

$$a:=(a^1,\dots,a^m)^T,\quad A:=[A^{ij}]. \tag{2.4}$$

Note that a is an m-dimensional column vector and A is a $m\times m$-matrix.

Under the above assumptions we can also write $\int_0^t HdZ=\int_0^t Hdb+\int_0^t Hdc$. Moreover, we can describe the space D^n as a set consisting of all n-dimensional adapted stochastic processes on $[0,\infty)$, the trajectories of which are right continuous and have left hand limits for all $t\in[0,\infty)$ and almost all ω (the so-called "cadlag processes"). In addition, the following estimate holds:

$$\Big(\mathbf{E}\Big|\int_0^t HdZ\Big|^{2p}\Big)^{\frac{1}{2p}}\le\Big(\mathbf{E}\Big(\int_0^t |Ha|d\lambda\Big)^{2p}\Big)^{\frac{1}{2p}}+c_p\Big(\mathbf{E}\Big(\int_0^t \|HAH^T\|d\lambda\Big)^{p}\Big)^{\frac{1}{2p}}, \tag{2.5}$$

where c_p is a certain positive constant depending on p (see e. g. [11, p. 65]).

Given $H=[H^{ij}]\in L^n(Z)$ and a, A defined in (2.4), we will write

$$a^+:=(|a^1|,\dots,|a^m|)^T,\quad A^+:=[|A^{ij}|],\quad H^+:=[|H^{ij}|]. \tag{2.6}$$

Studying different kinds of stochastic stability requires different spaces of stochastic processes which are listed below.

Main spaces. Assume that we are given:
a scalar nonnegative function ξ, defined on $[0,\infty)$ and locally integrable with respect to the measure λ^* generated by a nondecreasing function λ (λ is the same as in (2.2));
a positive scalar function $\gamma(t)$ $(t \in [0,\infty))$.

Remark 2.1. In what follows we silently adopt the following convention: if in a definition, a theorem etc. $\gamma(t)$ is mentioned without any comments, then it is only assumed to be a positive scalar function. Otherwise, additional properties of γ will be explicitly described.

These functions are involved in the definitions of almost all spaces we are going to use in the sequel. Both are crucial for our considerations as they are responsible for the asymptotic behavior of the solutions.

$$k_p^n = \{\alpha : \alpha \in k^n, \|\alpha\|_{k_p^n} := (\mathbf{E}|\alpha|^p)^{1/p} < \infty\};$$
$$M_p^\gamma = \{x : x \in D^n, \|x\|_{M_p^\gamma} := (\sup_{t\geq 0} \mathbf{E}|\gamma(t)x(t)|^p)^{1/p} < \infty\} \quad (M_p^1 = M_p);$$
$$\begin{aligned}\Lambda_{p,q}^n(\xi) = \{H : H \in L^n(Z), &(\mathbf{E}|Ha|^p)^{1/p}\xi^{q^{-1}-1} \\ &+ (\mathbf{E}\|HAH^\top\|^{p/2})^{1/p}\xi^{q^{-1}-0.5} \in L_q^\lambda\};\end{aligned}$$
$$\begin{aligned}\Lambda_{p,q}^{n+}(\xi) =\{H : H \in L^n(Z), &(\mathbf{E}|H^+a^+|^p)^{1/p}\xi^{q^{-1}-1} \\ &+ (\mathbf{E}\|H^+A^+(H^+)^\top\|^{p/2})^{1/p}\xi^{q^{-1}-0.5} \in L_q^\lambda\}.\end{aligned}$$

The following parameters are involved in the above definitions: The numbers p,q are assumed to satisfy the inequalities $1 \leq p < \infty$, $1 \leq q \leq \infty$; a, A are defined by (2.4); a^+, A^+, H^+ are given by (2.6).

In the last two spaces the norms are given by

$$\|H\|_{\Lambda_{p,q}^n(\xi)} := \|(\mathbf{E}|Ha|^p)^{1/p}\xi^{q^{-1}-1}\|_{L_q^\lambda} + \|(\mathbf{E}\|HAH^\top\|^{p/2})^{1/p}\xi^{q^{-1}-0.5}\|_{L_q^\lambda},$$
$$\|H\|_{\Lambda_{p,q}^{n+}(\xi)} := \|(\mathbf{E}|H^+a^+|^p)^{1/p}\xi^{q^{-1}-1}\|_{L_q^\lambda} + \|(\mathbf{E}\|H^+A^+(H^+)^\top\|^{p/2})^{1/p}\xi^{q^{-1}-0.5}\|_{L_q^\lambda}.$$

Operators and equations.

Definition 2.1. An operator $V : D^n \to L^n(Z)$ is called Volterra (see [9]) if for any stopping time [11, p. 9] $\tau \in [0,\infty)$ a. s. and any $x, y \in D^n$ such that $x(t) = y(t)$ $(t \in [0,\tau]$ a.s.) one has $(Vx)(t) = (Vy)(t)$ $(t \in [0,\tau]$ a.s.).

Definition 2.2. An operator $V : D^n \to L^n(Z)$ is called k-linear if

$$V(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 V x_1 + \alpha_2 V x_2$$

for any $\alpha_i \in k$, $x_i \in D^n$, $i = 1, 2$.

This property exclude "global" operations, like expectation, from the coefficients of the equation, and therefore determines the pathwise way of describing solutions.

Remark 2.2. If V is continuous with respect to natural topologies in the spaces D^n and $L^n(Z)$, then one can show that k-linearity follows from the usual linearity (with respect to $\mathbb{R}$).

The central object of this paper is a stochastic functional differential equation

$$dx(t) = [(Vx)(t) + f(t)]dZ(t) \quad (t \geq 0), \tag{2.7}$$

where $f \in L^n(Z)$ and $V : D^n \to L^n(Z)$ is a k-linear and Volterra (in the sense of Definition 2.1) operator.

In [15] it is shown that (2.7) covers linear stochastic delay equations, linear stochastic integro-differential equations, linear stochastic neutral equations - all with driven semimartingales etc. It can look a little bit confusing as (2.7) does not depend on the values $x(t)$ for $t < 0$. In fact, this dependence can be incorporated into the right-hand side as it is demonstrated in the following example.

Example 2.1. Consider a linear scalar stochastic differential equation of the form

$$dx(t) = [a(t)(Tx)(t) + g(t)]dZ(t) \quad (t \geq 0) \tag{2.8}$$

with the prehistory condition

$$x(s) = \varphi(s) \quad (s < 0), \tag{2.9}$$

where $(Tx)(t) = \int_{(-\infty,t)} d_s R(t,s)x(s)$ is the distributed delay operator.

Under natural assumptions on the right-hand side (see [15]) this equation can be reduced to the form (2.7) if one sets

$$(Vx)(t) := a(t) \int_{[0,t)} d_s R(t,s)x(s) \quad \text{and} \quad f(t) := a(t) \int_{(-\infty,0)} d_s R(0,s)\varphi(s) + g(t). \tag{2.10}$$

In addition to (2.7) we consider the associated homogeneous equation ($f \equiv 0$).

$$dx(t) = (Vx)(t)dZ(t) \ \ (t \geq 0). \tag{2.11}$$

Using k-linearity of the operator V, we immediately obtain the following result.

Lemma 2.1. *Let for any $x(0) \in k^n$ there exists the only solution (up to a $\mathbf{P}$-null set) $x(t)$ of* (2.7). *Then one has the following representation ("the Cauchy representation") of the solutions*

$$x(t) = X(t)x(0) + (Kf)(t) \quad (t \geq 0), \tag{2.12}$$

where $X(t)$ ($X(0) = \bar{E}$) is an $n \times n$-matrix, the columns of which are the solutions of the linear homogeneous equation (2.11) ("the fundamental matrix"), while $K : L^n(Z) \to D^n$ is a k-linear operator ("the Cauchy operator") such that $(Kf)(0) = 0$ and Kf satisfies (2.7).

In what follows we will always consider equation (2.7) under the uniqueness assumption, i.e. existence, for any $x(0) \in k^n$, of the unique (up to a P-null set) solution $x(t)$ of this equation. In other words, according to Lemma 2.1, the representation (2.12) is silently assumed to be fulfilled in all further considerations.

3. $\mathbf{M}_\mathbf{p}^\gamma$ - STABILITY

Definition 3.1. The zero solution of the linear homogeneous equation (2.11) is called

(a) p-stable if for an arbitrary $\varepsilon > 0$ there exist $\eta = \eta(\varepsilon) > 0$ such that
$$\mathbf{E}|X(t)x(0)|^p \leq \varepsilon \ \ (t \geq 0)$$
for all $x(0) \in \mathbb{R}^n$, $|x(0)| < \eta$.

(b) Asymptotically p-stable if it is p-stable and $\lim_{t\to+\infty} \mathbf{E}|X(t)x(0)|^p = 0$ for all $x(0) \in \mathbb{R}^n$.

(c) Exponentially p-stable if there exist $\bar{c} > 0$, $\beta > 0$ such that
$$\mathbf{E}|X(t)x(0)|^p \le \bar{c}|x(0)| \exp\{-\beta t\} \quad (t \ge 0)$$
for all $x(0) \in \mathbb{R}^n$.

Similarly, we can define stability of solutions to the nonhomogeneous equation (2.7). Clearly, the representation (2.12) implies that all solutions to (2.7) are p-stable (asymptotically p-stable, exponentially p-stable) if and only if the zero solution to the homogeneous equation (2.11) is p-stable (asymptotically p-stable, exponentially p-stable). In the sequel we shall therefore say that the nonhomogeneous equation (2.7) is stable (in a proper sense) if the zero solution to the homogeneous equation (2.11) is stable in the same sense.

Theorem 3.1. **(A)** . *Equation* (2.7) *is p-stable if and only if $X(\cdot)x(0) \in M_p$ for all $x(0) \in \mathbb{R}^n$.*

(B) . *Equation* (2.7) *is asymptotically p-stable if and only if there exists a function $\gamma(t)$, for which $\gamma(t) \ge \delta > 0$ and $\lim_{t\to+\infty} \gamma(t) = +\infty$, so that $X(\cdot)x(0) \in M_p^\gamma$ for all $x(0) \in \mathbb{R}^n$.*

(C) *Equation* (2.7) *is exponentially p-stable if and only if there exists a number $\beta > 0$ such that $X(\cdot)x(0) \in M_p^\gamma$ for all $x(0) \in \mathbb{R}^n$, where $\gamma(t) = \exp\{\beta t\}$.*

Proof. The proof is based on the ideas developed in [6].

(A). Letting (2.7) be p-stable we suppose that for some $x_0 \in \mathbb{R}^n$ the solution $X(\cdot)x_0$ is not in M_p, i. e. for any $K > 0$ there exists $t(K) > 0$ such that $\mathbf{E}|X(t(K))x_0|^p > K$. Taking an arbitrary $\varepsilon > 0$ we find $\eta > 0$, which satisfies the definition of p-stability from Definition 3.1, and put $K_0 = \varepsilon|2x_0/\eta|^p$, $t_0 = t(K_0)$, $x_0' = \eta|x_0|^{-1}x_0$. Then for the solution $x(t) = X(t)x_0'$ we have $\mathbf{E}|x(t_0)|^p > \varepsilon$, although $|x(0)| < \eta$. This contradicts the assumption.

To prove the converse we first note that for each individual $x(0) \in \mathbb{R}^n$ one has
$$\sup_{t\in[0,\infty)} \mathbf{E}|X(t)x(0)|^p \le K = K(x(0)).$$
As $X(t)$ is linear and $\mathbb{R}^n$ is finite dimensional, then there is a constant K' which provides the uniform estimate: $\sup_{t\in[0,\infty)} \mathbf{E}\|X(t)\|^p \le K'$. Hence for any $\varepsilon > 0$ we may put $\eta = (\epsilon/K')^{1/p}$ so that $|x(0)| < \eta$ implies $\mathbf{E}|X(t)x(0)|^p \le |x(0)|^p\mathbf{E}\|X(t)\|^p \le \varepsilon$ for all $t \in [0,\infty)$.

(B) Assume that (2.7) is asymptotically p-stable. We have to find $\gamma(t)$ satisfying conditions listed in Part (B) of the theorem. First we find a function $\bar{\gamma}(t)$ for which a) $0 < \bar{\gamma}(t) < M$ $(t \in [0,\infty))$ and b) $\bar{\gamma}^{-1}(t)\mathbf{E}\|X(t)\|^p \to 1$ as $t \to +\infty$. This is possible due to p-stability of (2.7) and the boundedness of the function $\mathbf{E}\|X(t)\|^p$ (see Part A of the proof).

We set now $\gamma(t) = 1/\bar{\gamma}(t)$ and check directly that $X(\cdot)x(0) \in M_p^\gamma$ for all $x(0) \in \mathbb{R}^n$.

The converse to Part (B) of the theorem is evident: for any $\gamma(t)$, satisfying conditions listed in Part a) of Definition 3.1, the space M_p^γ will be a subspace of the space M_p and $\lim_{t\to+\infty} \mathbf{E}|X(t)x(0)|^p = 0$ for all $x(0) \in \mathbb{R}^n$.

(C) The exponential p-stability trivially implies that $X(\cdot)x(0) \in M_p^\gamma$ for all $x(0) \in \mathbb{R}^n$, where $\gamma(t) = \exp\{\beta t\}$. Conversely, if $X(\cdot)x(0) \in M_p^\gamma$ for all $x(0) \in \mathbb{R}^n$, where $\gamma(t) = \exp\{\beta t\}$, then
$$\sup_{t\in[0,\infty)} (\exp\{\beta t\}\mathbf{E}|X(t)x(0)|^p) \le c'$$

for some $c' = c'(x(0))$. As $X(t)$ is linear and $\mathbb{R}^n$ is finite dimensional we, as in Part (A), find a constant $\bar{c}$ such that

$$\sup_{t\in[0,\infty)} (\exp\{\beta t\}\mathbf{E}\|X(t)\|^p) \le \bar{c}.$$

Clearly, this implies the exponential p-stability of (2.7). □

This theorem says that to prove p-stability (asymptotic, exponential p-stability) of (2.7) we can check that the solutions of the homogeneous equation (2.11) belong to a certain space of stochastic processes (M_p or M_p^γ).

Minding Theorem 3.1, we introduce now a new definition of stability which is more convenient for our purposes.

Definition 3.2. *Equation* (2.7) *is called M_p^γ-stable, if for any $x(0) \in k_p^n$ we have $X(\cdot)x(0) \in M_p^\gamma$.*

Due to Theorem 3.1 we can now say that:

- M_p-stability (i. e. M_p^γ-stability with $\gamma = 1$) of equation (2.7) implies the Lyapunov p-stability of (2.7);
- M_p^γ-stability of (2.7) with γ satisfying $\gamma(t) \ge \delta > 0$ and $\lim_{t\to+\infty} \gamma(t) = +\infty$ implies the asymptotic p-stability of (2.7);
- M_p^γ-stability of (2.7) with $\gamma(t) = \exp\{\beta t\}$ (for some $\beta > 0$) implies the exponential p-stability of (2.7).

Thus, we have replaced stability analysis of (2.7) by the problem of how resolve this equation in a certain space of stochastic processes. This observation is crucial for applying the method based on the W-transform, which we are going to describe now.

As we already have mentioned any W-transform comes from an auxiliary equation, which we call *a reference equation*. That is why we assume given another equation, similar to (2.7), but "simpler". In addition, we assume the asymptotic properties of the reference equation to be known.

Let the reference equation have the form

$$dx(t) = [(Qx)(t) + g(t)]dZ(t) \quad (t \ge 0), \tag{3.1}$$

where $Q : D^n \to L^n(Z)$ is a k-linear Volterra operator, and $g \in L^n(Z)$. Also for equation (3.1) it is always assumed the existence and uniqueness assumption, i. e. for any $x(0) \in k^n$ there is the only (up to a $\mathbf{P}$-null set) solution $x(t)$ of (3.1). Then, according to Lemma 2.1, for this solution we have "the Cauchy representation" $x(t) = U(t)x(0) + (Wg)(t)$ $(t \ge 0)$, where $U(t)$ is the fundamental matrix of the associated homogeneous equation, and W is the corresponding Cauchy operator.

Let us rewrite equation (2.7) in the form

$$dx(t) = [(Qx)(t) + ((V - Q)x)(t) + f(t)]dZ(t) \quad (t \ge 0),$$

or, alternatively,

$$x(t) = U(t)x(0) + (W(V - Q)x)(t) + (Wf)(t) \quad (t \ge 0).$$

Denoting $W(V - Q) = \Theta_l$, we obtain the operator equation

$$((I - \Theta_l)x)(t) = U(t)x(0) + (Wf)(t) \quad (t \ge 0). \tag{3.2}$$

Here and in the sequel by invertibility of the operator $(I - \Theta_l) : M_p^\gamma \to M_p^\gamma$ we mean that this operator is a bijection on the space M_p^γ.

Remark 3.1. The letter "l" in Θ_l stands for "left", which indicates that the W-transform is applied to the equation from the left hand side. Equivalently, one can apply the W-transform from the right. This will lead to a similar theory and a similar operator Θ_r. This approach is not studied in this paper. However, we are planning to develop it in one of the forthcoming papers.

The following important result is proved in [9].

Theorem 3.2 ([9]). *Let the reference equation* (3.1) *be* M_p^γ*-stable and the operator* Θ_l *act in the space* M_p^γ*. Then, if the operator* $(I-\Theta_l): M_p^\gamma \to M_p^\gamma$ *is invertible, then the equation* (2.7) *is* M_p^γ*-stable.*

Let us stress that, in general, there is no direct dependence between stability of the equation in question and invertibility of the operator equation (3.2). For instance, in certain situations the operator Θ_l even does not act in the corresponding space of stochastic processes, while both equations (2.7) and (3.1) are stable in this space.

Nevertheless, if (2.7) is stable, then there always is at least one (in fact, infinitely many) stable reference equations, for which the operator Θ_l will act in the related space of stochastic processes and $I - \Theta_l$ will be invertible there. For instance, one can choose equation (3.1) be identical to (or be in the vicinity of) the initial equation (2.7).

This observation and Theorem 3.2 imply the following stability criterion based on the W-transform.

Corollary 3.1. *Equation* (2.7) *is* M_p^γ*-stable if and only if there exists a reference equation* (3.1)*, which is* M_p^γ*-stable and which gives rise to the invertible operator* $(I-\Theta_l): M_p^\gamma \to M_p^\gamma$*.*

Among the assumptions imposed on the initial equation (2.7) and the reference equation (3.1) one is more involved than the others when applying Theorem 3.2. It is invertibility of the operator $(I-\Theta_l): M_p^\gamma \to M_p^\gamma$. A reasonable method to check this requirement in practice is to estimate the norm of the operator Θ_l in the space M_p^γ.

Thus, from Theorem 3.2 we obtain the following simple result.

Corollary 3.2. *Assume that there is a* M_p^γ*-stable reference equation* (3.1)*, for which the operator* Θ_l *act in the space* M_p^γ *and* $\|\Theta_l\|_{M_p^\gamma} < 1$*. Then* (2.7) *is* M_p^γ*-stable.*

In what follows we will need a more explicit description of the W-transform (and the corresponding reference equation (3.1)), which is summarized in the assumptions below:

- **(R1)** The fundamental matrix $U(t)$ to (3.1)) satisfies $\|U(t)\| \le \bar{c}$, where $\bar{c} \in \mathbb{R}_+$.
- **(R2)** The W-transform coming from (3.1)) has the form

$$(Wg)(t) = \int_0^t C(t,s)g(s)dZ(s) \quad (t \ge 0), \tag{3.3}$$

where $C(t,s)$ is an $n \times n$-matrix defined on $G := \{(t,s) : t \in [0,\infty),\ 0 \le s \le t\}$, and satisfies

$$\|C(t,s)\| \le \bar{\bar{c}} \exp\{-\alpha \Delta v\}, \tag{3.4}$$

where $v(t) = \int_0^t \xi(\zeta) d\lambda(\zeta)$, $\Delta v = v(t) - v(s)$ for some $\alpha > 0, \bar{\bar{c}} > 0$.

Example 3.1. Let a reference equation be given by

$$dx(t) = (A(t)x(t-) + g_0(t))\, d\lambda(t) + \sum_{i=2}^{m} g_i(t) dz^i(t) \quad (t \geq 0), \tag{3.5}$$

where $A(t)$ is an $n \times n$-matrix with locally λ-integrable entries (in this case $(Qx)(t) = (A(t)x(t), \bar{0}, \dots, \bar{0})$). In this case it is straightforward that the kernel $C(t,s)$ in (3.3) is of the form $C(t,s) = U(t)U^{-1}(s)$. Note also that if we set $A(t) = -\alpha\xi\bar{E}$, then conditions (R1) and (R2) will be fulfilled.

A more involved example of a reference equation is given by

$$dx(t) = \Big(\int_{[0,t)} d_s \mathcal{R}(t,s)x(s) + g_0(t) \Big) d\lambda(t) + \sum_{i=2}^{m} g_i(t) dz^i(t) \quad (t \geq 0), \tag{3.6}$$

where the entries $r_{jk}(t,s)$ of a (non-random) $n \times n$-matrix $\mathcal{R}(t,s)$, defined on the set G from (R2), are of bounded variation in s and, in addition, $\bigvee_{s \in [0,t]} r_{jk}(t,s)$ ($t \in [0,\infty)$) are locally λ-integrable for all $1 \leq j,k \leq n$. In this case, the representation (3.3) is again valid, but there is no direct relations between $C(t,s)$ and the fundamental matrix $U(t)$. The estimate (3.4) can be obtained in special cases (see [2] and [4] for details). One particular case of this reference equations is used in this paper (see Section 5).

For more examples of reference equations (3.1), giving rise to the W-transforms of the form (3.3) and which satisfy all the additional assumptions listed above, see [7]. In the rest of this section we will be concerned with the M_p^γ-stability of (2.7) with

$$\gamma(t) = \exp\Big\{ \beta \int_0^t \xi(s) d\lambda(s) \Big\} \quad (t \geq 0), \tag{3.7}$$

where β is some positive number satisfying $\beta < \alpha$ (see (R2) for the notation). This specific weight γ comes naturally from the W-transform satisfying (R1)-(R2). We wish to use such a W-transform and the corresponding weight γ in order to prove two main results of this section (Theorems 3.3 and 3.4). The first theorem justifies the W-method in connection with M_p^γ-stability (and by this to the Lyapunov stability of (2.7) with respect to the initial value $x(0)$). The second theorem deals with the following fundamental problem which is also well-known for deterministic functional differential equations (see e. g. [2]): find conditions, under which the p-stability implies the exponential p-stability. We shall prove that it is the case if the delay function satisfies the so-called "Δ-condition" (see Definition 3.4 below). The Δ-condition is fulfilled if for instance the delays are bounded (see Lemma 3.2). Apart from the importance of these two general facts for the theory of stochastic functional differential equations, the technique we use to prove them is itself a good illustration of how the W-transform works in practice.

For further purposes we will need the following technical lemma.

Lemma 3.1. *If the reference equation* (3.1) *satisfies the assumption (R2), then W, given by* (3.3)*, is a continuous operator from $(\Lambda^n_{2p,q}(\xi))^\gamma$ to M^γ_{2p}, where $2p \leq q \leq \infty$ and γ is defined by* (3.7) *for all β $(0 < \beta < \alpha)$, the number α is the same as in* (3.4).

Proof. To prove the lemma it suffices to check that

$$\|Wg\|_{M^\gamma_{2p}} \leq \bar{c}\|g\|_{(\Lambda^n_{2p,q}(\xi))^\gamma} \quad (\bar{c} \in \mathbb{R}_+), \tag{3.8}$$

if $g \in (\Lambda^n_{2p,q}(\xi))^\gamma$. From the definition of the space M^γ_{2p},

$$\|Wg\|_{M^\gamma_{2p}} = \|\gamma Wg\|_{M_{2p}} = \|\gamma \int_0^\cdot C(\cdot,s)g(s)dZ(s)\|_{M_{2p}}.$$

We are to show that

$$lg := \|\int_0^\cdot C(\cdot,s)g(s)dZ(s)\|_{M_{2p}} \le \bar{c}\|\gamma g\|_{\Lambda^n_{2p,q}(\xi)}, \tag{3.9}$$

where $\bar{c}$ is some positive number. We have

$$\begin{aligned}
lg &\le \bar{\bar{c}}(\sup_{t\ge0}(\mathbf{E}(\int_0^t \exp\{-(\alpha-\beta)\Delta v\}|\gamma(s)g(s)a(s)|d\lambda(s))^{2p})^{1/(2p)}\\
&\quad + c_p \sup_{t\ge0}(\mathbf{E}(\int_0^t \exp\{-2(\alpha-\beta)\Delta v\}|\gamma(s)g(s)A(s)(\gamma(s)g(s))^\top|d\lambda(s))^p)^{1/(2p)})\\
&\le \bar{\bar{c}}(\sup_{t\ge0}(\int_0^t \exp\{-(\alpha-\beta)\Delta v\}dv(s))^{(2p-1)/2p}\\
&\quad \times (\int_0^t \exp\{-(\alpha-\beta)\Delta v\}(\xi(s))^{1-2p}\mathbf{E}|\gamma(s)g(s)a(s)|^{2p}d\lambda(s))^{1/(2p)}\\
&\quad + c_p \sup_{t\ge0}(\int_0^t \exp\{-2(\alpha-\beta)\Delta v\}dv(s))^{(p-1)2p}\\
&\quad \times (\int_0^t \exp\{-2(\alpha-\beta)\Delta v\}(\xi(s))^{1-p}\mathbf{E}\|\gamma(s)g(s)A(s)(\gamma(s)g(s))^\top\|^p d\lambda(s))^{1/(2p)})\\
&\le \hat{c}\Big\{\sup_{t\ge0}(\int_0^t \exp\{-(\alpha-\beta)\Delta v\}(\xi(s))^{1-2p}\mathbf{E}|\gamma(s)g(s)a(s)|^{2p}d\lambda(s))^{1/(2p)}\\
&\quad + c_p \sup_{t\ge0}(\int_0^t \exp\{-2(\alpha-\beta)\Delta v\}(\xi(s))^{1-p}\mathbf{E}\|\gamma(s)g(s)A(s)(\gamma(s)g(s))^\top\|^p\\
&\quad \times d\lambda(s))^{1/(2p)}\Big\},
\end{aligned}$$

where $\hat{c}$ is some positive number. Here we have used the inequality

$$\begin{aligned}
&\Big(\mathbf{E}\Big|\gamma\int_0^t C(t,s)g(s)dZ(s)\Big|^{2p}\Big)^{1/(2p)}\\
&\le \bar{\bar{c}}\Big(\mathbf{E}\Big(\int_0^t \exp\{-(\alpha-\beta)\Delta v\}|\gamma(s)g(s)a(s)|d\lambda(s)\Big)^{2p}\Big)^{1/(2p)}\\
&\quad + c_p\bar{\bar{c}}\Big(\mathbf{E}\Big(\int_0^t \exp\{-2(\alpha-\beta)\Delta v\}\|\gamma(s)g(s)A(s)|(\gamma(s)g(s))^\top\|d\lambda(s)\Big)^p\Big)^{1/(2p)},
\end{aligned}$$

which follows directly from the estimates (2.5) and (3.4).

To obtain further estimates we have to consider three cases separately: (1) $q > 2p$, $q \ne \infty$; (2) $q = 2p$; (3) $q = \infty$.

Let first $q > 2p$, $q \ne \infty$. Then

$$\begin{aligned}
lg &\le \hat{c}\{\sup_{t\ge0}[(\int_0^t \exp\{(-(\alpha-\beta)q/(q-2p))\Delta v\}dv(s))^{(q-2p)/2pq}\\
&\quad \times (\int_0^t ((\mathbf{E}|\gamma(s)g(s)a(s)|^{2p})^{1/(2p)}(\xi(s))^{q^{-1}-1})^q d\lambda(s))^{1/q}]
\end{aligned}$$

$$
\begin{aligned}
&+ c_p \sup_{t\geq 0}[(\int_0^t \exp\{(-2(\alpha-\beta)q/(q-2p))\Delta v\}dv(s))^{(q-2p)/2pq}\\
&\times (\int_0^t ((\mathbf{E}\|\gamma(s)g(s)A(s)(\gamma(s)g(s))^\top\|^p)^{1/(2p)}(\xi(s))^{q^{-1}-0.5})^q d\lambda(s))^{1/q}]\}\\
&\leq \bar{c}\|\gamma g\|_{\Lambda^n_{2p,q}(\xi)}.
\end{aligned}
$$

Assume now that $q=2p$. In this case we derive the estimate

$$
\begin{aligned}
lg &\leq \hat{c}\{\sup_{t\geq 0}(\int_0^t ((\mathbf{E}|\gamma(s)g(s)a(s)|^{2p})^{1/(2p)}(\xi(s))^{q^{-1}-1})^q d\lambda(s))^{1/q}\\
&\quad + c_p \sup_{t\geq 0}(\int_0^t ((\mathbf{E}\|\gamma(s)g(s)A(s)(\gamma(s)g(s))^\top\|^p)^{1/(2p)}(\xi(s))^{q^{-1}-0.5})^q d\lambda(s))^{1/q}\}\\
&\leq \bar{c}\|\gamma g\|_{\Lambda^n_{2p,q}(\xi)}.
\end{aligned}
$$

Finally, if $q=\infty$, then we have

$$
\begin{aligned}
lg &\leq \hat{c}\{\sup_{t\geq 0}(\int_0^t \exp\{-(\alpha-\beta)\Delta v\}\xi(s)[(\mathbf{E}|\gamma(s)g(s)a(s)|^{2p})^{1/(2p)}(\xi(s))^{-1}]^{2p}d\lambda(s))^{\frac{1}{2p}}\\
&\quad + c_p \sup_{t\geq 0}\int_0^t \exp\{-2(\alpha-\beta)\Delta v\}\xi(s)[(\mathbf{E}\|\gamma(s)g(s)A(s)(\gamma(s)g(s))^\top\|^p\\
&\quad \times (\xi(s))^{-0.5})^{1/(2p)}]^{2p}d\lambda(s))^{1/(2p)}\}\\
&\leq \hat{c}\{\underset{0\leq t\leq\infty}{\mathrm{v}rai\,sup}[(\mathbf{E}|\gamma(t)g(t)a(t)|^{2p})^{1/(2p)}(\xi(t))^{-1}](1/\alpha)^{1/(2p)}\\
&\quad + c_p\underset{0\leq t\leq\infty}{\mathrm{v}rai\,sup}[(\mathbf{E}\|\gamma(s)g(s)A(s)(\gamma(s)g(s))^\top\|^p)^{1/(2p)}(\xi(t))^{-0.5}](1/2\alpha)^{1/(2p)}\}\\
&\leq \bar{c}\|\gamma g\|_{\Lambda^n_{2p,q}(\xi)}.
\end{aligned}
$$

The proof of the lemma is completed. □

Corollary 3.3. *Assume that the reference equation* (3.1) *satisfies (R2). Then W given by* (3.3) *is a continuous operator from $\Lambda^n_{2p,q}(\xi)$ to M_{2p}, where $2p\leq q\leq\infty$.*

From Lemma 3.1 and Theorem 3.2 we obtain the following result.

Theorem 3.3. *Let $\gamma(t)$ be given by* (3.7) *for some β (0 < β < α, where α is taken from (3.4)). Assume that the reference equation* (3.1) *is M^γ_{2p}-stable. Assume also that the operators V and Q from* (2.7)*) and* (3.1)*, respectively, act from M^γ_{2p} to $(\Lambda^n_{2p,q}(\xi))^\gamma$. Then the estimate $\|\Theta\|_{M^\gamma_{2p}} < 1$ implies the M^γ_{2p}-stability of* (2.7)*, where $2p\leq q\leq\infty$.*

This theorem offers a formal justification of stability analysis in the case (rather general) when the W-transform is given by (3.3).

To formulate and prove the second main result of this section we need some preparations. Below m_p stands for the space M_p in the scalar case. We assume that the k-linear operator V in (2.7) satisfies $V: M_p\to\Lambda^n_{p,q}(\xi)$. We will also use the following notation related to the operator V:

- $Vx=(V_1x,\dots,V_mx)$;
- $(V^\beta x)(t) := \gamma(t)(V(x/\gamma))(t)$, where $\gamma(t)$ is defined in (3.7).

Definition 3.3. We say that a k-linear Volterra operator $\bar{V} : m_p \to \Lambda^{1+}_{p,q}(\xi)$ dominates a Volterra operator $V : M_p \to \Lambda^n_{p,q}(\xi)$, if 1) $\bar{V}$ is positive, i. e. $x \geq 0$ a. s. implies $\bar{V}x \geq 0$ a. s., and 2) $(|V_1 x|, \dots, |V_m x|) \leq \bar{V}|x|$ a. s. for any $x \in M_p$.

Definition 3.4. We say that a k-linear Volterra operator $V : M_p \to \Lambda^n_{p,q}(\xi)$ satisfies the Δ-condition, if V is dominated by some k-linear Volterra operator $\bar{V} : m_p \to \Lambda^{1+}_{p,q}(\xi)$ with the following additional assumption: there exists a number $\beta > 0$, for which the operator

$$(\bar{V}^\beta x)(t) := \gamma(t)(\bar{V}(x/\gamma))(t)$$

acts continuously from the space m_p to the space $\Lambda^{1+}_{p,q}(\xi)$.

Definition 3.5. Let X, Y be two linear spaces consisting of predictable stochastic processes on $[0, \infty)$, and $T : X \to Y$ be a k-linear Volterra operator. We say that the operator T satisfies δ**-condition** if there exist two positive numbers δ', δ'', $\delta' > \delta''$, providing the following implication for all $t \in [0, \infty)$: any $x \in X$, satisfying $x(\zeta') = 0$ for all $\zeta' \in [0, t]$ such that $\int_{\zeta'}^t \xi(s)d\lambda(s) < \delta'$, also satisfies $(Tx)(\zeta'') = 0$ for all $\zeta'' \in [0, t]$ such that $\int_{\zeta''}^t \xi(s)d\lambda(s) < \delta''$.

Lemma 3.2. *Assume that a k-linear Volterra operator $V : M_p \to \Lambda^n_{p,q}(\xi)$ is dominated by a k-linear, bounded and positive operator $\bar{V} : m_p \to \Lambda^{1+}_{p,q}(\xi)$ satisfying the δ-condition. Then the operator V satisfies the Δ-condition.*

Proof. According to our notation

$$(\bar{V}^{\beta_0} x)(t) = \Big(\bar{V}\Big(\exp\{\beta_0 \int_\cdot^t \xi(s)d\lambda(s)\}x\Big)\Big)(t).$$

The δ-condition from Definition 3.5 implies that the value $(\bar{V}y)(t)$ depends only on the values $y(\zeta'')$, where $\int_{\zeta''}^t \xi(s)d\lambda(s) < \delta''$ (here δ'' is again taken from Definition 3.5 and $\zeta'' \in [0, t]$), and for these ζ'' we have $\exp\{\beta_0 \int_{\zeta''}^t \xi(s)d\lambda(s)\} \leq \exp\{\beta_0\delta''\}$. This leads to the following estimate

$$\bar{V}^{\beta_0} x \leq \bar{V}(\exp\{\beta_0\delta''\}|x|) = \exp\{\beta\delta''\}\bar{V}|x|$$

almost everywhere. □

In examples below we use equations with a discrete delay as reference equations. The next definition describes the corresponding operators.

Definition 3.6. Given a measurable function $g : [0, \infty) \to \mathbb{R}$ such that $g(t) \leq t$ $(t \in [0, \infty))$ and a row vector $G = (G_1, G_2, \dots, G_m)$, where $G_i = G_i(t)$ are all predictable and nonnegative stochastic processes, we define the weighted shift operator GS_g by $(GS_g x)(t) = G(t)(S_g x)(t)$, where

$$(S_g x)(t) = \begin{cases} x(g(t)), & \text{if } g(t) \geq 0\,, \\ 0, & \text{if } g(t) < 0\,. \end{cases} \tag{3.10}$$

Clearly, $GS_g : D^1 \to L^1(Z)$. To check the δ-condition from Definition 3.5 for weighted shifts we will use special conditions on g. We will also need some new notation: for a given measurable function $g : [0, \infty) \to \mathbb{R}$ we will write

$$\chi_g(t) = \begin{cases} 1, & \text{if } g(t) \geq 0\,, \\ 0, & \text{if } g(t) < 0\,. \end{cases} \tag{3.11}$$

Definition 3.7. We say that a measurable function $g : [0,\infty) \to \mathbb{R}$ satisfies the δ-condition if there exists $\delta > 0$ such that $\int_{\chi_g(t)g(t)}^t \xi(s)d\lambda(s) < \delta$ for all $t \in [0,\infty)$.

Example 3.2. If a measurable function $g : [0,\infty) \to \mathbb{R}$ satisfies the δ-condition from Definition 3.7, then the weighted shift operator $GS_g : D^1 \to L^1(Z)$ satisfies the δ-condition from Definition 3.5. To see this, we notice that according to Definition 3.7 there exists $\delta > 0$ such that

$$\int_{\chi_g(t)g(t)}^t \xi(s)d\lambda(s) < \delta \quad \text{for all} \quad t \in [0,\infty).$$

Setting $\delta' = 2\delta$, $\delta'' = \delta$ and taking arbitrary $t \in [0,\infty)$ and $x \in D^1$, for which $y(\zeta') = 0$ for all $\zeta' \in [0,t]$ satisfying $\int_{\zeta'}^t \xi(s)d\lambda(s) < \delta'$, we have to check that $(GS_g x)(\zeta'') = 0$ a.s. for all $\zeta'' \in [0,t]$ such that $\int_{\zeta''}^t \xi(s)d\lambda(s) < \delta''$. This follows from the equality $(S_g x)(\zeta'') = 0$ a.s., or equivalently, from the estimate $\int_{\chi_g(\zeta'')g(\zeta'')}^t \xi(s)d\lambda(s) < \delta'$. But this is implied by

$$\int_{\chi_g(\zeta'')g(\zeta'')}^t \xi(s)d\lambda(s) = \int_{\chi_g(\zeta'')g(\zeta'')}^{\zeta''} \xi(s)d\lambda(s) + \int_{\zeta''}^t \xi(s)d\lambda(s) < \delta + \delta'' = \delta'.$$

Note also that if $\lambda = t$ (i.e. λ^* is the standard Lebesgue measure), then $\xi(t) \equiv 1$, and the δ-condition for g takes the following form: $t - g(t) \le \delta$ $(t \ge 0)$, i. e. the delay will be bounded.

The concluding result of this section explains when the usual stability of solutions implies the exponential and asymptotic stability. We present here only a general principle, postponing all further discussions and examples until the last section.

Theorem 3.4. *Let equation* (2.7) *and the reference equation* (3.1) *satisfy the following assumptions:*

- *The operators V, Q act as follows: $V, Q : M_{2p} \to \Lambda^n_{2p,q}(\xi)$, where $2p \le q < \infty$*
- *The reference equation* (3.1) *is M_{2p}-stable and satisfies condition (R2)*
- *The operator V satisfies the Δ-condition.*

If now the operator $(I - \Theta_l) : M_{2p} \to M_{2p}$ is continuously invertible, then (2.7) *is M^γ_{2p}- stable, where γ is defined in (3.7) with some $\beta > 0$.*

Remark 3.2. Note that under the assumptions of Theorem 3.4 the operator Θ_l does act in the space M_{2p} (see Corollary 3.3).

Proof of Theorem 3.4. First of all, we notice that (2.7) is M^γ_{2p}-stable if and only if the equation

$$\begin{aligned} dy(t) = \exp\{\beta \int_0^t \xi(\zeta)d\lambda(\zeta)\}\Big[\Big(V\Big(\exp\big\{-\beta\int_0^{\cdot} \xi(\zeta)d\lambda(\zeta)\big\}y\Big)\Big)(t) + f(t)\Big]dZ(t) \\ + \beta\xi(t)y(t)d\lambda(t) \quad (t \ge 0) \end{aligned} \tag{3.12}$$

is M_{2p}-stable. Hence, in order to prove the theorem it is sufficient to show the existence of a positive number β, for which (3.12) will be M_{2p}-stable. From Theorem 3.2 it follows that if the operator Θ^β_l acts in the space M_{2p}, and the operator

$(I - \Theta_l^\beta) : M_{2p} \to M_{2p}$ is invertible for some $\beta > 0$, then (3.12) will be M_{2p}-stable for this β. Here Θ_l^β is a k-linear operator defined, according to our previous notational agreements, by

$$(\Theta_l^\beta x)(t) := \gamma(t)(\Theta_l(x/\gamma))(t), \tag{3.13}$$

so that, in particular, $\Theta_l^0 = \Theta_l$.

Using the assumption of the theorem saying that the operator V satisfies the Δ-condition, we obtain a number $\beta_0 > 0$, for which the operator Θ_l^β acts continuously in the space M_{2p} for all $0 \le \beta \le \beta_0$. This fact follows from Corollary 3.3 and a simple observation that if the operator V satisfies the Δ-condition, then the operator $\bar V^\beta$ acts from the space m_p to the space $\bar\Lambda^{1+}_{2p,\infty}(\xi)$ and, moreover, it is bounded for all $0 \le \beta \le \beta_0$. If now we check that

$$\|\Theta_l^\beta - \Theta_l\|_{M_{2p}} \to 0, \tag{3.14}$$

when $\beta \to 0$, then the operator $(I - \Theta_l^\beta) : M_{2p} \to M_{2p}$ will also be invertible for some $\beta > 0$. Clearly, in this case the continuous extension of the operator $(I - \Theta_l^\beta) : M_{2p} \to M_{2p}$ to the completion of the space M_{2p} will be invertible as well. The fact that the operator $(I - \Theta_l^\beta) : M_{2p} \to M_{2p}$ will be invertible can be derived from the observation that the solution of the equation $(I - \Theta_l^\beta)x = g$ belongs to the space D^n if $g \in M_{2p}$, while the intersection of the space D^n with the completion of the space M_{2p} coincides with the space M_{2p}.

Note that the operator $(\Theta_l^\beta - \Theta_l)$ is given by

$$((\Theta_l^\beta - \Theta_l)x)(t) = \int_0^t C(t,s)(V(\gamma(t)/\gamma(.) - 1)x)(s)dZ(s) + \int_0^t C(t,s)\beta\xi(s)x(s)d\lambda(s).$$

Then

$$\begin{aligned}
&\|(\Theta_l^\beta - \Theta_l)x\|_{M_{2p}} \\
&\le \sup_{t\ge 0}\Big(\mathbf{E}\Big(\int_0^t |C(t,s)(V(\gamma(t)/\gamma(.) - 1)x)(s)a(s)|d\lambda(s)\Big)^{2p}\Big)^{1/(2p)} \\
&\quad + c_p \sup_{t\ge 0}(\mathbf{E}(\int_0^t \|C(t,s)(V(\gamma(t)/\gamma(.) - 1)x)(s)A(s) \\
&\quad \times (C(t,s)(V(\gamma(t)/\gamma(.) - 1)x)(s))^\top\|d\lambda(s))^p)^{1/(2p)} \\
&\quad + \sup_{t\ge 0}\Big(\mathbf{E}\Big(\int_0^t |C(t,s)\beta\xi(s)x(s)|d\lambda(s)\Big)^{2p}\Big)^{1/(2p)} \\
&\le \sup_{t\ge 0}\Big(\mathbf{E}\Big(\int_0^t \exp\{-\alpha\Delta v\}|(V(\gamma(t)/\gamma(.) - 1)x)(s)a(s)|d\lambda(s)\Big)^{2p}\Big)^{1/(2p)} \\
&\quad + c_p \sup_{t\ge 0}\Big(\mathbf{E}\Big(\int_0^t \exp\{-2\alpha\Delta v\}\|(V(\gamma(t)/\gamma(.) - 1)x)(s)A(s) \\
&\quad \times (V(\gamma(t)/\gamma(.) - 1)x)(s)^\top\|d\lambda(s)\Big)^p\Big)^{1/(2p)} \\
&\quad + \sup_{t\ge 0}\Big(\mathbf{E}\Big(\int_0^t \exp\{-\alpha\Delta v\}|\beta\xi(s)x(s)|d\lambda(s)\Big)^{2p}\Big)^{1/(2p)}.
\end{aligned}$$

Here we have used the inequality (2.5). The next estimation step is based on the Δ-condition for the operator V and the inequality

$$\gamma(t)/\gamma(s) - 1 \leq (\beta/\beta_0)\exp\{\beta_0\Delta v\} \quad (s \in [0,t],\ 0 \leq \beta \leq \beta_0),$$

following from the estimate

$$\begin{aligned}\beta\nu + \beta^2\nu^2/2! + \beta^3\nu^3/3! + \dots &\leq \beta/\beta_0 + \beta\nu + \beta^2\nu^2/2! + \dots \\ &\leq (\beta/\beta_0)(1 + \beta_0\nu + \beta_0^2\nu^2/2! + \dots) \quad (\nu > 0).\end{aligned}$$

Using this estimate, we obtain

$$\begin{aligned}
&\|(\Theta_l^\beta - \Theta_l)x\|_{M_{2p}} \\
&\leq \sup_{t\geq 0}(\mathbf{E}(\int_0^t \exp\{-\alpha\Delta v\}|(\bar{V}(|\gamma(t)/\gamma(.) - 1\|x|))(s)(a(s))^+|d\lambda(s))^{2p})^{1/(2p)} \\
&\quad + c_p\sup_{t\geq 0}(\mathbf{E}(\int_0^t \exp\{-2\alpha\Delta v\}\|(\bar{V}(|\gamma(t)/\gamma(.) - 1\|x|))(s) \\
&\quad \times (A(s))^+(\bar{V}(|\gamma(t)/\gamma(.) - 1\|x|))(s)^\top\|d\lambda(s))^p)^{1/(2p)} \\
&\quad + \sup_{t\geq 0}(\mathbf{E}(\int_0^t \exp\{-\alpha\Delta v\}|\beta\xi(s)x(s)|d\lambda(s))^{2p})^{1/(2p)} \\
&\leq \sup_{t\geq 0}(\mathbf{E}(\int_0^t \exp\{-\alpha\Delta v\}|(\beta/\beta_0)|(\bar{V}_{\beta_0}|x|)(s)(a(s))^+|d\lambda(s))^{2p})^{1/(2p)} \\
&\quad + c_p\sup_{t\geq 0}(\mathbf{E}(\int_0^t \exp\{-2\alpha\Delta v\}(\beta^2/\beta_0^2)\| \\
&\quad \times (\bar{V}_{\beta_0}|x|)(s)(A(s))^+(\bar{V}^{\beta_0}|x|)(s)^\top\|d\lambda(s))^p)^{1/(2p)} \\
&\quad + \sup_{t\geq 0}(\mathbf{E}(\int_0^t \exp\{-\alpha\Delta v\}|\beta\xi(s)x(s)|d\lambda(s))^{2p})^{1/(2p)} \\
&\leq \sup_{t\geq 0}(\int_0^t \exp\{-\alpha\Delta v\}dv(s))^{(2p-1)/2p}(\beta/\beta_0) \\
&\quad \times \int_0^t \exp\{-\alpha\Delta v\}(\xi(s))^{1-p}\mathbf{E}|(\bar{V}^{\beta_0}|x|)(s)(a(s))^+|^{2p}d\lambda(s))^{1/(2p)} \\
&\quad + c_p\sup_{t\geq 0}(\int_0^t \exp\{-2\alpha\Delta v\}dv(s))^{(p-1)/2p}(\beta/\beta_0) \\
&\quad \times \int_0^t \exp\{-2\alpha\Delta v\}(\xi(s))^{1-p}\mathbf{E}\|(\bar{V}^{\beta_0}|x|)(s)(A(s))^+(\bar{V}_{\beta_0}|x|)(s)^\top\|^p d\lambda(s))^{1/(2p)} \\
&\quad + \beta\sup_{t\geq 0}(\int_0^t \exp\{-\alpha\Delta v\}dv(s))^{(p-1)/2p}(\int_0^t \exp\{-\alpha\Delta v\}\xi(s)\mathbf{E}|x(s)|^{2p}d\lambda(s))^{1/(2p)} \\
&\leq (\beta/\beta_0)(1/\alpha)^{(2p-1)/2p}\sup_{t\geq 0}(\int_0^t \exp\{-\alpha\Delta v\}(\xi(s))^{1-p}\mathbf{E}|(\bar{V}^{\beta_0}|x|)(s)(a(s))^+|^{2p} \\
&\quad d\lambda(s))^{1/(2p)} \\
&\quad + (c_p\beta/\beta_0)(1/\alpha)^{(p-1)/2p}\sup_{t\geq 0}\int_0^t \exp\{-2\alpha\Delta v\}(\xi(s))^{1-p} \\
&\quad \times \mathbf{E}\|(\bar{V}^{\beta_0}|x|)(s)(A(s))^+(\bar{V}_{\beta_0}|x|)(s)^\top\|^p d\lambda(s))^{1/(2p)}
\end{aligned}$$

$$+\beta(1/\alpha)^{(2p-1)/2p}\sup_{t\geq 0}\Big(\int_0^t \exp\{-\alpha\Delta v\}\xi(s)\mathbf{E}|x(s)|^{2p}d\lambda(s)\Big)^{1/(2p)}.$$

To proceed, we have to consider three cases: (1) $q>2p, q\neq\infty$; (2) $q=2p$; (3) $q=\infty$. Treating each case separately and making use of the last estimate we obtain, as in the proof of Lemma 3.1, that

$$\|(\Theta_l^\beta-\Theta_l)x\|_{M_{2p}}\leq(\beta/\alpha)\|x\|_{M_{2p}}+\beta d\|\bar V^{\beta_0}\|_{\Lambda^{1+}_{2p,q}(\xi)}$$

for some positive number d. Hence, due to the boundedness of the operator

$$\bar V^{\beta_0}: m_{2p}\to\Lambda^{1+}_{2p,q}(\xi),$$

we get

$$\|(\Theta_l^\beta-\Theta_l)x\|_{M_{2p}}\leq(\beta/\alpha)\|x\|_{M_{2p}}+\beta\bar d\|x\|_{M_{2p}},$$

where $\bar d$ is a positive number. From this we deduce that $\|\Theta_l^\beta-\Theta_l\|_{M_{2p}}\to 0$ as $\beta\to 0$. This proves (3.14), and as it is was mentioned above, this suffices to complete the proof of the theorem. □

4. Admissible pairs of spaces and stability with respect to the initial function

Another name for admissibility of pairs of spaces is stability under constantly acting perturbations. Roughly speaking, given a pair (B_1, B_2) of spaces of stochastic processes, one calls it *admissible* for a linear stochastic differential equation if any solution of the equation lies in B_1 as soon as the right-hand side of the equation ("perturbation") lies in B_2. This terminology goes back to Massera and Schäfer [13] who studied admissibility for ordinary deterministic differential equations in Banach spaces. The main idea of this theory is to connect admissibility and Lyapunov stability (or the dichotomy of solution spaces). This approach proved to be particularly useful for deterministic *functional differential equations* [2]. Stochastic functional differential equations admissibility was studied in [7, 9], and in this paper we continue those studies.

To outline this method in brief, let us again look at Example 2.1. Suppose we want to study Lyapunov stability of the solutions of (2.8) with respect to the initial function (2.9). The usual Lyapunov-Krasovskii-Razumikhin method suggests that we rewrite (2.8) as an equation in a Banach space of all initial functions φ (usually it is the space $C[-h,0]$). A detailed description of this approach in the case of stochastic differential equation can e.g. be found in the monographs [12, 16].

Another way is presented in [2] and developed in [7, 9] for the case of stochastic delay differential equations. The idea is to rewrite (2.8) in a different manner, namely in the form (2.7) with V and f defined in (2.10), as it is described in Example 2.1. By this, the initial function φ will be included in the right-hand side of the equation, and stability of (2.8) with respect to φ will be reduced to a particular case of the general admissibility problem for the functional differential equation (2.7). This approach is flexible and efficient, especially in the case of linear equations. In its practical use, it is common to exploit the W-transform as an additional tool.

The main objective of this section is to demonstrate how this approach, in combination with the general results and techniques developed in the previous section, can be utilized to derive stability of stochastic delay differential equations with respect to the initial function φ.

We start with some more notation. Let B be a linear subspace of the space $L^n(Z)$ (defined in Section 1). The space B is assumed to be equipped with a norm $\|.\|_B$. Given a weight $\gamma(t)$ ($t \in [0,\infty)$) we set $B^\gamma = \{f : f \in B, \gamma f \in B\}$, which is a linear space with the norm $\|f\|_{B^\gamma} := \|\gamma f\|_B$.

For the sake of convenience we will also write $x_f(t, x_0)$ for the (unique) solution of (2.7). Here f is the right-hand side of (2.7) and x_0 is the initial value of the solution, i. e. $x_f(0, x_0) = x_0$.

Definition 4.1. We say that the pair (M_p^γ, B) is *admissible* for equation (2.7) if there exists $\bar{c} > 0$, for which $x_0 \in k_p^n$ and $f \in B$ imply $x_f(., x_0) \in M_p^\gamma$ and the following estimate:

$$\|x_f(\cdot, x_0)\|_{M_p^\gamma} \le \bar{c}(\|x_0\|_{k_p^n} + \|f\|_B).$$

By definition the solutions belong to M_p^γ whenever $f \in B$ and $x_0 \in k_p^n$ and depend continuously on f and x_0 in the appropriate topologies. The choice of spaces is closely related to the kind of stability we are interested in. The first two results in this section describe assumptions on the reference equation that are to be checked if one wants to exploit the W-transform to study admissibility.

Theorem 4.1. *Assume that the reference equation* (3.1) *satisfies conditions (R1)-(R2). If the operator $I - \Theta_l$ acts in the space M_p^γ and has a bounded inverse in this space, then the pair (M_p^γ, B^γ) is admissible for* (2.7).

Proof. Under the above assumptions, $U(\cdot)x_0 \in M_p^\gamma$ whenever $x_0 \in k_p^n$ and

$$x_f(t, x_0) = ((I - \Theta_l)^{-1}(U(\cdot)x_0))(t) + ((I - \Theta_l)^{-1}Wf)(t) \quad (t \ge 0)$$

for an arbitrary $x_0 \in k_p^n$, $f \in B^\gamma$. Taking the norms and using (R1)-(R2) for the reference equation, we arrive at the inequality

$$\|x_f(\cdot, x_0)\|_{M_p^\gamma} \le \bar{c}(\|x_0\|_{k_p^n} + \|f\|_{B^\gamma}),$$

which holds for any $x_0 \in k_p^n$, $f \in B^\gamma$. Here $\bar{c}$ is some positive number. This means that the pair (M_p^γ, B^γ) is admissible for (2.7). $\square$

If, in addition, we have the Δ-condition from Definition 3.4, then we can prove more.

Theorem 4.2. *Let γ be defined by (3.7), the assumptions of Theorem 3.4 be fulfilled and the reference equation* (3.1) *satisfy the condition (R1). Then the pair $(M_{2p}^\gamma, (\Lambda_{2p,q}^n(\xi))^\gamma)$ is admissible for* (2.7) *for some $\beta > 0$.*

Proof. First we note that the pair $(M_{2p}^\gamma, (\Lambda_{2p,q}^n(\xi))^\gamma)$ is admissible for (2.7) if and only if the pair $(M_{2p}, (\Lambda_{2p,q}^n(\xi))$ is admissible for the modified equation (3.12). The latter can be proved if we check that the assumptions of Theorem 4.2 imply the assumptions of Theorem 4.1 for the modified equation (3.12), where we put $\gamma = 1$, $B = \Lambda_{2p,q}^n(\xi)$ and use $2p$ instead of p (so that M_p becomes M_{2p}).

Then we check that there exists $\beta_0 > 0$ such that the operator

$$I - \Theta_l^\beta : M_{2p} \to M_{2p},$$

where Θ_l^β was defined in (3.13), has a bounded inverse for all $0 < \beta < \beta_0$. According to the proof of Theorem 3.4 we have $\|\Theta_l^\beta - \Theta_l\|_{M_{2p}} \to 0$ as $\beta \to 0$, so that the operator $I - \Theta_l^\beta$ is invertible for sufficiently small $\beta > 0$.

To see that the operator W is continuous from the space $\Lambda^n_{2p,q}(\xi)$ to the space M_{2p} we apply Corollary 3.3. Summarizing, we conclude that for some $\beta > 0$ the pair $(M^\gamma_{2p}, (\Lambda^n_{2p,q}(\xi))^\gamma)$ is admissible for (2.7). □

We are now ready to investigate Lyapunov stability with respect to the initial function φ. In the previous section we studied stability with respect to the initial value $x(0)$. The difference between these two stabilities can again be explained by virtue of Example 2.1. In the initial condition (2.9) there is no formal difference between all the "prehistory" values of the solution $x(s), s \le 0$. In fact, if we change the value of the initial function $\varphi(s)$ for one (or even countably many) $s < 0$, then the solution $x(t), t > 0$ will not be changed. If we, however, change the value $\varphi(0)$, then the solution will be different, that is the instants $s = 0$ and $s < 0$ *are* different. This observation explains roughly why it is reasonable to treat the function $\varphi(s), s < 0$ and $\varphi(0) = x(0)$ separately. That is why we rewrite the delay equation (2.8) with the initial condition (2.9) as the functional differential equation (2.7). This idea proved to be fruitful in many cases (see e. g. [2, 7] and references therein).

In our paper we exploit this approach to study Lyapunov stability wit respect to the initial function with the help of the theory of admissible pairs of spaces and the W-method. Generalizing Example 2.1, we consider a linear stochastic differential equation with distributed delay of the form

$$\begin{gathered} dx(t) = (\hat{V}x)(t)dZ(t) \quad (t \ge 0), \\ x(\nu) = \varphi(\nu) \quad (\nu < 0), \end{gathered} \tag{4.1}$$

where

$$(\hat{V}x)(t) = \Big(\int_{(-\infty,t)} d_s\mathcal{R}_1(t,s)x(s), \dots, \int_{(-\infty,t)} d_s\mathcal{R}_m(t,s)x(s)\Big),$$

$$\mathcal{R}_i(t,s) = \sum_{j=0}^{m_i} Q_{ij}(t) r_{ij}(t,s).$$

Equation (4.1) can be rewritten in the form (2.7) by putting

$$\begin{gathered} (Vx)(t) = \Big(\int_{[0,t)} d_s\mathcal{R}_1(t,s)x(s), \dots, \int_{[0,t)} d_s\mathcal{R}_m(t,s)x(s)\Big), \\ f(t) = \Big(\int_{(-\infty,0)} d_s\mathcal{R}_1(t,s)\varphi(s), \dots, \int_{(-\infty,0)} d_s\mathcal{R}_m(t,s)\varphi(s)\Big), \end{gathered} \tag{4.2}$$

where Q_{ij} are $n \times n$-matrices with the entries being predictable stochastic processes and r_{ij} are scalar functions defined on $\{(t,s) : t \in [0,\infty), \ -\infty < s \le t\}$ for $i = 1, \dots, m$; $j = 0, \dots, m_0$. Let

$$H^i_0(t) = \sum_{j=0}^{m_i} \|Q_{ij}(t)\| \bigvee_{s \in (-\infty,0)} r_{ij}(t,s),$$

$$H^i_1(t) = \sum_{j=0}^{m_i} \|Q_{ij}(t)\| \bigvee_{s \in [0,t]} r_{ij}(t,s) \quad (i = 1, \dots, m),$$

$$H_j = (H^1_j, \dots, H^m_j) \quad (j = 0, 1).$$

Equation (4.1) will be considered under the assumption

$$\int_0^t (|H_j a^+| + \|H_j A^+ H_j^\top\|) d\lambda < \infty \quad \text{a. e. for any } t \geq 0,\ j = 0, 1.$$

This implies, in particular, that $H_j \in L^n(Z)$ (compare the last inequality with (2.3)). The initial function φ will be a stochastic process such that $f \in L^n(Z)$. An example of such φ is given by a stochastic process on $(-\infty, 0)$ which is independent of the semimartingale $Z(t)$ and which has a. s. essentially bounded trajectories with respect to to the measure λ^*, generated by the function $\lambda(t)$. If these assumptions are satisfied, then the operator V in equation (2.7), defined by the first formula in (4.2), will be k-linear and Volterra and act from the space D^n to the space $L^n(Z)$. In addition, for any $x(0) \in k^n$ there will be the unique (up to a P-null set) solution of (2.7) (remember that (2.7) is equivalent to (4.1)). For the proof of these results see [7].

As a particular case of equation (4.1) we obtain stochastic differential equations with "ordinary", or concentrated delay. Another name is difference-differential stochastic equations. By this we mean the following object:

$$\begin{aligned} dx(t) &= (\tilde{V}x)(t)dZ(t) \quad (t \geq 0), \\ x(\nu) &= \varphi(\nu) \quad (\nu < 0), \end{aligned} \tag{4.3}$$

where

$$(\tilde{V}x)(t) = \Big(\sum_{j=0}^{m_1} \tilde{Q}_{1j}(t) x(h_{1j}(t)), \dots, \sum_{j=0}^{m_m} \tilde{Q}_{mj}(t) x(h_{mj}(t)) \Big). \tag{4.4}$$

Here h_{ij} are λ^*-measurable functions, for which

$$h_{ij}(t) \leq t\ (\lambda^* - \text{a. e.}) \text{ for } t \in [0, \infty),\ i = 1, \dots, m,\ j = 0, \dots, m_i;$$

$\tilde{Q}_{ij}$ are $n \times n$-matrices with the entries that are predictable stochastic processes for all $i = 1, \dots, m,\ j = 0, \dots, m_i$; φ is a stochastic process which is independent of the semimartingale $Z(t)$.

The assumptions imposed on the general delay equation (4.1) can easily be adjusted to its particular case (4.3). The details can be found in [7]. Here we just outline briefly how equation (4.3) can be represented in the form (4.1) and then formulate the assumptions on the coefficients. We set

$$\mathcal{R}_i(t, s) = \sum_{j=0}^{m_i} \tilde{Q}_{ij}(t) r_{ij}(t, s),$$

where $\tilde{Q}_{ij}$ are the matrices from (4.4) and r_{ij} is the indicator (the characteristic function) of the set

$$\{(t, s) : t \in [0, \infty),\ h_{ij}(t) \leq s \leq t\},$$

defined on $t \in [0, \infty),\ s \in (-\infty, t]$ for $i = 1, \dots, m$, $j = 0, \dots, m_i$. By this, equation (4.3) is rewritten in the form (4.1) and this leads automatically to the following assumptions on the coefficients of (4.3):

$$\int_0^t (|HA^+| + \|HA^+H^\top\|) d\lambda < \infty \quad \text{a. s. for any } t \geq 0,$$

where

$$H = (H^1, \dots, H^m), \quad H^i := \sum_{j=0}^{m_i} \|\tilde{Q}_{ij}\| \quad (i = 1, \dots, m);$$

the initial function φ is a stochastic process with trajectories which are a. s. essentially bounded on $[0,\infty)$ with respect to the measure λ^*.

In what follows we treat equation (4.3) as a special case of (4.1).

Remark 4.1. The assumptions on the initial function φ do not imply, in general, that φ should be cadlag. It is an important observation for what follows as we are going to use a weaker topology (the L^p-topology) on the set of all φ. Moreover, we do not treat the solution $x(t)$ on $t \in [0,\infty)$ as a continuation of the stochastic process φ. This is an essential feature of the theory of functional differential equations presented in [2] as it offers more possibilities to choose a suitable topology in the space of initial functions. A similar idea was also used in [14] to define the Lyapunov exponents for stochastic flows associated with certain linear stochastic functional differential equations. This was motivated by the fact that Ruelle's multiplicative ergodic theorem, which is needed to define the Lyapunov exponents, requires the topology of a Hilbert space instead of the uniform topology on the space of initial functions.

If we, nevertheless, want the solutions $x(t)$ of (4.1) (or (4.3)) to be continuations of the initial functions $\varphi(t)$, then we can easily treat this situation as a particular case of the more general setting described above. First of all we have to require that $\varphi(t)$ should be cadlag (or continuous, if the semimartingale $Z(t)$ is continuous). In addition, we set the continuity condition at $t = 0$, i. e. we demand that

$$x(0) = \lim_{\delta\to 0-} \varphi(\delta).$$

By this, the solution will be cadlag (or continuous) for all t.

Now we describe different kinds of stability of solutions of (4.1) and (4.3) which we intend to study in this paper. The definitions below are classical, up to some small adjustments, and can be found in many monographs (see e. g. [10, 14, 16]).

In the next definition we use the following notation: $x(t, x_0, \varphi)$ stands for the solution of (4.1), with the initial function φ, such that $x(0, x_0, \varphi) = x_0$.

Definition 4.2. The zero solution of (4.1) (resp. of (4.3)) is called:

- *p-stable* with respect to the initial function, if for any $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that the inequality
$$E|x_0|^p + \operatorname*{vrai\,sup}_{\nu<0} \mathbf{E}|\varphi(\nu)|^p < \eta$$
(vrai sup is the essential sup with respect to the measure λ^*) implies the estimate
$$\mathbf{E}|x(t, x_0, \varphi)|^p \le \varepsilon \quad (t \ge 0)$$
for any $\varphi(\nu)$, $\nu < 0$ and $x_0 \in k_p^n$;
- *asymptotically p-stable* with respect to the initial function, if it is p-stable with respect to the initial function and, in addition, for any $\varphi(\nu)$, $\nu < 0$ and $x_0 \in k_p^n$ such that
$$\mathbf{E}|x_0|^p + \operatorname*{vrai\,sup}_{\nu<0} \mathbf{E}|\varphi(\nu)|^p < \infty$$
one has $\lim_{t\to+\infty} \mathbf{E}|x(t, x_0, \varphi)|^p = 0$;
- *exponentially p-stable* with respect to the initial function, if there exist positive constants $\bar{c}$, β such that
$$\mathbf{E}|x(t, x_0, \varphi)|^p \le \bar{c}(\mathbf{E}|x_0|^p + \operatorname*{vrai\,sup}_{\nu<0} \mathbf{E}|\varphi(\nu)|^p) \exp\{-\beta t\} \quad (t \ge 0)$$

for any $\varphi(\nu)$, $\nu < 0$ and $x_0 \in k_p^n$.

It is easy to see that p-stability (resp. asymptotic p-stability, exponential p-stability) of the zero solution of (4.1) with respect to the initial function implies p-stability (resp. asymptotic p-stability, exponential p-stability) of the zero solution of the homogeneous equation (2.11), corresponding to (4.1), with respect to the initial value $x(0)$. The converse is, in general, not true, even in the case of deterministic delay equations (see e. g. [2]).

The notions of admissibility and stability with respect to the initial function are close to each other. In the following lemma we assume, when treating admissibility, that (4.1) is rewritten in the form (2.7).

Lemma 4.1. *Assume that for any φ such that $\operatorname{vrai}\sup_{\nu<0}\mathbf{E}|\varphi(\nu)|^p < \infty$ the stochastic process f defined in (4.2) belongs to a normed subspace B of the space $L^n(Z)$, the norm satisfying*

$$\|f\|_B \le K \operatorname*{vrai\,sup}_{\nu<0}(\mathbf{E}|\varphi(\nu)|^p)^{1/p},$$

where K is a positive constant. If the pair (M_p, B) is admissible for (2.7), corresponding to (4.1), then the zero solution of (4.1) is p-stable with respect to the initial function.

Proof. Under the assumptions of the lemma, we have

$$\begin{aligned}\|x_f(\cdot, x_0)\|_{M_p} &\le \hat{c}(\|x_0\|_{k_p^n} + \|f\|_B)\\ &\le \hat{c}(\|x_0\|_{k_p^n} + K \operatorname*{vrai\,sup}_{\nu<0}(\mathbf{E}|\varphi(\nu)|^p)^{1/p})\\ &\le \bar{c}(\|x_0\|_{k_p^n} + \operatorname*{\mathit{vrai}\,\mathit{sup}}_{\nu<0}(\mathbf{E}|\varphi(\nu)|^p)^{1/p}),\end{aligned}$$

where $\hat{c}, \bar{c}, K$ are some positive numbers. From this, using the estimate $x(t, x_0, \varphi) = x_f(t, x_0)$, we obtain

$$\sup_{t\ge0}(\mathbf{E}|x(t, x_0, \varphi)|^p)^{1/p} \le \bar{c}(\|x_0\|_{k_p^n} + \operatorname*{vrai\,sup}_{\nu<0}(\mathbf{E}|\varphi(\nu)|^p)^{1/p}).$$

This implies p-stability of the zero solution of (4.1) with respect to the initial function. $\square$

Remark 4.2. Evidently, in Lemma 4.1 one can replace the space B by the space B^γ for any reasonable weight γ. Then admissibility of the pair (M_p^γ, B^γ) for (2.7) with $\gamma(t) = \exp\{\beta t\}$, $\beta > 0$ will imply the exponential p-stability of the zero solution of (4.1) with respect to the initial function. The asymptotic p-stability of the zero solution of (4.1) with respect to the initial function can be derived from admissibility of the pair (M_p^γ, B^γ) for the corresponding equation (2.7) , if $\lim_{t\to+\infty}\gamma(t) = +\infty$ and $\gamma(t) \ge \delta > 0$, $t \in [0, \infty)$ for some δ.

Definition 4.3. We say that the semimartingale $Z(t)$ satisfies condition (Z):

- **(Z)** If $< c^i, c^j >= 0$ for $i \ne j$, so that $\lambda^* \times P$- almost everywhere $A^{ij} = 0$ ($i \ne j$, $i, j = 1, \dots, m$).

We will subsequently use only semimartingales with condition (Z). We first treat equation (4.1) including distributed delays. Wishing to use the W-transform and the related operator Θ_l we have to rewrite (4.1) in the form (2.7). It is easily done via the formulas (4.2).

We begin by listing some technical conditions:

(D1) $1 \le p < \infty$, $2p \le q \le \infty$; $\sup_{t\in[1,\infty)}(v(t)-v(t-1)) < \infty$, where $v(t) = \int_0^t \xi(s)d\lambda(s)$;
$\|Q_{ij}\||a^i| \le a^i_j, \quad \|Q_{ij}\||A^{ii}|^{0.5} \le h^i_j \quad (\lambda^* \times \mathbf{P})$-almost everywhere,
$$a^i_j \times \bigvee_{(-\infty,\cdot]} r_{ij}(\cdot,s)\xi^{q^{-1}-1} \in L^\lambda_q, \quad h^i_j \times \bigvee_{(-\infty,\cdot]} r_{ij}(\cdot,s)\xi^{q^{-1}-0.5} \in L^\lambda_q$$
$(i = 1,\dots,m,\ j = 0,\dots,m_i)$.

Theorem 4.3. *Let the semimartingale $Z(t)$ satisfy condition (Z), the reference equation* (3.1) *satisfy (R1)-(R2), and equation* (4.1) *satisfy (D1). If now the operator $(I-\Theta_l) : M_{2p} \to M_{2p}$ (constructed for* (2.7) *corresponding to* (4.1)*) has a bounded inverse, then the zero solution of* (4.1) *is* $2p$*-stable with respect to the initial function.*

Remark 4.3. Due to Corollary 3.3 the operator W, under the assumptions of Theorem 4.3, are continuous from the space $\Lambda^n_{2p,q}(\xi)$ to the space M_{2p}, while the operator Θ_l, defined in (3.2), acts in the space M_{2p}.

Proof of Theorem 4.3. We first go over to the form (2.7) of equation (4.1), where V and f are specified by the formulas (4.2).

Applying Theorem 4.1 gives us, under the assumptions of Theorem 4.3, admissibility of the pair $(M_{2p}, \Lambda^n_{2p,q}(\xi))$ for (2.7). The property of $2p$-stability of the zero solution of (4.1) with respect to the initial function follows now from Lemma 4.1 if we manage to prove the following property: For any φ such that $\operatorname{vrai\,sup}_{\nu<0} \mathbf{E}|\varphi(\nu)|^p < \infty$ the function f in equation (2.7) belongs to the normed space $B := \Lambda^n_{2p,q}(\xi)$ and the following estimate holds $\|f\|_B \le K \operatorname{vrai\,sup}_{\nu<0}(E|\varphi(\nu)|^{2p})^{1/(2p)}$, where K is a positive number.

To prove this property, we observe that
$$\begin{aligned}
\|f\|_B &= \|(E|fa|^{2p})^{1/(2p)}\xi^{q^{-1}-1}\|_{L^\lambda_q} + \|(E\|fAf^\top\|^p)^{1/(2p)}\xi^{q^{-1}-0.5}\|_{L^\lambda_q}\\
&\le \|\sum_{i=1}^m\sum_{j=0}^{m_i}\Big(\mathbf{E}\Big(\int_{(-\infty,0)} a^i_j(\cdot)|\varphi(\tau)|d_\tau \bigvee_{s\in(-\infty,\tau]} r_{ij}(\cdot,s)\Big)^{2p}\Big)^{1/(2p)}\xi^{q^{-1}-1}\|_{L^\lambda_q}\\
&\quad + \|\sum_{i=1}^m\sum_{j=0}^{m_i}\Big(\mathbf{E}\Big(\int_{(-\infty,0)} (h^i_j(\cdot)|\varphi(\tau)|)^2 d_\tau \bigvee_{s\in(-\infty,\tau]} r_{ij}(\cdot,s)\Big)^{p}\Big)^{\frac{1}{2p}}\xi^{q^{-1}-0.5}\|_{L^\lambda_q}\\
&\le \operatorname*{vrai\,sup}_{\nu<0}(\mathbf{E}|\varphi(\nu)|^{2p})^{1/(2p)}\Big(\sum_{i=1}^m\sum_{j=0}^{m_i}\|a^i_j \times \bigvee_{s\in(-\infty,0)} r_{ij}(\cdot,s)\xi^{q^{-1}-1}\|_{L^\lambda_q}\\
&\quad + \sum_{i=1}^m\sum_{i=1}^{m_i}\|h^i_j \times \bigvee_{s\in(-\infty,0)} r_{ij}(\cdot,s)\xi^{q^{-1}-0.5}\|_{L^\lambda_q}\Big)\\
&\le K \operatorname*{vrai\,sup}_{\nu<0}(\mathbf{E}|\varphi(\nu)|^{2p})^{1/(2p)},
\end{aligned}$$
where K is a positive constant, then $f \in B$ and
$$\|f\|_B \le K \operatorname*{vrai\,sup}_{\nu<0}(E|\varphi(\nu)|^{2p})^{1/(2p)}.$$

This completes the proof. □

Let us now consider discrete delays, that is equation (4.3) with the operators (4.4). The following assumption will be used.

(D2) $1 \le p < \infty$, $2p \le q \le \infty$; $\sup_{t\in[1,\infty)}(v(t) - v(t-1)) < \infty$, where $v(t) = \int_0^t \xi(s)d\lambda(s)$;

$\|\tilde{Q}_{ij}\||a^i| \le \tilde{a}^i_j, \quad \|\tilde{Q}_{ij}\||A^{ii}|^{0.5} \le \tilde{h}^i_j \quad (\lambda^* \times \mathbf{P})$-almost everywhere,

$\tilde{a}^i_j \xi^{q^{-1}-1} \in L^\lambda_q, \quad \tilde{h}^i_j \xi^{q^{-1}-0.5} \in L^\lambda_q \quad (i = 1, \dots, m, j = 0, \dots, m_i)$.

From Theorem 4.3 we have the following result.

Corollary 4.1. *Let the semimartingale $Z(t)$ satisfy condition (Z), the reference equation* (3.1) *satisfy (R1)-(R2), and* (4.3) *satisfy (D2). If now the operator $(I - \Theta_l) : M_{2p} \to M_{2p}$ (constructed for equation* (2.7) *corresponding to* (4.3)*) has a bounded inverse, then the zero solution of* (4.3) *is* $2p$*-stable with respect to the initial function.*

Definition 4.4. Equation (4.1) (Equation (4.3)) is called M^γ_p*-stable* with respect to the initial function, if for all $x_0 \in k^n_p$ and φ such that $\operatorname{vrai\,sup}_{\nu<0} E|\varphi(\nu)|^p < \infty$ one has $x(\cdot, x_0, \varphi) \in M^\gamma_p$ and

$$\|x(\cdot, x_0, \varphi)\|_{M^\gamma_p} \le \bar{c}(\|x_0\|_{k^n_p} + \operatorname*{vrai\,sup}_{\nu<0}(\mathbf{E}|\varphi(\nu)|^p)^{1/p}),$$

where $\bar{c} \in \mathbb{R}_+$.

Let us stress that, as before, the notion of M^γ_p-stability of (4.1) with respect to the initial function covers the classical notions of p-stability, exponential p-stability and asymptotical p-stability of the zero solution with respect to the initial function. It is also evident that M^γ_p-stability of (4.1) with respect to the initial function implies M^γ_p-stability of the associated equation in the form (2.7).

Theorem 4.4. *Let the semimartingale $Z(t)$ satisfy condition (Z), the reference equation* (3.1) *satisfy (R1)-(R2), and equation* (4.1) *satisfy **D1**. If now the operator $(I - \Theta_l) : M_{2p} \to M_{2p}$ (constructed for equation* (2.7) *corresponding to* (4.1)*) has a bounded inverse and there exist numbers $\delta_{ij} > 0$ such that $r_{ij}(t, s) = 0$, where $-\infty < s \le t - \delta_{ij} < \infty$, $t \in [0, \infty)$, $i = 1, \dots, m$, $j = 0, \dots, m_i$, then* (4.1) *is M^γ_{2p}-stable with respect to the initial function, where $\gamma(t) = \exp\{\beta v(t)\}$ for some $\beta > 0$.*

Proof. As in the previous theorem, we first rewrite (4.1) in the form (2.7). Then we observe that under the assumptions of the theorem, the operator V in (2.7) will act from M_{2p} to $\Lambda^n_{2p,q}(\xi)$. Due to Lemma 3.2, the operator $V : M_{2p} \to \Lambda^n_{2p,q}(\xi)$ satisfies the Δ-condition. Hence the assumptions of Theorem 4.2 are satisfied.

We proceed now as in the proof of the preceding theorem, i.e. we show that for any φ such that $\operatorname{vrai\,sup}_{\nu<0} \mathbf{E}|\varphi(\nu)|^{2p} < \infty$ the function f in equation (4.1), given by the formulas (4.2), belongs to the normed space B^γ, where $B := \Lambda^n_{2p,q}(\xi)$, and the following estimate holds

$$\|f\|_{B^\gamma} \le K \operatorname*{vrai\,sup}_{\nu<0}(\mathbf{E}|\varphi(\nu)|^{2p})^{1/(2p)},$$

K being a positive number. In this case the M^γ_{2p}-stability of (4.1) with $\gamma(t) = \exp\{\beta \int_0^t \xi(\nu)d\lambda(\nu)\}$ (for some $\beta > 0$) is implied Theorem 4.2 and Lemma 4.1.

To check the above estimate on f we observe that

$$\|f\|_{B^\gamma} = \|(\mathbf{E}|\gamma f a|^{2p})^{1/(2p)}\xi^{q^{-1}-1}\|_{L_q^\lambda} + \|(\mathbf{E}\|\gamma f A(\gamma f)^\top\|^p)^{1/(2p)}\xi^{q^{-1}-0.5}\|_{L_q^\lambda}.$$

Now we have

$$\begin{aligned}
&\|(\mathbf{E}|\gamma f a|^{2p})^{1/(2p)}\xi^{q^{-1}-1}\|_{L_q^\lambda} + \|(\mathbf{E}\|\gamma f A(\gamma f)^\top\|^p)^{1/(2p)}\xi^{q^{-1}-0.5}\|_{L_q^\lambda}\\
&\le \|\sum_{i=1}^m\sum_{j=0}^{m_i}\Big(\mathbf{E}\Big(\int_{(-\infty,0)}\gamma(\cdot)a_j^i(\cdot)|\varphi(\tau)|d_\tau \bigvee_{s\in(-\infty,\tau]} r_{ij}(\cdot,s)\Big)^{2p}\Big)^{1/(2p)}(\xi(\cdot))^{q^{-1}-1}\|_{L_q^\lambda}\\
&\quad + \|\sum_{i=1}^m\sum_{j=0}^{m_i}\Big(\mathbf{E}\Big(\int_{(-\infty,0)}(\gamma(\cdot)h_j^i(\cdot)|\varphi(\tau)|)^2 d_\tau \bigvee_{s\in(-\infty,\tau]} r_{ij}(\cdot,s)\Big)^{p}\Big)^{1/(2p)}\\
&\quad \times (\xi(\cdot))^{q^{-1}-0.5}\|_{L_q^\lambda}\\
&\le \Big[\sum_{i=1}^m\sum_{j=0}^{m_i}\exp\{\beta\int_0^{\delta_{ij}}\xi(\nu)d\lambda(\nu)\}\|a_j^i(\cdot)\times \bigvee_{s\in(-\infty,0)} r_{ij}(\cdot,s)(\xi(\cdot))^{q^{-1}-1}\|_{L_q^\lambda}\\
&\quad + \sum_{i=1}^m\sum_{j=0}^{m_i}\exp\{\beta\int_0^{\delta_{ij}}\xi(\nu)d\lambda(\nu)\}\|h_j^i(\cdot)\times \bigvee_{s\in(-\infty,0)} r_{ij}(\cdot,s)(\xi(\cdot))^{q^{-1}-0.5}\|_{L_q^\lambda}\Big]\\
&\quad \times \operatorname*{vrai\,sup}_{\nu<0}(\mathbf{E}|\varphi(\nu)|^{2p})^{1/(2p)}\\
&\le K \operatorname*{vrai\,sup}_{\nu<0}(\mathbf{E}|\varphi(\nu)|^{2p})^{1/(2p)},
\end{aligned}$$

where K is some positive number. This gives $f \in B^\gamma$ and

$$\|f\|_{B^\gamma} \le K \operatorname*{vrai\,sup}_{\nu<0}(\mathbf{E}|\varphi(\nu)|^{2p})^{1/(2p)}.$$

The theorem is proved. □

From Theorem 4.4 for (4.3) we obtain the following result.

Corollary 4.2. *Let the semimartingale $Z(t)$ satisfy condition (Z), the reference equation* (3.1) *satisfy (R1)-(R2), and equation* (4.3) *satisfy (D2). Assume that the operator $(I-\Theta_l): M_{2p}\to M_{2p}$ (constructed for* (2.7) *corresponding to* (4.3)*) has a bounded inverse, and there exist numbers $\bar\delta_{ij}>0$ such that*

$$\int_{\chi_{h_{ij}}(t)h_{ij}(t)}^t \xi(\nu)d\nu \le \bar\delta_{ij} \quad (t\in[0,\infty)),$$

where $i=1,\dots,m$, $j=0,\dots,m_i$, and $\chi_g(t)$ was defined in (3.11). Then (4.3) *is M_{2p}^γ-stable with respect to the initial function $\gamma(t)=\exp\{\beta\int_0^t\xi(\nu)d\lambda(\nu)\}$ for some* $\beta>0$.

Proof. To apply Theorem 4.4 we notice that under the assumptions, listed in Corollary 4.2, equation (4.3) in the form (4.1) has the following properties: $r_{ij}(t,s)=0$ if $-\infty<s\le t-\delta_{ij}<\infty$ $(t\in[0,\infty))$, where $\delta_{ij}=\inf\{t\in[0,\infty): \int_0^t\xi(\nu)d\nu>\bar\delta_{ij}\}$ $(i=1,\dots,m,\ j=0,\dots,m_i)$. □

For the sake of completeness we also observe that the estimates on $h_{ij}(t)$ in the corollary imply the δ-condition on $h_{ij}(t)$ (see Definition 3.7).

5. Some sufficient conditions for stability of stochastic delay equations

In this section we use the developed theory to derive certain stability results for specific classes of equations (4.1) and (4.3). We stress, however, that all the examples below are of an illustrative character. That is why we do not formally compare them with the known stability criteria (e.g. those presented in [10] and [12] as well as in other papers not listed in the bibliography). The aim of this paper is to describe and illustrate an alternative method of studying stability. A more careful analysis of specific classes is therefore left to forthcoming papers. Here we only mention that our approach normally covers more general classes of linear stochastic functional differential equations than the Lyapunov-Krasovskii-Razumikhin method does in this case. Moreover, our method treats different kinds of stability in an unified framework. Finally, the W-approach seems to give stability criteria which are *different*, i.e. not exactly comparable, with those which can be obtained with the help of other techniques. This observation suggests that the W-method should be one of the additional instruments in “the stability analysis toolbox”.

To study equation (4.1), we intend to use a special reference equation of the form (3.1) with

$$(Qx)(t) = \big(-\int_{[0,t)} d_s\mathcal{R}(t,s)x(s)d\lambda(s), \bar{0}, \dots, \bar{0}\big), \quad \mathcal{R}(t,s) = \sum_{j=0}^{l} Q_{1j}(t)r_{1j}(t,s). \tag{5.1}$$

Note that we use here the same Q_{1j}, r_{1j} $(j = 0, \dots, l)$as in (4.1). As we also want the assumptions (R1)-(R2) to be fulfilled, we require that the $n \times n$-matrices Q_{1j} should be non-random for $j = 1, \dots, l$ (the matrices Q_{1j} $(j > l)$ are still allowed to be random).

Let us now introduce to important constants which are used in what follows. Assuming that the formula (3.3) from condition (R2) in Section 2 is valid we put

$$C_1 = \sup_{t\ge 0} \int_0^t \xi(s)\|C(t,s)\|d\lambda(s), \quad C_2 = \sup_{t\ge 0} \int_0^t \xi(s)\|C(t,s)\|^2 d\lambda(s). \tag{5.2}$$

We will also use the following notation: if M is an $n \times n$-matrix function, then we write $|||M|||_{L_q^\lambda} := \| \, \|M\| \, \|_{L_q^\lambda}$. We proceed with describing the main assumptions on the semimartingale $Z(t)$.

Definition 5.1. For a semimartingale $Z(t)$ we difne the conndition

(Z0) The condition (Z) from Definition 4.3 holds and, $a^1 = 1$, $A^{11} = 0$, $a^i = 0$ $(i = 2, \dots, m)$ $\lambda^* \times P$-almost everywhere (see (2.2)).

We note that, in fact, we can always deduce condition (Z0) from condition (Z) by increasing the number of the components of the semimartingale $Z(t)$ (and adjusting the operator $\hat{V}$ appropriately). This means that (Z) and (Z0) are equivalent. But in this section we choose to use (Z0) as it simplifies our calculations. A typical example we have in mind is given by the semimartingale coming from Itô equations.

In what follows we will also need some hypotheses on the coefficients of (4.1).

(D3) $1 \le p < \infty$, $2p \le q \le \infty$; $\sup_{t\in[1,\infty)} (v(t) - v(t-1)) < \infty$, where $v(t) = \int_0^t \xi(s) d\lambda(s)$; Q_{1j} $(j = 0, \dots, l)$ are non-random;

$$\|Q_{1j}\| \le a_j^1 \quad (\lambda^* \times \mathbf{P})\text{-almost everywhere},$$
$$a_j^1 \times \bigvee_{s\in(-\infty,0)} r_{1j}(\cdot, s)\xi^{q^{-1}-1} \in L_q^\lambda,$$
$$\hat{a}_j^1(\cdot) := a_j^1(\cdot) \times \bigvee_{s\in[0,\cdot]} r_{1j}(\cdot, s)\xi^{q^{-1}-1} \in L_q^\lambda, \quad (j = 0, \dots, m_1),$$
$$\|Q_{ij}\||A^{ii}|^{0.5} \le h_j^i \quad (\lambda^* \times \mathbf{P})\text{-almost everywhere},$$
$$h_j^i \times \bigvee_{s\in(-\infty,0)} r_{ij}(\cdot, s)\xi^{q^{-1}-0.5} \in L_q^\lambda,$$
$$\hat{h}_j^i(\cdot) = h_j^i(\cdot) \times \bigvee_{s\in[0,\cdot]} r_{ij}(\cdot, s)\xi^{q^{-1}-0.5} \in L_q^\lambda \quad (i = 2, \dots, m, j = 0, \dots, m_i).$$

We remark that if we replace condition (Z) by condition (Z0), then condition (D1) becomes condition (D3).

Our first theorem in this section is of general character and will in the sequel be used to more specific studies.

Theorem 5.1. *Let the semimartingale $Z(t)$ satisfy condition (Z0), equation* (4.1) *satisfy condition (D3), the reference equation* (3.1)*, where Q is given by* (5.1)*, satisfy (R1)-(R2). Assume also that for some l ($0 \le l \le m_1$) the following estimate holds:*

$$\rho := C_1^{1-q^{-1}} \sum_{j=l+1}^{m_1} \|\hat{a}_j^1\|_{L_q} + c_p C_2^{0.5-q^{-1}} \sum_{i=2}^{m} \sum_{j=0}^{m_i} \|\hat{h}_j^i\|_{L_q} < 1,$$

where C_1, C_2 are given by (5.2). Then (4.1) *is M_{2p}-stable with respect to the initial function. If, in addition, there exist positive numbers δ_{ij}, $i= 1, \dots, m$, $j = 0, \dots, m_i$ such that $r_{ij}(t, s) = 0$, where $-\infty < s \le t - \delta_{ij} < \infty$, $t \in [0, \infty)$, $i = 1, \dots, m$, $j = 0, \dots, m_i$, then* (4.1) *will be M_{2p}^γ-stable with respect to the initial function, where $\gamma(t) = \exp\{\beta v(t)\}$ for some $\beta > 0$.*

Proof. The proof of the first statement in the theorem is based on Theorem 4.3, while the second statement exploits Theorem 4.4.

According to the assumptions of the theorem the operator Θ_l for (4.1) in the form (2.7) acts in the space M_{2p}. Now, if we manage to show that the operator $(I - \Theta_l) : M_{2p} \to M_{2p}$ has a bounded inverse, then applying Theorems 4.3 and 4.4 will prove Theorem 5.1.

To prove the invertibility of the operator $(I - \Theta_l)$ we check that, under the assumptions of the theorem, the norm of the operator Θ_l in the space M_{2p} is less than 1. In this case the only continuous extension of the operator $(I - \Theta_l) : M_{2p} \to M_{2p}$ to the completion of the space M_{2p} in its own norm will be invertible. To see this, we observe that the equation $(I - \Theta_l)x = g$ will have the unique solution in the space D^n for all $g \in M_{2p}$, while the intersection of the completion of the space M_{2p} with the space D^n coincides with the space M_{2p} by definition. This will imply the existence of a bounded inverse of the operator $(I - \Theta_l) : M_{2p} \to M_{2p}$.

In the rest of the proof we estimate the norm of the operator Θ_l, which is given by

$$(\Theta_l x)(t) = \int_0^t C(t,s)[(Vx)(s)dZ(s) - \int_{[0,s)} d_\tau \mathcal{R}(s,\tau)x(\tau)d\lambda(s)],$$

in the space M_{2p}. We have

$$\begin{aligned}
&\|\Theta_l x\|_{M_{2p}} \\
&\le \Big(\sup_{t\ge 0}\mathbf{E}\Big|\int_0^t C(t,s)\Big(\sum_{j=l+1}^{m_1} Q_{1j}(s)\int_{[0,s)} d_\nu r_{1j}(s,\nu)x(\nu)\Big)d\lambda(s)\Big|^{2p}\Big)^{1/(2p)} \\
&\quad + c_p\Big(\sup_{t\ge 0}\mathbf{E}\Big(\int_0^t \|C(t,s)\|^2 \sum_{i=2}^{m} |A^{ii}(s)|\Big|\int_{[0,s)} d_\nu \mathcal{R}_i(s,\nu)x(\nu)\Big|^2 d\lambda(s)\Big)^p\Big)^{1/(2p)} \\
&\le C_1^{(2p-1)/2p}(\|x\|_{M_{2p}}\sum_{j=l+1}^{m_1}(\sup_{t\ge 0}\mathbf{E}\int_0^t \|C(t,s)\| \\
&\times (\xi(s))^{1-2p/q}(\hat{a}_j^1(s))^{2p}d\lambda(s))^{1/(2p)}) + c_pC_2^{(p-1)/2p}\|x\|_{M_{2p}} \\
&\times \sum_{i=2}^{m}\sum_{j=0}^{m_i}\Big(\sup_{t\ge 0}\mathbf{E}\int_0^t \|C(t,s)\|(\xi(s))^{1-2p/q}(\hat{h}_j^i(s))^{2p}d\lambda(s)\Big)^{1/(2p)}.
\end{aligned}$$

Now we, as in the proof of Lemma 3.1, should consider three cases separately. We omit the corresponding calculations here as they are identical with those in Lemma 3.1. Accepting this we then obtain, using the above estimate on $\|\Theta_l x\|_{M_{2p}}$, that $\|\Theta_l x\|_{M_{2p}} \le \rho\|x\|_{M_{2p}}$. Since $\rho < 1$, we conclude that $\|\Theta_l\|_{M_{2p}} < 1$, and the theorem is proved. □

Corollary 5.1. *Let the semimartingale $Z(t)$ satisfy condition (Z0) and reference equation* (3.1)*, where Q is given by* (5.1)*, satisfy (R1)-(R2). Equation* (4.1) *is supposed to have the following property:*

- *The functions ξ^{-1}, A^{ii}, the entries of the matrix Q_{ij} and the variation $\bigvee_{s\in[0,\cdot]} r_{ij}(\cdot,s)$ are all from the space L_∞^λ for $i = 1,\dots,m$, $j = 0,\dots,m_i$.*

Also assume that for some l $(0 \le l \le m_1)$ one has the estimate

$$\begin{aligned}
&\sum_{j=l+1}^{m_1} \||Q_{1j} \times \bigvee_{s\in[0,\cdot]} r_{1j}(\cdot,s)\xi^{-1}\||_{L_\infty^\lambda} \\
&+ c_p(\sqrt{C_2}/C_1)\sum_{i=2}^{m}\sum_{j=0}^{m_i} \||Q_{ij}(A^{ii})^{0.5} \times \bigvee_{s\in[0,\cdot]} r_{ij}(\cdot,s)\xi^{-0.5}\||_{L_\infty^\lambda} < 1/C_1,
\end{aligned}$$

where C_1, C_2 are given by (5.2). Then (4.1) *is M_{2p}-stable with respect to the initial function. If, in addition, there exist positive numbers δ_{ij}, $i = 1,\dots,m$, $j = 0,\dots,m_i$ such that $r_{ij}(t,s) = 0$, where $-\infty < s \le t - \delta_{ij} < \infty$, $t \in [0,\infty)$, $i = 1,\dots,m$, $j = 0,\dots,m_i$, then* (4.1) *will be M_{2p}^γ-stable with respect to the initial function, where $\gamma(t) = \exp\{\beta v(t)\}$ for some $\beta > 0$.*

We remark that the property assumed in Corollary 5.1, which describes the assumptions on (4.1), implies the property (D3). The next proposition is a particular case of Corollary 5.1 if we put $\xi(t) \equiv 1$ $(t \in [0,\infty)$.

Corollary 5.2. *Assume that the semimartingale $Z(t)$ satisfies condition (Z0) and reference equation* (3.1)*, where Q is given by* (5.1) *with $l = 0$, satisfies (R1)-(R2). Equation* (4.1) *has the following property:*

- $z^i = 0$ *a. s.* $(i = 3, \dots, m)$, $m_1 = 0$, $m_2 = 0$, *the entries of the matrices* Q_{i0} *and* $\bigvee_{s\in[0,\cdot]} r_{i0}(\cdot, s)$ *belong to the space* L^λ_∞ $(i = 1, 2)$.

Also assume that one has the estimate

$$c_p(\sqrt{C_2}/C_1)\||d_2\||_{L^\lambda_\infty} < 1/C_1,$$

where C_1, C_2 *are given by (5.2) and* $d_2 = Q_{20} \times \bigvee_{s\in[0,\cdot]} r_{20}(\cdot, s)|A^{22}|^{0.5}$. *Then* (4.1) *is M_{2p}-stable with respect to the initial function. If, in addition, there exist positive numbers δ_i $(i = 1, 2)$ such that $r_{i0}(t, s) = 0$, where $-\infty < s \le t - \delta_i < \infty$, $t \in [0,\infty)$, $i = 1, 2$, then* (4.1) *will be M^γ_{2p}-stable with respect to the initial function, where $\gamma(t) = \exp\{\beta(\lambda(t) - \lambda(0))\}$ for some $\beta > 0$.*

In the rest of the paper we are concerned with the stability analysis of equation (4.3) with discrete time delays. As this equation is a particular case of equation (4.1), we can use Theorem 5.1 to obtain sufficient conditions of M^γ_{2p}-stability of (4.3) with respect to the initial function.

As we wish the reference equation to be a part of the studied equation, we define Q in (3.1) as follows:

$$(Qx)(t) = (\sum_{j=0}^{l} \tilde{\tilde{Q}}_{1j}(t)(S_{h_{1j}}x)(t), \bar{0}, \dots, \bar{0}), \quad \tilde{\tilde{Q}}_{ij}(t) = \tilde{Q}_{ij}(t)\chi_{h_{ij}}(t). \tag{5.3}$$

Shift operators of the form S_g are described in (3.10), while $\chi_g(t)$ is defined by (3.11). In (5.3) we again require that the matrices $\tilde{\tilde{Q}}_{1j}$ $(j = 1, \dots, l)$ should be non-random, while the matrices $\tilde{\tilde{Q}}_{1j}$ $(j > l)$ can be random). This is to ensure the assumptions (R1)-(R2).

The assumptions on the coefficients of (4.3) are summarized in the following condition

(D4) $1 \le p < \infty$, $2p \le q \le \infty$; $\sup_{t\in[1,\infty)}(v(t) - v(t-1)) < \infty$, where $v(t) = \int_0^t \xi(s)d\lambda(s)$;

$$\tilde{\tilde{Q}}_{ij}(t) := \tilde{Q}_{ij}(t)\chi_{h_{ij}}(t) \quad (i = 1, \dots, m, j = 0, \dots, m_i),$$

where χ_g is given by (3.11), $\tilde{\tilde{Q}}_{1j}(t)$ are non-random for $j = 0, \dots, l$;

$$\|\tilde{\tilde{Q}}_{1j}\| \le \tilde{a}^1_j \quad (\lambda^* \times \mathbf{P})\text{-almost everywhere},$$

$$\hat{a}^1_j = \tilde{a}^1_j \xi^{q^{-1}-1} \in L^\lambda_q \text{ for } j = 0, \dots, m_1,$$

$$\|\tilde{\tilde{Q}}_{ij}\|\,|A^{ii}|^{0.5} \le \tilde{h}^i_j \quad (\lambda^* \times P)\text{-almost everywhere},$$

$$\hat{\tilde{h}}^i_j = \tilde{h}^i_j \xi^{q^{-1}-0.5} \in L^\lambda_q \text{ for } i = 2, \dots, m,\ j = 0, \dots, m_i.$$

Clearly, if condition (Z) is replaced by condition (Z0), then condition (D2) becomes condition (D4).

From Theorem 5.1 we now deduce the following corollary.

Corollary 5.3. *Let the semimartingale $Z(t)$ satisfy condition (Z0), equation* (4.3) *satisfy condition (D4), the reference equation* (3.1)*, where Q is given by* (5.3)*, satisfy (R1)-(R2). Assume also that for some l ($0 \le l \le m_1$) the following estimate holds:*

$$C_1^{1-q^{-1}} \sum_{j=l+1}^{m_1} \|\hat{\hat{a}}_j^1\|_{L_q^\lambda} + c_p C_2^{0.5-q^{-1}} \sum_{i=2}^{m} \sum_{j=0}^{m_i} \|\hat{\hat{h}}_j^i\|_{L_q^\lambda} < 1,$$

where C_1, C_2 are given by (5.2)*. Then* (4.3) *is M_{2p}-stable with respect to the initial function. If, in addition, there exist positive numbers δ_{ij}, $i = 1, \dots, m$, $j = 0, \dots, m_i$ such that $r_{ij}(t, s) = 0$, where $-\infty < s \le t - \delta_{ij} < \infty$, $t \in [0, \infty)$, $i = 1, \dots, m$, $j = 0, \dots, m_i$, then* (4.3) *will be M_{2p}^γ-stable with respect to the initial function, where $\gamma(t) = \exp\{\beta v(t)\}$ for some $\beta > 0$.*

As an important particular case of Corollary 5.3 we obtain:

Corollary 5.4. *Assume that the semimartingale $Z(t)$ satisfies condition (Z0) and the reference equation* (3.1)*, where Q is given by (5.3), satisfies (R1)-(R2). Then* (4.3) *has the following property:*

- *The entries of the matrices $\tilde{\tilde{Q}}_{ij}$ and the functions A^{ii}, ξ^{-1} belong to the space L_∞^λ for $i = 1, \dots, m$, $j= 0, \dots, m_i$.*

Assume also that for some l ($0 \le l \le m_1$) the following estimate holds:

$$\sum_{j=l+1}^{m_1} \||\tilde{\tilde{Q}}_{1j}\xi^{-1}|\|_{L_\infty^\lambda} + c_p(\sqrt{C_2}/C_1) \sum_{i=2}^{m} \sum_{j=0}^{m_i} \||\tilde{\tilde{Q}}_{ij}(A^{ii})^{0.5}\xi^{-0.5}|\|_{L_\infty^\lambda} < 1/C_1,$$

where C_1, C_2 are given by (5.2). Then equation (4.3) *is M_{2p}-stable with respect to the initial function. If, in addition, there exist positive numbers δ_{ij}, $i= 1, \dots, m$, $j = 0, \dots, m_i$ such that $r_{ij}(t, s) = 0$, where $-\infty < s \le t - \delta_{ij} < \infty$, $t \in [0, \infty)$, $i = 1, \dots, m$, $j = 0, \dots, m_i$, then* (4.3) *will be M_{2p}^γ-stable with respect to the initial function, where $\gamma(t) = \exp\{\beta v(t)\}$ for some $\beta > 0$.*

Remark 5.1. It is convenient to apply Theorem 5.1 and Corollaries 5.1 - 5.4 if it is known that a part of the drift operator gives rise to a stable deterministic equation. In this case we use this deterministic equation as a reference equation. In order to achieve best possible stability results we have to find the constants C_1 and C_2 from (5.2), or at least good estimates on these constants. The exact values are only known in exceptional cases (like for diagonal ordinary differential systems). But good estimates on C_1 and C_2 can easily be found if the constants α and $\bar{\bar{c}}$ in (3.4) are known or estimated.

In what follows we will restrict ourselves to the case of Itô delay equations. In this case the semimartingale $Z(t)$ has the form.

- **(B)** $Z(t) = (t, B^1(t), \dots, B^{m-1}(t))^T$, where $B^i, i = 1, \dots, m-1$ are independent standard Wiener processes.

When this condition is satisfied, $\lambda(t) = t$ and the associated measure λ^* becomes the Lebesgue measure which we denote by μ.

Remark 5.2. It is easy to see that the semimartingale, described in (B), satisfies condition (Z0) as $a = (1, 0, \dots, 0)^T$, and the $m \times m$-matrix A is given by $A^{ii} = 1$ if $i = 2, \dots, m$, and $A^{ij} = 0$ otherwise, i.e. if $i = j = 1$ or $i \ne j$, $i, j = 1, \dots, m$

(see (2.4)). In addition, we have that $L^n(Z)$ is a linear space consisting of $n \times m$-matrices, where all the entries are stochastic processes on $[0,\infty)$ that are adapted with respect to the given filtration, and the first column in the matrix are a.s. locally (Lebesgue) integrable, while the other columns are a.s. locally squire (Lebesgue) integrable. The space D^n consists now of adapted stochastic processes on $[0,\infty)$ with a.s. continuous trajectories.

We will also use an adjusted reference equation (3.1) with

$$\begin{gathered} Qx)(t) = (-\kappa(S_h x)(t), \bar{0}, \dots, \bar{0}), \quad \kappa \text{ is an } n \times n\text{-matrix;} \\ h(t) \text{ is a } \mu\text{-measurable function such that } h(t) \le t \ (t \in [0,\infty)). \end{gathered} \tag{5.4}$$

Recall that here μ is the Lebesgue measure, and the operator S_h is given by (3.10).

As before, we introduce a new condition to summarize assumptions on the coefficients.

(D5) $1 \le p < \infty$, $2p \le q \le \infty$; $\sup_{t\in[1,\infty)}(v(t) - v(t-1)) < \infty$, where $v(t) = \int_0^t \xi(s)ds$;

$$\tilde{\tilde{Q}}_{ij}(t) := \tilde{Q}_{ij}(t)\chi_{h_{ij}}(t) \quad (i = 1,\dots,m, j = 0,\dots,m_i),$$

where χ_g is given by (3.11), $\tilde{\tilde{Q}}_{1j}(t)$ $(j = 0,\dots,l)$ are non-random,

$$\|\tilde{\tilde{Q}}_{1j}\| \le \tilde{a}_j^1 \quad (\mu \times \mathbf{P})\text{-almost everywhere}$$

$$\hat{a}_j^1 = \tilde{a}_j^1 \xi^{q^{-1}-1} \in L_q \text{ for } j = 0,\dots,m_1,$$

$$\|\tilde{\tilde{Q}}_{ij}\| \le \tilde{h}_j^i \quad (\mu \times P)\text{-almost everywhere,}$$

$$\hat{\hat{h}}_j^i = \tilde{h}_j^i \xi^{q^{-1}-0.5} \in L_q \text{ for } i = 2,\dots,m, \quad j = 0,\dots,m_i.$$

Note that if the semimartingale $Z(t)$ satisfies condition (B), then (D4) becomes (D5).

Recall that we use the following notation (adjusted for the case $\lambda(t) = t$, $t \in [0,\infty)$): if M is an $n \times n$-matrix function, then we write $|||M|||_{L_q} := \| \, \|M\| \, \|_{L_q}$.

Theorem 5.2. *Let the semimartingale $Z(t)$ satisfy condition (B), equation* (4.3) *satisfy condition (D5), reference equation* (3.1)*, where Q is given by* (5.4)*, satisfy (R1)-(R2). Assume also that there exist a natural number l $(0 \le l \le m_1)$ and positive constants ϑ_j $(j = 0,\dots,l)$ such that*

$$\int_{[\chi_h(t)h(t),\ \chi_{h_{1j}}(t)h_{1j}(t)]} \xi(s)ds \le \vartheta_j \quad (j = 0,\dots,l).$$

Finally, the following estimate is supposed to hold:

$$\begin{aligned} \rho := \ & C_1^{1-q^{-1}}\{|||\Big(\sum_{j=0}^{l} \tilde{\tilde{Q}}_{1j} + \xi\kappa\Big)\xi^{q^{-1}-1}|||_{L_q} + \sum_{j=0}^{l} \|\hat{a}_j^1\|_{L_q}(\vartheta_j^{1-q^{-1}} \sum_{j=0}^{m_1} \|\hat{a}_j^1\|_{L_q} \\ & + c_p \vartheta_j^{0.5-q^{-1}} \sum_{i=2}^{m}\sum_{j=0}^{m_i} \|\hat{\hat{h}}_j^i\|_{L_q}) \\ & + \sum_{j=l+1}^{m_1} \|\hat{a}_j^1\|_{L_q}\} + c_p C_2^{0.5-q^{-1}} \sum_{i=2}^{m}\sum_{j=0}^{m_i} \|\hat{\hat{h}}_j^i\|_{L_q} < 1, \end{aligned}$$

where C_1, C_2 are given by (5.2). Then (4.3) *is M_{2p}-stable with respect to the initial function. If, in addition, there exist positive numbers δ_{ij}, $i= 1,\dots,m$, $j = 0,\dots,m_i$ such that the functions $h_{ij}(t)$ satisfy the δ_{ij}-condition for $i = 1,\dots,m; j = 0,\dots,m_i$ (see Definition 3.7), then* (4.3) *will be M_{2p}^{γ}-stable with respect to the initial function, where $\gamma(t) = \exp\{\beta v(t))\}$ for some $\beta > 0$.*

Proof. As in Theorem 5.1, to prove the first part we use Corollary 4.1, while to prove the second part we apply Corollary 4.2. Evidently, that under the assumptions of Theorem 5.2 the operator V for (4.3) in the form (2.7) acts from M_{2p} to $\Lambda^n_{2p,q}(\xi)$. By Corollary 3.3 this implies that $\Theta_l : M_{2p} \to M_{2p}$.

Due to Corollaries 4.1 and 4.2 it suffices to prove that the operator $I - \Theta_l$ has a bounded inverse in the space M_{2p}. We show that the norm of the operator Θ_l in the space M_{2p} is less than 1. As in the proof of Theorem 5.1, we then observe that the only continuous extension of the operator $(I - \Theta_l) : M_{2p} \to M_{2p}$ to the completion of the space M_{2p} in its own norm is invertible. Indeed, the equation $(I - \Theta_l)x = g$ has the unique solution in the space D^n for all $g \in M_{2p}$, while the intersection of the completion of the space M_{2p} with the space D^n coincides with the space M_{2p} by definition. This implies the existence of a bounded inverse of the operator $(I - \Theta_l) : M_{2p} \to M_{2p}$.

In our case, the operator Θ_l is given by

$$(\Theta_l x)(t) = \int_0^t C(t,s)((Vx)(s)dZ(s) - \kappa\xi(s)(S_hx)(s)ds) \quad (t \ge 0).$$

For short notation, let us write $\sigma(t) = \chi_h(t)h(t)$, $\sigma_j(t) = \chi_{h_{1j}}(t)h_{1j}(t)$ $(j = 0,\dots,l)$. Then estimating the norm of the operator Θ_l in the space M_{2p} gives

$$\begin{aligned}
&\|\Theta_l x\|_{M_{2p}}\\
&\le (\sup_{t\ge0}\mathbf{E}|\int_0^t C(t,s)[\sum_{j=0}^{l}(\tilde{\tilde{Q}}_{1j}(s) + \xi(s)\kappa)(S_hx)(s)\\
&\quad + \sum_{j=0}^{l}\tilde{\tilde{Q}}_{1j}(s)\int_{\sigma_j(s)}^{\sigma(s)} dx(\tau) + \sum_{j=l+1}^{m_1}\tilde{\tilde{Q}}_{1j}(s)(S_{h_{1j}}x)(s)]ds|^{2p})^{1/(2p)}\\
&\quad + c_p(\sup_{t\ge0}\mathbf{E}(\int_0^t \|C(t,s)\|^2\sum_{i=2}^{m}|\sum_{j=0}^{m_i}\tilde{\tilde{Q}}_{ij}(s)(S_{h_{ij}}x)(s)|^2ds)^p)^{1/(2p)}\\
&\le C_1^{(2p-1)/2p}[(\sup_{t\ge0}\int_0^t \|C(t,s)\|\xi(s)\|\sum_{j=0}^{l}(\tilde{\tilde{Q}}_{1j}(s)\xi^{-1}(s) + \kappa\|^{2p}ds)^{1/(2p)}\|x\|_{M_{2p}}\\
&\quad + \sum_{j=0}^{l}(\sup_{t\ge0}(\mathbf{E}|\int_{\sigma_j(t)}^{\sigma(t)} dx(\tau)|^{2p})^{1/(2p)})(\sup_{t\ge0}\int_0^t \|C(t,s)\|\xi(s)(\tilde{a}^1_j(s)\xi^{-1}(s))^{2p}ds)^{1/(2p)}\\
&\quad + \|x\|_{M_{2p}}\sum_{j=l+1}^{m_1}(\sup_{t\ge0}\int_0^t \|C(t,s)\|\xi(s)(\tilde{a}^1_j(s)\xi^{-1}(s))^{2p}ds)^{1/(2p)}]\\
&\quad + c_p(C_2)^{(p-1)/2p}\|x\|_{M_{2p}}\sum_{i=2}^{m}\sum_{j=0}^{m_i}(\sup_{t\ge0}\int_0^t \|C(t,s)\|^2\xi(s)(\tilde{h}^i_j(s)\xi^{-0.5}(s))^{2p}ds)^{1/(2p)}.
\end{aligned}$$

Since x is a solution of equation (4.3),

$$\begin{aligned}
\Gamma_k &:= \sup_{t\geq 0}(\mathbf{E}|\int_{\sigma_j(t)}^{\sigma(t)} dx(\tau)|^{2p})^{1/(2p)} \\
&\leq \sum_{j=0}^{m_1} \sup_{t\geq 0}(\mathbf{E}|\int_{\sigma_j(t)}^{\sigma(t)} \xi(s)|\tilde{\tilde{Q}}_{ij}(s)\xi^{-1}(s)(S_{h_{1j}}x)(s)ds|^{2p})^{1/(2p)} \\
&\quad + c_p \sum_{i=2}^{m}\sum_{j=0}^{m_i} \sup_{t\geq 0}(\mathbf{E}|\int_{\sigma_j(t)}^{\sigma(t)} \xi(s)|\tilde{\tilde{Q}}_{ij}(s)\xi^{-0.5}(s)(S_{h_{ij}}x)(s)|^2 ds|^{p})^{1/(2p)} \\
&\leq \vartheta_k^{(2p-1)/2p} \sum_{j=0}^{m_1} \sup_{t\geq 0}(\mathbf{E}|\int_{\sigma_j(t)}^{\sigma(t)} \xi(s)(\|\tilde{\tilde{Q}}_{ij}(s)\xi^{-1}(s)\|\,|(S_{h_{1j}}x)(s)|)^{1-2p} ds|)^{1/(2p)} \\
&\quad + c_p\vartheta_k^{(p-1)/2p} \sum_{i=2}^{m}\sum_{j=0}^{m_i} \sup_{t\geq 0}(\mathbf{E}|\int_{\sigma_j(t)}^{\sigma(t)} \xi(s)(\|\tilde{\tilde{Q}}_{ij}(s)\xi^{-0.5}(s)\| \\
&\quad \times |(S_{h_{ij}}x)(s)|)^{1-p} ds|)^{1/(2p)},
\end{aligned}$$

where $k = 0, \dots, l$.

Now we have to consider three different cases: 1) $q > 2p$, $q \neq \infty$, 2) $q = 2p$, 3) $q = \infty$. Fortunately, they can be treated in a similar way. Let us therefore restrict ourselves to the first case.

Assuming $q > 2p$, $q \neq \infty$ we obtain

$$\Gamma_k \leq \Big(\vartheta_k^{1-q^{-1}} \sum_{j=0}^{m_1} \|\hat{a}_j^1\|_{L_q} + c_p\vartheta_k^{0.5-q^{-1}} \sum_{i=2}^{m}\sum_{j=0}^{m_i} \|\hat{\hat{h}}_j^i\|_{L_q}\Big)\|x\|_{M_{2p}}$$

where $k = 0, \dots, l$. From this and from the estimates for $\|\Theta_l x\|_{M_{2p}}$ we conclude that $\|\Theta_l x\|_{M_{2p}} \leq \rho\|x\|_{M_{2p}}$. Since $\rho < 1$, we have $\|\Theta_l\|_{M_{2p}} < 1$. This completes the proof. $\square$

We apply now Theorem 5.2 to an Itô equation with unbounded delays. In (4.3), we therefore assume that

$$h_{ij}(t) = t/\tau_{ij}, \quad \tau_{ij} \geq 1 \ (i = 1, \dots, m,\ j = 0, \dots, m_i). \tag{5.5}$$

Such equation are known to have a number of "strange" properties, for instance they are exponentially stable only in exceptional cases. Applying our general scheme gives, however, asymptotic stability of such equations in a natural way. This is shown in Corollaries 5.5-5.7 below.

Remark 5.3. In the case of the delays given by (5.5), the initial function in (4.3) disappears as $h_{ij}(t) \geq 0$ for all $t \in [0,\infty)$, $i = 1, \dots, m$, $j = 0, \dots, m_i$. That is why for (4.3) with the delays (5.5) it is natural to study its M_p^γ-stability in the sense Definition 3.2 (which, in turn, for certain γ implies stability properties from Definition 3.1).

Below we use the following function which determines asymptotical properties of the equation we are interested in:

$$\xi(t) = \mathbf{1}_{[0,r]}(t) + \mathbf{1}_{[r,\infty]}(t)(1/t) \quad (t \in [0,\infty)), \tag{5.6}$$

where $r > 0$ is some number and $\mathbf{1}_e$ is the indicator of a set e. From Theorem 5.2 we obtain the following result.

Corollary 5.5. *Let the semimartingale $Z(t)$ satisfy condition (B), equation* (4.3) *with (5.5) satisfy condition (D5), the reference equation* (3.1)*, where Q is given by (5.4), satisfy (R1)-(R2). Assume also that there exist a natural number l $(0 \le l \le m_1)$ and a positive real number r such that*

$$\begin{aligned}\rho := \ & C_1^{1-q^{-1}}\{|||(\sum_{j=0}^{l}\tilde{\tilde{Q}}_{1j} + \xi\kappa)\xi^{q^{-1}-1}|||_{L_q} + \sum_{j=0}^{l}\|\hat{a}_j^1\|_{L_q}(\vartheta_j^{1-q^{-1}}\sum_{j=0}^{m_1}\|\hat{a}_j^1\|_{L_q} \\ & + c_p\vartheta_j^{0.5-q^{-1}}\sum_{i=2}^{m}\sum_{j=0}^{m_i}\|\hat{\hat{h}}_j^i\|_{L_q}) \\ & + \sum_{j=l+1}^{m_1}\|\hat{a}_j^1\|_{L_q}\} + c_pC_2^{0.5-q^{-1}}\sum_{i=2}^{m}\sum_{j=0}^{m_i}\|\hat{\hat{h}}_j^i\|_{L_q} < 1,\end{aligned}$$

where ξ are defined in (5.6), C_1, C_2 are given by (5.2) and $\vartheta_j = \max\{\log\tau_{1j}, r(1-\tau_{1j}^{-1})\}$, $j = 0, \dots, l$. Then (4.3) *is M_{2p}^{γ}-stable, where $\gamma(t) = \mathbf{1}_{[0,r]}(t) + \mathbf{1}_{[r,\infty]}(t)(t/r)^{\beta}$ for some $\beta > 0$.*

Remark 5.4. In fact, Corollary 5.5 gives us the usual asymptotic $2p$-stability in the sense of Definition 3.1.

Proof of Corollary 5.5. Using (5.6) we easily check that the delay functions $h_{ij}(t) = t/\tau_{ij}$ satisfy the δ_{ij}-condition with $\delta_{ij} = \max\{\log\tau_{ij}, r(1-\tau_{ij}^{-1})\}$ $(i = 1, \dots, m,\ j = 0, \dots, m_i)$. This enables us to use Theorem 5.2 directly. $\square$

Applying this corollary to the equation

$$dx(t) = Q(t)\xi^{-1}(t)x(t)dt + \sum_{i=2}^{m}\sum_{j=0}^{m_i}Q_{ij}(t)x(t/\tau_{ij})dB^{i-1}(t) \quad (t \ge 0;\ \tau_{ij} \ge 1), \tag{5.7}$$

where $B^i(t)$ $(i = 2, \dots, m)$ are independent standard Wiener processes, the $n \times n$-matrix $Q(t)$ has entries from the space L_∞ and

$$\|Q_{ij}(t)\| \le q_{ij}(t)\sqrt{\xi(t)} \quad (t \ge 0,\ 2 = 1, \dots, m,\ j = 0, \dots, m_i)$$

for some $q_{ij} \in L_\infty$ (ξ is again given by (5.6)) we obtain from Theorem 5.2 the following result.

Corollary 5.6. *Assume that there exists $\bar{\alpha} > 0$ such that*

$$|||Q + \bar{\alpha}\bar{E}|||_{L_\infty} + c_p\sqrt{0.5\bar{\alpha}}\sum_{i=2}^{m}\sum_{j=0}^{m_i}\|q_{ij}\|_{L_\infty} < \bar{\alpha}.$$

Then (5.7) is M_{2p}^{γ}-stable with respect to the initial function, where

$$\gamma(t) = \mathbf{1}_{[0,r]}(t) + \mathbf{1}_{[r,\infty]}(t)(t/r)^{\beta}$$

for some $\beta > 0$.

Proof. As the reference equation we can take

$$dx(t) = (\mathrm{diag}[-\bar{\alpha}, \dots, -\bar{\alpha}]\xi(t)x(t) + g_1(t))\,dt + \sum_{i=2}^{m}g_i(t)dB^{i-1}(t) \quad (t \ge 0). \tag{5.8}$$

It is straightforward that conditions (R1)-(R2) are satisfied in this case. It is also easy to see that (D5) is fulfilled. $\square$

The example below illustrates Corollary 5.6.

Example 5.1. The equation

$$dx(t) = \big(a\xi^{-1}(t)x(t) + b\xi^{-1}(t)x(t/\tau_0)\big)\,dt + c\xi^{-0.5}(t)x(t/\tau_1)dB(t) \quad (t \geq 0), \tag{5.9}$$

where ξ is again given by (5.6), $B(t)$ is a scalar Wiener process, a, b, c, τ_0, τ_1 are real numbers ($\tau_0 \geq 1, \tau_1 \geq 1$) is M^{γ}_{2p}-stable, where $\gamma(t) = \mathbf{1}_{[0,r]}(t) + \mathbf{1}_{[r,\infty]}(t)(t/r)^{\beta}$ for some $\beta > 0$, provided there exists $\bar{\alpha} > 0$ such that

$$|a + b + \bar{\alpha}| + c_p|c|\sqrt{0.5\bar{\alpha}} + (|ab| + b^2)\delta_0 + c_p|bc|\sqrt{\delta_0} < \bar{\alpha},$$

and $\delta_0 = \max\{\log h_0, (1 - h_0^{-1})r\}$.

Some situations where such a number α does exist are found in the dissertation [7, Sect. 3.3], where the results are formulated in terms of coefficients of (5.9).

Let us now consider the case of a scalar equation of the form

$$\begin{gathered} dx(t) = [ax(t) + bx(h_0(t))]dt + cx(h_1(t))dB(t) \quad (t \geq 0), \\ x(\nu) = \varphi(\nu) \quad (\nu < 0), \end{gathered} \tag{5.10}$$

where $B(t)$ is a scalar Wiener process, a, b, c are real numbers, h_0, h_1 are μmeasurable functions such that $h_i(t) \leq t$ ($t \in [0, \infty)$) for $i = 0, 1$, φ is a stochastic process, which is independent of the (scalar) standard Wiener process $B(t)$.

We will now exploit the following reference equation:

$$dx(t) = (-\bar{\alpha}(S_h x)(t) + g_1(t))\,dt + g_2(t)dB(t) \quad (t \geq 0), \tag{5.11}$$

where $\bar{\alpha} > 0$ and $h(t)$ is μ-measurable and $h(t) \leq t$ for all $t \in [0, \infty)$. We remark that in the works [6, 7], the M^{γ}_{p}-stability of (5.10) was studied with the help of the reference equations (5.11) which was ordinary differential equations, i.e. when $h(t) \equiv t$.

From Theorem 5.2, we obtain the following corollary for equation (5.10).

Corollary 5.7. *(1) Assume that there exist positive numbers $\bar{\alpha}$ and δ such that the reference equation* (5.11) *satisfies $t - h(t) \leq \delta$ (for all $t \in [0, \infty)$) and conditions (R1)-(R2). If*

$$|a + \bar{\alpha}| + \delta(a^2 + |ab|) + c_p\sqrt{\delta}|ac| + |b| + c_p|c|C_1^{-1}\sqrt{C_2} < C_1,$$

where C_1 and C_2 are given by (5.2)*, then the zero solution of* (5.10) *is $2p$-stable with respect to the initial function. If, in addition, there exist positive numbers δ_0, δ_1, for which $t - h_i(t) \leq \delta_i$ ($i = 0, 1$, $t \in [0, \infty)$), then the zero solution of the* (5.10) *exponentially $2p$-stable with respect to the initial function.*
(2). Assume that there exist positive numbers $\bar{\alpha}$ and δ, δ_0 such that the reference equation (5.11) *satisfies $t - h(t) \leq \delta$, $h_0(t) - h(t) \leq \delta_0$ (for all $t \in [0, \infty)$) and conditions (R1)-(R2). If*

$$|a + b + \delta(a^2 + |ab|) + c_p\sqrt{\delta}|ac| + \delta_0(|ba| + b^2) + c_p\sqrt{\delta_0}|bc| + c_p|c|(\sqrt{C_2}/C_1) < C_1,$$

where C_1, C_2 are given by (5.2)*, then the zero solution of* (5.10) *is $2p$-stable with respect to the initial function. If, in addition, there exist a positive number δ_1, for which $t - h_1(t) \leq \delta_1$ ($t \in [0, \infty)$), then the zero solution of the* (5.10) *exponentially $2p$-stable with respect to the initial function.*

Remark 5.5. Formally, the inequalities in the lemma does not include the constant $\bar{\alpha}$. However, the constants C_1 and C_2 depend on $\bar{\alpha}$ through the formulas (5.2).

If the reference equation (5.11) is an ordinary differential equation, then it is easy to see that conditions (R1)-(R2) are satisfied. If equation (5.11) is not ordinary, then (R1)-(R2) may fail, and the problem of how to find exponential estimates on the fundamental matrix $U(t)$ and the matrix kernel $C(t,s)$ from the representation (3.3) for the reference equation (5.11) is rather difficult. This was discussed in [2]. The problem of how one can numerically estimate the constant C_1 in some particular cases was studied in [4]. Similar algorithms cane be applied to the constant C_2. Using these estimates we derive below some stability results for (5.10).

Consider the following Itô delay equation:

$$\begin{gathered} dx(t) = bx(t-\epsilon)dt + cx(\tilde{h}(t))dB(t) \quad (t \geq 0), \\ x(\nu) = \varphi(\nu) \quad (\nu < 0), \end{gathered} \tag{5.12}$$

b, c, ϵ are real numbers, $b < 0$, $\epsilon > 0$, $-b\epsilon < \pi/2$; $\tilde{h}(t)$ is a μ-measurable function such that $\tilde{h}(t) \leq t$ for all $t \in [0,\infty)$, φ is a stochastic process, which is independent of the (scalar) standard Wiener process $B(t)$.

Taking in the reference equation (5.11) with $\bar{\alpha} = -b$, $h(t) = t - \epsilon$, we refer to [2], where the assumptions (R1)-(R2) are verified.

Corollary 5.7 yields now the following result.

Corollary 5.8. *(1). If* $2|b| + |c|(\sqrt{C_2}/C_1)c_p < C_1$, *where* C_1, C_2 *are given by* (5.2), *then the zero solution of* (5.12) *is* $2p$*-stable with respect to the initial function. If, in addition, there exists a number* $\delta > 0$ *such that* $t - \tilde{h}(t) \leq \delta$ *(*$t \in [0,\infty)$*), then the zero solution of (5.12) is exponentially* $2p$*-stable with respect to the initial function.*
(2). If $|c|(\sqrt{C_2}/C_1)c_p < C_1$, *where* C_1, C_2 *are given by* (5.2), *then the zero solution of* (5.12) *is* $2p$*-stable with respect to the initial function. If, in addition, there exists a number* $\delta > 0$ *such that* $t - \tilde{h}(t) \leq \delta$ *(*$t \in [0,\infty)$*), then the zero solution of* (5.12) *is exponentially* $2p$*-stable with respect to the initial function.*

The constants C_1 and C_2 can only be estimated numerically. An algorithm of how to find C_1 with an arbitrary precision is presented in [4]. There are some estimates from this paper in Tables 1 and 2.

TABLE 1. Estimates for C_1

$-b\tau$	0.4	0.5	0.6	0.7	0.8	0.9	1.2	1.4
C_1	1.001	1.164	1.262	1.510	1.840	2.290	4.620	9.740

TABLE 2. Estimates for C_2

$-b\tau$	0.4	0.5	0.6	0.7	0.8	0.9	1.2	1.4
C_2	0.754	0.843	0.948	1.075	1.233	1.434	2.666	5.833

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