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SEMILINEAR ELLIPTIC BOUNDARY-VALUE PROBLEMS ON BOUNDED MULTICONNECTED DOMAINS

WENJIE GAO, JUNYU WANG, ZHONGXIN ZHANG

Abstract. A semilinear elliptic boundary-value problem on bounded multiconnected domains is studied. The authors prove that under suitable conditions, the problem may have no solutions in certain cases and many have one or two nonnegative solutions in some other cases. The radial solutions were also studied in annular domains.

1. Introduction

This paper consists of two parts. The first part deals with a semilinear elliptic boundary-value problem of the form

$$
\begin{aligned}
-\Delta u = f(u,x) \quad &\in \Omega,\\
u = 0 \quad &\text{on } B_0 := \cup_{j=0}^{k-1}\Gamma_j,\\
u = \rho \quad &\text{on } B := \cup_{j=k}^{m}\Gamma_j\,,
\end{aligned}
\tag{1.1}
$$

where $f(u,x) \in C(\mathbb{R}_+ \times \overline{\Omega};\mathbb{R})$, $\mathbb{R}_+ := [0,+\infty)$, Ω is a bounded multiconnected domain in $\mathbb{R}^n$, $n \geq 2$, Γ_0 is its outer boundary, $\cup_{j=1}^{m}\Gamma_j$ its inner boundary, Γ_0, $\Gamma_1,\dots,\Gamma_m$ are all sufficiently smooth closed surfaces so that the Green's function $G(x,y)$, for $-\Delta$ with zero Dirichlet boundary conditions is existent (See, e.g., [2, p.112]), $k \in \{1,2,\dots,m\}$, and the constant $\rho \geq 0$ is given. Some particular cases of Problem (1.1) were considered by Bandle and Peletier [3], by Lee and Lin [5] and by Hai [4], in which $f \in (\mathbb{R}_+;\mathbb{R}_+)$ and Ω is a domain with a "hole".

The second part is devoted to the semilinear elliptic boundary-value problem, namely,

$$
\begin{gathered}
-\Delta u = f(u,|x|) \quad 0<\alpha<|x|<\beta<+\infty,\\
u = 0 \quad \text{on } |x| = \alpha, \quad u = \rho \quad \text{on } |x| = \beta
\end{gathered}
\tag{1.2}
$$

where $f \in C(\mathbb{R}_+ \times J;\mathbb{R}_+)$, $J = [\alpha,\beta]$, $|x| = \sqrt{x_1^2+\dots+x_n^2}$. Problem (1.2) was studied by Lee and Lin [5] and by Hai [4] in the case $f \in C(\mathbb{R}_+;\mathbb{R}_+)$.

We study Problems (1.1) and (1.2) motivated by the following results recently established by Hai [4].

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Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with a hole and f in $C(\mathbb{R}_+;\mathbb{R}_+)$ satisfy*

$$\lim_{u\to 0+}\frac{f(u)}{u} = 0 \quad \textit{and} \qquad \lim_{u\to +\infty}\frac{f(u)}{u} = +\infty.$$

Then there exists a positive number ρ^* such that Problem (1.1) *has a positive solution for $\rho \in (0,\rho^*)$ and no solution for $\rho > \rho^*$.*

Theorem 1.2. *Let $f \in C(\mathbb{R}_+;\mathbb{R}_+)$ be convex and satisfy*

$$\liminf_{u\to 0+}\frac{\int_0^u f(s)ds}{u^2} = 0 \quad \textit{and} \qquad \lim_{u\to +\infty}\frac{f(u)}{u} = +\infty.$$

Then there exists a positive number ρ^* such that Problem (1.2) *has at least two positive radial solutions for $\rho \in (0,\rho^*)$, at least one for $\rho = \rho^*$ and none for any $\rho > \rho^*$.*

The above results extend and complement the corresponding results in [3, 5].

As in [4], our purpose is to extend, improve and complement the corresponding results in [3, 4, 5]. In the first part, we will prove three theorems. The second theorem extends and complements Theorem 1.1, in which the function f is allowed to contain variable x and is not necessarily nonnegative, also, $f(u,x)$ is not necessarily superlinear at $u = 0$ and at $u = +\infty$. In the second part, we obtain a similar result as Theorem 1.2 where the function f may depends on the variable $|x|$ and the convexity constrain to f is removed, also f is not required to be superlinear at $u = 0$ and $u = +\infty$.

2. Results in multiconnected domains

For the first part, we make the following assumptions:

(A1) $f \in C(\mathbb{R}_+ \times \overline{\Omega};\mathbb{R})$.

(A2) $M := \sup\{\lambda_1 u - f(u,x) : u \in \mathbb{R}_+,\ x \in \overline{\Omega}\} < +\infty$.

Here and throughout this section, λ_1 denotes the first eigenvalue of the problem

$$\begin{gathered} -\Delta\phi = \lambda\phi \quad \text{in } \Omega \\ \phi = 0 \quad \text{on } \partial\Omega = B_0 \cup B. \end{gathered} \tag{2.1}$$

The positive eigenfunction corresponding to λ_1 is denoted by $\phi_1(x)$ with $\|\phi_1\| = 1$, where $\|\cdot\|$ stands for the supremum norm.

(A3)

$$\limsup_{u\to 0+}\frac{\max\{|f(u,x)| : x \in \overline{\Omega}\}}{u} < \frac{1}{\|g\|},$$

where

$$g(x) := \int_\Omega G(x,y)\,dy, \quad x \in \overline{\Omega}. \tag{2.2}$$

(A4) $f(0,x) \geq 0$ for all $x \in \overline{\Omega}$ and

$$\limsup_{u\to +\infty}\frac{\max\{|f(u,x)| : x \in \overline{\Omega}\}}{u} < \frac{1}{\|g\|}.$$

Concerning problem (1.1), we can establish the following results.

Theorem 2.1. *Let* (A1) *and* (A2) *hold. Then problem* (1.1) *has no solution for*

$$\rho > \tilde{\rho} := \frac{M \int_\Omega \phi_1(x)dx}{\lambda_1 \int_\Omega \phi_1(x)h(x)\,dx}. \tag{2.3}$$

Here $h(x)$ is the harmonic function defined on $\overline{\Omega}$ satisfying

$$h(x) = 0 \quad \text{on } B_0 \quad \text{and} \quad h(x) = 1 \quad \text{on } B. \tag{2.3}$$

Theorem 2.2. *Let* (A1), (A2) *and* (A3) *hold. Then there exists a positive number ρ^* such that problem* (1.1) *has a nonnegative solution for $\rho \in [0, \rho^*)$ and no solution for $\rho > \rho^*$.*

Theorem 2.3. *Let* (A1) *and* (A4) *hold. Then problem* (1.1) *has a nonnegative solution for all $\rho \geq 0$. In addition, assume that $f(0, x) \equiv 0$ for all $x \in \overline{\Omega}$ and there exists a $\delta > 0$ such that $f(u, x) \geq \lambda_1 u$ for all $u \in [0, \delta]$ and all $x \in \overline{\Omega}$. Then Problem* (1.1) *with $\rho = 0$ has at least one positive solution.*

Theorems 2.1 and 2.3 are new and Theorem 2.2 extends and complements the corresponding theorems in [3, 4, 5]. Before proving these theorems, we make several remarks.

Remark 2.4. A function $u \in C(\overline{\Omega}; \mathbb{R})$ is a solution to Problem (1.1), if and only if it is a solution to the integral equation

$$u(x) = \int_\Omega G(x, y) f(u(y), y)\,dy + \rho h(x), \quad x \in \overline{\Omega}. \tag{2.4}$$

Clearly, each solution to Problem (1.1) is positive when $f \in C(\mathbb{R}_+ \times \overline{\Omega}; \mathbb{R}_+)$ and $\rho > 0$.

Remark 2.5. ¿From the maximum principle, we know that $0 < h(x) < 1$ in Ω.

Remark 2.6. According to (2.1) and (2.4), we have

$$0 < \phi_1(x) = \lambda_1 \int_\Omega G(x, y)\phi_1(x)dx \leq 1 \quad \text{in } \Omega$$

and hence $\lambda_1 > 1/\|g\|$.

Remark 2.7. It is obvious that Theorems 2.1, 2.2 and 2.3 are still valid if the boundary conditions in (1.1) are replaced by $u = \rho$ on B_0, $u = 0$ on B.

Proof of Theorem 2.1. Suppose to the contrary that Problem (1.1) has a solution,

$$u(x) = \int_\Omega G(x, y) f(u(y), y)\,dy + \rho h(x) =: w(x) + \rho h(x), \quad x \in \overline{\Omega},$$

for some $\rho > \tilde{\rho}$. In this case, we have $-\Delta w(x) = f(u(x), x)$ in $\overline{\Omega}$. Consequently,

$$\begin{aligned}
\rho\lambda_1 \int_\Omega \phi_1(x)h(x)dx &= \int_\Omega \lambda_1\phi_1(x)u(x)dx - \int_\Omega \lambda_1\phi_1(x)w(x)dx \\
&= \int_\Omega \lambda_1\phi_1(x)u(x)dx + \int_\Omega \phi_1(x)\Delta w(x)dx \\
&= \int_\Omega \phi_1(x)(\lambda_1 u(x) - f(u(x), x)\,dx \\
&\leq M \int_\Omega \phi_1(x)dx,
\end{aligned}$$

i.e., $\rho \leq \tilde{\rho}$, a contradiction. Theorem 2.1 is thus proved. ☐

Proof of Theorem 2.2. To prove Theorem 2.2, we first define a mapping $K : E \mapsto E$ by setting

$$(Kw)(x) := \int_\Omega G(x,y)f^*(w(y),y)dy + \rho h(x), \quad \forall w \in E, \tag{2.5}$$

where $E := C(\overline{\Omega};\mathbb{R})$ and

$$f^*(u,x) := \begin{cases} f(0,x) & \text{if } u < 0, \\ f(u,x) & \text{if } u \geq 0. \end{cases}$$

It is easy to check that K is completely continuous on E. ¿From (A3), we know that there exists an $\epsilon > 0$ such that

$$\limsup_{u\to 0+} \frac{\max\{|f(u,x)| : x \in \overline{\Omega}\}}{u} < \frac{1}{\|g\|+\epsilon} < \frac{1}{\|g\|}$$

and hence there exists a $\sigma > 0$ such that

$$|f(u,x)| \leq \frac{u}{\|g\|+\epsilon} \quad \text{for all } u \in [0,\sigma] \text{ and all } x \in \overline{\Omega},$$

from which it follows that

$$f(0,x) \equiv 0 \quad \text{for all } x \in \overline{\Omega}. \tag{2.6}$$

Clearly, $u(x) \equiv 0$ is a trivial solution to Problem (1.1) with $\rho = 0$. Put

$$D_\sigma := \{w \in E : \|w\| \leq \sigma\} \quad \text{and} \quad \sigma^* := \sigma\big(1 - \frac{\|g\|}{\|g\|+\epsilon}\big).$$

Then for each fixed $w \in D_\sigma$ and each fixed $\rho \in [0,\sigma^*]$, we have

$$\|Kw\| \leq \frac{\|g\|}{\|g\|+\epsilon}\sigma + \sigma^* = \sigma$$

which means that K is a compactly continuous mapping from D_σ into itself. The Schauder fixed point theorem tells us that K has a fixed point $u \in D_\sigma$, i.e.,

$$\begin{gathered} -\Delta u = f^*(u(x),x) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } B_0, \quad u = \rho \text{on } B. \end{gathered}$$

¿From (2.6) and the maximum principle, we know that $u(x) \geq 0$ on $\overline{\Omega}$, which shows that the fixed point $u(x) \in D_\sigma$ is also a nonnegative solution to Problem (1.1) with $\rho \in [0,\sigma^*]$.

Let

$$\rho^* = \sup\{\rho \geq 0 : \text{ Problem (1.1) has a nonnegative solution}\}.$$

¿From the previous results, $\rho^* \in [\sigma^*, \tilde{\rho}]$. We are now going to prove that Problem (1.1) has a nonnegative solution for all $\rho \in [0,\rho^*)$.

For each given $\rho \in [0,\rho^*)$, from the definition of ρ^*, we can choose a $\bar{\rho} \in (\rho,\rho^*)$ such that Problem $(1.1)_{\bar{\rho}}$ has a nonnegative solution $\bar{u}(x)$. Let $\xi(x) \equiv 0$ and $\eta(x) = \bar{u}(x)$. Then

$$\begin{gathered} -\Delta\xi(x) \equiv 0 \equiv f(\xi(x),x) \quad \in \Omega, \\ \xi(x) = 0 \quad \text{on} \quad B_0, \quad \xi(x) = 0 \leq \rho \quad \text{on } B, \end{gathered}$$

and

$$-\Delta\eta(x) \equiv f(\eta(x), x) \quad \text{in } \Omega,$$
$$\eta(x) = 0 \quad \text{on } B_0, \quad \eta(x) = \bar{\rho} > \rho \quad \text{on } B.$$

i.e., $\xi(x)$ is a lower solution to Problem (1.1) and $\eta(x)$ is an upper solution. Employing the method of upper and lower solutions, we can find a solution $u(x)$ to Problem (1.1) with

$$0 \equiv \xi(x) \le u(x) \le \eta(x) = \bar{u}(x) \quad on \overline{\Omega}.$$

The proof of Theorem 2.2 is complete. □

Proof of Theorem 2.3. From (A4), we know that there exists an $\epsilon > 0$ such that

$$\limsup_{u\to+\infty} \frac{\max\{|f(u,x)| : x \in \overline{\Omega}\}}{u} \le \frac{1}{\|g\| + \epsilon} < \frac{1}{\|g\|}$$

and hence there exists an $N > 0$ such that

$$|f(u,x)| \le \frac{u}{\|g\| + \epsilon} \quad \text{for all } \ u \ge N \text{ and all } x \in \overline{\Omega}.$$

For each given $\rho \ge 0$, we put

$$D_\beta := \{w \in E : \|w\| \le \beta\},$$
$$\beta := \big(1 - \frac{\|g\|}{\|g\| + \epsilon}\big)^{-1} \big(N + \rho + \|g\| \max\{|f(u,x)| : x \in \overline{\Omega},\ 0 \le u \le N\}\big).$$

We define a mapping $K : E \mapsto E$ by (2.5). For each fixed $w \in D_\beta$, we have

$$\|Kw\| < \|g\| \Big(\max\{|f(u,x)| : 0 \le u \le N,\ x \in \overline{\Omega}\} + \frac{\beta}{\|g\| + \epsilon} \Big) + \rho + N = \beta,$$

which implies that K is a compactly continuous mapping from D_β into itself. The Schauder fixed point theorem tells us that K has a fixed point $u \in D_\beta$, i.e.,

$$-\Delta u(x) = f^*(u(x), x) \quad \text{in } \Omega,$$
$$u(x) = 0 \quad \text{on } B_0, \quad u(x) = \rho \text{on } B.$$

From the maximum principle and the assumption that $f(0,x) \ge 0$ for all $x \in \overline{\Omega}$, we know that

$$u(x) \ge 0 \quad \text{on } \overline{\Omega}.$$

This shows that the fixed point $u \in D_\beta$ is a nonnegative solution of (1.1).

We now assume that $f(0,x) \equiv 0$ for all $x \in \overline{\Omega}$ and there exists a $\delta > 0$ such that

$$f(u,x) \ge \lambda_1 u \text{ for all } u \in [0,\delta] \text{ and all } x \in \overline{\Omega}.$$

In this case, $u(x) \equiv 0$ is a trivial solution to Problem $(1.1)_0$. Then we consider the modified boundary-value problem

$$\begin{gathered} -\Delta u(x) = \bar{f}(u(x), x) \quad \text{in } \Omega, \\ u(x) = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.7}$$

where the function

$$\bar{f}(u,x) := \begin{cases} f(\delta\phi_1(x), x) & \text{if } u < \delta\phi_1(x), \\ f(u,x) & \text{if } u > \delta\phi_1(x) \end{cases}$$

satisfies (A1) and (A4) again. From the above discussion, we know that Problem (2.7) has a solution $u \in D_\beta$. The maximum principle tells us that

$$u(x) \geq \delta\phi_1(x) \quad \text{on } \overline{\Omega}.$$

This shows that the solution $u(x)$ is also a positive solution to Problem (1.1) with $\rho = 0$. Theorem 2.3 is thus proved. □

3. Results in annular domains

In this section, we restrict our attention to the multiplicity of radial solutions to (1.2). When we seek for a radial solution to (1.2), the problem can be rewritten as

$$\begin{gathered} -(k(t)u'(t))' = k(t)f(u(t),t), \quad \alpha < t < \beta, \\ u(\alpha) = 0, \quad u(\beta) = \rho, \end{gathered} \tag{3.1}$$

where $k(t) = t^{n-1}$, $n \geq 2$. Concerning Problem (3.1), we make the following hypotheses:

(H1) $f \in C(\mathbb{R}_+ \times J; \mathbb{R}_+)$, $J := [\alpha, \beta]$.

(H2)
$$\limsup_{u \to 0} \frac{\max\{f(u,t) : t \in J\}}{u} < \frac{1}{\|g\|}, \tag{3.2}$$
where
$$\begin{gathered} g(t) = \int_\alpha^\beta G(t,s)ds, \quad t \in J; \\ G(t,s) = \begin{cases} \frac{k(s)}{P(\beta)}(P(\beta) - P(t))P(s), & \alpha \leq s \leq t \leq \beta, \\ \frac{k(s)}{P(\beta)}(P(\beta) - P(t))P(t), & \alpha \leq t \leq s \leq \beta; \end{cases} \\ P(t) = \int_\alpha^t \frac{dr}{k(r)}, \quad t \in J. \end{gathered} \tag{3.3}$$

(H3) There exists an interval $[a, b]$ with $\alpha < a < b < \beta$, such that
$$\liminf_{u \to +\infty} \frac{\min\{f(u,t) : a \leq t \leq b\}}{u} > \frac{1}{m},$$
where
$$\begin{gathered} m = \delta \max\{\int_a^b G(t,s)\,ds : a \leq t \leq b\}, \\ \delta = \min\{q(t) : a \leq t \leq b\}, \\ q(t) = \min\big\{\frac{P(t)}{P(\beta)}, \frac{P(\beta) - P(t)}{P(\beta)}\big\}, \quad t \in J\,. \end{gathered} \tag{3.4}$$

(H4) $f(u,t)$ is nondecreasing in $u \geq 0$ for each fixed $t \in J$.

(H4)* $f(u,t)$ is locally Lipschitz continuous in $u \geq 0$ for each fixed $t \in J$.

Theorem 3.1. *Let* (H1), (H2), (H3) *and* (H4) *(or* (H4)**) hold. Then there exists a positive number ρ^* such that Problem* (1.2) *has at least two nonnegative radial solutions for $\rho \in [0, \rho^*)$, at least one for $\rho = \rho^*$ and none for any $\rho > \rho^*$.*

Remark 3.2. Theorem 3.1 is still valid when the boundary conditions in (3.1) are replaced by

$$u(\alpha) = \rho, \quad u(\beta) = 0.$$

Clearly, Theorem 3.1 is an extension and improvement of the results in [4, Theorem 1.2]. We need the following lemmas.

Lemma 3.3. *Let $f \in C(\mathbb{R}_+ \times J;\ \mathbb{R}_+)$ and $u(t)$ a nonnegative solution to Problem* (3.1). *Then*

$$u(t) \geq \|u\| q(t), \quad t \in J\,.$$

Proof. Note that Problem (3.1) is equivalent to the integral equation

$$u(t) = \int_\alpha^\beta G(t,s) f(u(s),s)\,ds + \rho h(t), \quad t \in J, \tag{3.5}$$

where

$$h(t) := P(t)/P(\beta), \quad t \in J. \tag{3.6}$$

Let $\|u\| = u(t_0)$. Then $t_0 \in (\alpha, \beta]$. If $t_0 = \beta$, then $\|u\| = \rho$ and hence $u(t) = \rho h(t) \geq \|u\| q(t)$, $t \in J$. If $t_0 \in (\alpha,\beta)$, then

$$\begin{aligned} u(t) &= \int_\alpha^\beta \frac{G(t,s)}{G(t_0,s)} G(t_0,s) f(u(s),s)\,ds + \rho h(t_0)\frac{h(t)}{h(t_0)} \\ &\geq q(t) \int_\alpha^\beta G(t_0,s) f(u(s),s)\,ds + \rho h(t_0) q(t) \\ &= q(t)\|u\| \quad \text{for all } t \in J. \end{aligned}$$

Here we have used the fact that

$$\frac{G(t,s)}{G(t_0,s)} \geq q(t) \quad \text{for all } t,s \text{ in } (\alpha,\beta),$$

the proof of which can be found in [1]. ☐

Lemma 3.4. *Let* (H1) *and* (H3) *hold. Then there exists a positive number M such that all solutions $u(t)$ to* (3.1) *satisfy $\|u\| < M$.*

Proof. By Lemma 3.3, we know that if $u(t)$ is a nonnegative solution to Problem (3.1), then

$$u(t) \geq \delta \|u\| \quad \text{for all } t \in [a,b],$$

where the constant δ is defined by (3.4). From (H3), we know that there exists an $\epsilon > 0$ such that

$$\liminf_{u \to +\infty} \frac{\min\{f(u,t);\ t \in [a,b]\}}{u} > \frac{1}{m-\epsilon} > \frac{1}{m},$$

and hence there exists an $M > 0$ such that

$$f(u,t) \geq \frac{u}{m-\epsilon} \quad \text{for all } u \geq \delta M \text{ and all } t \in [a,b]. \tag{3.7}$$

We now claim that $\|u\| < M$ for all nonnegative solution $u(t)$ to (3.1), where the constant M satisfies (3.7). If the claim is false, then Problem (3.1) has a nonnegative solution $u(t)$ with $\|u\| \geq M$. In this case, we have

$$u(t) \geq \int_a^b G(t,s) f(u(s),s)\,ds \geq \frac{\delta \|u\|}{m-\epsilon} \int_a^b G(t,s)\,ds$$

for all $t \in [a,b]$; i.e.,

$$\|u\| \geq \frac{m}{m-\epsilon} \|u\|.$$

This is a contradiction which proves the claim. ☐

Proof of Theorem 3.1. We first define a mapping $K : E \mapsto E$ by setting

$$(Kw)(t) = \int_\alpha^\beta G(t,s) f^*(w(s),s)\,ds + \rho h(t),$$

where $E = C(J;\mathbb{R})$ and

$$f^*(u,t) = \begin{cases} f(0,t) & \text{if } u < 0,\\ f(u,t) & \text{if } u \geq 0. \end{cases}$$

It is easy to check that K is completely continuous on E. ¿From (H2), we know that there exists an $\epsilon > 0$ such that

$$\limsup_{u\to 0+} \frac{\max\{f(u,t) : t \in J\}}{u} < \frac{1}{\|g\|+\epsilon} < \frac{1}{\|g\|}$$

and hence there exists a $\sigma > 0$ such that

$$0 \leq f(u,t) \leq \frac{u}{\|g\|+\epsilon}$$

for all $u \in [0,\sigma]$ and all $t \in J$, which implies that $f(0,t) \equiv 0$ for all $t \in J$. We now put

$$D_\sigma = \{w \in E : \|w\| \leq \sigma\} \quad \text{and} \quad \sigma^* = \sigma\big(1 - \frac{\|g\|}{\|g\|+\epsilon}\big).$$

Then for each fixed $w \in D_\sigma$ and each fixed $\rho \in [0,\sigma^*]$, we have

$$\|Kw\| \leq \frac{\|g\|}{\|g\|+\epsilon} + \sigma^* = \sigma,$$

which implies that K is a completely continuous mapping from D_σ into itself. The Schauder fixed point theorem tells us that K has a fixed point $u \in D_\sigma$, i.e.,

$$\begin{gathered} -(k(t)u'(t))' = k(t) f^*(u(t),t), \quad \alpha < t < \beta,\\ u(\alpha) = 0, \quad u(\beta) = \rho. \end{gathered}$$

We now claim that $u(t) \geq 0$ for all $t \in J$. If the claim is false, then there exists an interval $[a,b]$, $\alpha \leq a < b \leq \beta$, such that

$$u(t) < 0 \quad \text{in } (a,b) \quad \text{and} \quad u(a) = u(b) = 0.$$

Consequently,

$$\begin{gathered} -(k(t)u'(t))' = 0, \quad a < t < b,\\ u(a) = u(b) = 0. \end{gathered}$$

which implies that $u(t) \equiv 0$ on $[a,b]$. This is a contradiction and hence the claim is true. As a result, the fixed point $u \in D_\sigma$ is a nonnegative solution to (3.1).

We now put

$$\rho^* = \sup\{\rho \geq 0 : \text{ (3.1) has a nonnegative solution}\}.$$

Then $\rho^* \in [\sigma^*, M)$. Here we have used Lemma 3.4.

From the definition of ρ^*, we can choose a sequence $\{\rho_j\}_{j=1}^\infty$ such that $\rho_j < \rho_{j+1}$, $\rho_j \to \rho^*$ as $j \mapsto +\infty$, and Problem (3.1) with ρ_J has a nonnegative solution $u_j(t) \in E$. From Lemma 3.4 and the complete continuity of K on E, we know that $\{u_j(t)\}_{j=1}^\infty$ is uniformly bounded and equicontinuous on J. Without loss of generality, we may assume that $u_j(t) \to u^*(t)$ uniformly on J as $j \to +\infty$. Note that

$$u_j(t) = \int_\alpha^\beta G(t,s) f(u_j(s), s)\, ds + \rho_j h(t), \quad t \in J\,.$$

Letting $j \to +\infty$ in the above yields

$$u^*(t) = \int_\alpha^\beta G(t,s) f(u^*(s), s)\, ds + \rho^* h(t), \quad t \in J.$$

This shows that $u^*(t) \in E$ is a positive solution to (3.1) with ρ^*.

We are now in position to prove that (3.1) has at least two nonnegative solutions for all $\rho \in [0, \rho^*)$.

For each given $\rho \in [0, \rho^*)$, we set $\xi(t) \equiv 0$ and $\eta(t) = u^*(t)$. Then $\xi(t)$ is a lower solution to (3.1) and $\eta(t)$ an upper solution. Employing the method of upper and lower solutions, we can find a solution $u(t) \in E$ with

$$0 \equiv \xi(t) \le u(t) \le \eta(t) = u^*(t) \quad \text{on } J\,.$$

We now assume that (H4)* holds. Using the local Lipschitz continuity of $f(u,t)$ with respect to $u \in \mathbb{R}_+$ and the strong maximum principle, we deduce that

$$0 \le u(t) < u^*(t) \quad \text{for all } t \in (\alpha, \beta]. \tag{3.8}$$

If (H4) holds, i.e., $f(u,t)$ is nondecreasing in $u \in \mathbb{R}_+$ for each fixed $t \in J$. Then

$$u^*(t) - u(t) = \int_\alpha^\beta \big(f(u^*(s), s) - f(u(s), s)\big)\, ds + (\rho^* - \rho) h(t) > 0$$

for all $t \in (\alpha, \beta]$. i.e., (3.8) is also valid.

To obtain the existence of a second nonnegative solution to (3.1) for all $\rho \in [0, \rho^*)$, we define a mapping $\tilde{K} : E \mapsto E$ by setting

$$\big(\tilde{K} w\big)(t) := \int_\alpha^\beta G(t,s) \tilde{f}(w(s), s)\, ds + \rho h(t),$$

and another mapping $K : E \times \mathbb{R}_+ \mapsto E$ by setting

$$K(w,y)(t) := \int_\alpha^\beta G(t,s) f^*(w(s), s)\, ds + y h(t),$$

where

$$f^*(w,t) = \begin{cases} 0 & \text{if } w < 0, \\ f(w,t) & \text{if } w \ge 0, \end{cases}$$

and

$$\tilde{f}(w,t) := \begin{cases} f^*(w,t) & \text{if } w \le u^*(t), \\ f^*(u^*(t), t) & \text{if } w > u^*(t). \end{cases}$$

Note that for each fixed $w \in E$, we have

$$\|\tilde{K} w\| < \|g\| \max\{f(u,t) : 0 \le u \le \|u^*\|,\ t \in J\} + \rho^* =: N^*.$$

Picking $t_0 \in (\alpha, \beta)$ such that $u^*(t_0) > 0$, we define

$$v^*(t) = \begin{cases} u^*(t_0) & \text{if } \alpha \le t \le t_0, \\ u^*(t) & \text{if } t_0 < t \le \beta. \end{cases}$$

We now put

$$A := \{w \in E : -M - N^* < w(t) < v^*(t),\ t \in [\alpha, \beta]\},$$
$$B := \{w \in E : \|w\| < M + N^*\}$$

where the constant M is determined by Lemma 3.4. Then both A and B are open subsets of E and $u \in A \subset B$.

Clearly, $\tilde{K}$ has fixed points in A. In fact, $u \in A$ is a fixed point of $\tilde{K}$. We now consider whether $\tilde{K}$ has a fixed point in $B \setminus A$ or not. There are two possibilities.

Case (i). $\tilde{K}$ has no fixed point in $B \setminus A$. In this case, we have

$$\deg(I - K(\cdot, \rho), A, 0) = \deg(I - \tilde{K}, A, 0) = \deg(I - \tilde{K}, B, 0),$$

where I denotes the identity mapping from E into itself. We now claim that $\deg(I - \tilde{K}, B, 0) = 1$. To prove the claim, we consider the homotopic mapping

$$\Psi(w, \tau) := w - \tau \tilde{K} w \quad \forall (w, \tau) \in E \times [0, 1].$$

For any $(w, \tau) \in \partial B \times [0, 1]$, we have

$$\|\Psi(w, \tau)\| \geq \|w\| - \|\tilde{K} w\| > M + N^* - N^* = M.$$

i.e., $\Psi(w, \tau) \neq 0$ for any $(w, \tau) \in \partial B \times [0, 1]$. Consequently

$$\deg(I - \tilde{K}, B, 0) = \deg(\Psi(\cdot, 1), B, 0) = \deg(\Psi(\cdot, 0), B, 0) = \deg(I, B, 0) = 1.$$

On the other hand, we know that $\deg(I - K(\cdot, y), B, 0)$ is constant for all $y \geq 0$. From Lemma 3.4, we know that $K(\cdot, M)$ has no fixed point in E and hence the constant must be zero. Therefore,

$$\deg(I - K(\cdot, \rho), B, 0) = \deg(I - K(\cdot, M), B, 0) = 0.$$

By the excision property of the Leray-Schauder degree, we obtain

$$\deg(I - K(\cdot, \rho), B \setminus A, 0) = -1$$

which implies that $K(\cdot, \rho)$ has a fixed point in $B \setminus A$. The fixed point is a nonnegative solution to Problem (3.1), of course.

Case (ii). $\tilde{K}$ has a fixed point $\bar{u} \in B \setminus A$. By the maximum principle, we know that

$$0 \leq \bar{u}(t) \leq u^*(t) \quad \text{on } J.$$

This means that $\bar{u}(t)$ is also a second solution to (3.1). Since each nonnegative solution to (3.1) is also a nonnegative radial solution to (1.2), Theorem 3.1 is thus proved. $\square$

Finally, we consider the boundary-value problem

$$\begin{gathered} -\Delta u = f_j(u), \quad 0 < \alpha < |x| < \beta, \\ u = 0 \quad \text{on } |x| = \alpha, \quad u = \rho \quad \text{on } |x| = \beta, \end{gathered} \tag{3.9}$$

where

$$f_1(u) = \begin{cases} \xi u, & 0 \leq u \leq 1;\ 0 \leq \xi < 1/\|g\|, \\ 9(u-1)^{1/9} + \xi, & 1 \leq u \leq 2, \\ \eta(u-2) + 9 + \xi, & u \geq 2;\ \eta > 1/m \end{cases}$$

and

$$f_2(u) := \begin{cases} \sin^2 u, & 0 \leq u \leq 8\pi, \\ \eta(u - 8\pi), & u \geq 8\pi,\ \eta > 1/m, \end{cases}$$

the constant m and the function $g(t)$ are determined by (3.4) and (3.2), respectively.

Since $f_1(u)$ satisfies (H1), (H2), (H3) and (H4), $f_2(u)$ satisfies (H1), (H2), (H3) and (H4)*, according to Theorem 3.1, there exists a positive number ρ^* such that Problem (3.9), $j = 1,\ 2$, has at least two nonnegative radial solutions for $\rho \in [0, \rho^*)$, at least one for $\rho = \rho^*$ and none for $\rho > \rho^*$.

However, Theorem 1.2 cannot be applied in studying (3.9), $j = 1,\ 2$.

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Wenjie Gao
Institute of Mathematics, Jilin University, Changchun 130012, China
E-mail address: `wjgao@mail.jlu.edu.cn`

Junyu Wang
Institute of Mathematics, Jilin University, Changchun 130012, China

Zhongxin Zhang
Institute of Mathematics, Jilin University, Changchun 130012, China
E-mail address: `wx555555@163.net`