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SOLVABILITY OF QUASILINEAR ELLIPTIC EQUATIONS IN LARGE DIMENSIONS

OLEG ZUBELEVICH

Abstract. We study the solvability of quasilinear elliptic Dirchlet boundary-value problems. In particular, we show that if the dimension of the domain is large enough then the solution exists independent of the growth rate on right-hand side.

1. Introduction

It is well known the boundary-value problem

$$-\Delta u = |u|^{p-2}u, \quad u\big|_{\partial M} = 0,$$

where M is an m-dimensional star-shaped bounded domain, has nontrivial solutions in $H_0^1(M)$ provided that $p < 2m/(m-2)$; see for example [2]. It is also known that by the Pohozaev's identity, if $p > 2m/(m-2)$ there is no non-trivial solution; see for example [1]. This indicates the importance of the growth rate of the right-hand side.

On the other hand, the dimension of the domain M also plays a role on the existence of solutions. In this note, we show that if the dimension of the domain is large enough then the solution exists independent of the growth rate on the right-hand side.

2. Main theorem

Let M be a bounded domain in $\mathbb{R}^m$ with smooth boundary ∂M. For $x = (x_1, \dots, x_m)$, we use the standard Euclidian norm $|x|^2 = \sum_{i=1}^m x_i^2$. We assume that the domain M is contained in a ball of radius R:

$$M \subseteq B_R(x_0, \mathbb{R}^m) = \{x \in \mathbb{R}^m : |x - x_0| < R\}.$$

We use the Banach space

$$C_0^1(\overline{M}) = \{v \in C^1(\overline{M}) : v\big|_{\partial M} = 0\}.$$

For the right-hand side, we use function $f : C_0^1(\overline{M}) \to L^\infty(M)$ which is continuous. The our main objective is to show the existence of solutions to elliptic problem

$$-\Delta u = f(u), \quad u\big|_{\partial M} = 0. \tag{2.1}$$

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Theorem 2.1. *Suppose there exists a constant λ such that for any $v \in C_0^1(\overline{M})$ with $|v(x)| \le \lambda$, the inequality*

$$|f(v)| \le \frac{2m\lambda}{R^2} \tag{2.2}$$

holds almost everywhere (a.e.) in M. Then problem (2.1) has a solution

$$u \in \widetilde{H}^{2,r}(M) := H_0^{1,r}(M) \bigcap H^{2,r}(M), \quad r > m.$$

As an example of a right-hand side that satisfies the conditions above, we have $f(u) = (2 + \cos(|\nabla u|^2))e^u$.

Let $M_m \subset \mathbb{R}^m$, be a sequence of bounded domains with smooth boundaries and inscribed in Euclidian balls with a given radius R. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function and f be mapping $f(v) = g(v(x))$.

Consider problem (2.1) with f defined on the domains M_m. We claim that in such case problem (2.1) has a solution provided m is sufficiently large. Indeed, take a positive constant λ and observe that the function g is bounded in the closed interval $[-\lambda, \lambda]$, thus inequality (2.2) will certainly be fulfilled, when the number m is sufficiently large.

To illustrate this effect consider the example:

$$-\Delta u = ce^u, \quad u\big|_{\partial B_1(0,\mathbb{R}^m)} = 0, \tag{2.3}$$

where c is a positive constant.

In the one-dimensional case, equation (2.3) can be integrated explicitly. However the corresponding integrals can not be expressed by elementary functions. Numerical simulation of these integrals shows that the problem (2.3) has a solution if and only if

$$c \le 0,87845\dots$$

On the other hand, applying Theorem 2.1 with $|v(x)| \le \lambda$ one has:

$$ce^v \le ce^\lambda \le 2m\lambda. \tag{2.4}$$

If

$$c \le 2e^{-1}m = m \cdot 0.73575\dots$$

then the second inequality of (2.4) has a solution λ. So letting $c = 1$, we see that problem (2.3) has no solutions in the one-dimensional case, and by Theorem 2.1 it has a solution for $m \ge 2$.

To conclude, we note that by Proposition 3.1 (see below), the solution to (2.3) is nonnegative.

3. Proof of Main Theorem

The arguments presented here are quite standard: We use a version of the comparison principle. Denote by $\Delta^{-1}h$ the solution of the problem

$$\Delta w = h \in H^{s,p}(M), \quad w\big|_{\partial M} = 0, \quad s \ge 0, \ p > 1.$$

It is well known that the linear mapping $\Delta^{-1} : H^{s,p}(M) \to \widetilde{H}^{s+2,p}(M)$ is bounded. Now we construct a mapping

$$G(v) = -\Delta^{-1} f(v)$$

and look for a fixed point of this mapping.

By the assumptions above, $G : C_0^1(\overline{M}) \to \widetilde{H}^{2,r}(M)$ is continuous and by virtue of the embeddings:

$$\widetilde{H}^{2,r}(M) \sqsubset \widetilde{H}^{2-\delta,r}(M) \subset C_0^1(\overline{M}), \quad 0 < \delta < 1,\ (1-\delta)r > m, \tag{3.1}$$

(here $\sqsubset$ is a completely continuous embedding) the mapping $G : C_0^1(\overline{M}) \to C_0^1(\overline{M})$ is completely continuous.

Consider a function

$$U(x) = \frac{\lambda}{R^2}(R^2 - |x - x_0|^2).$$

This function takes positive values for $x \in B_R(x_0, \mathbb{R}^m)$, attains its maximum at x_0:

$$\max_{B_R(x_0,\mathbb{R}^m)} U = U(x_0) = \lambda,$$

and satisfy the Poisson equation

$$-\Delta U = \frac{2m\lambda}{R^2}. \tag{3.2}$$

Let us recall a version of the maximum principle.

Proposition 3.1 ([3]). *IF $v \in H^1(M)$ and $\Delta v \geq 0$ then inequality $v(x) \leq 0$ a.e. in ∂M implies that $v(x) \leq 0$ a.e. in M.*

Lemma 3.2. *The function G maps the set*

$$W = \{w \in C_0^1(\overline{M}) \mid |w(x)| \leq \lambda, \quad x \in M\}$$

to itself. Furthermore, the set $G(W)$ is bounded in $\widetilde{H}^{2,r}(M)$.

Proof. Since $-\Delta G(w) = f(w)$, by formula (3.2) one has

$$\Delta(G(w) - U) = -f(w) + \frac{2m\lambda}{R^2} \geq 0$$

a.e. in M. Observing that $(G(w) - U)\big|_{\partial M} = -U\big|_{\partial M} \leq 0$, by Proposition 3.1 we see that $G(w) \leq U$ a.e. in M. The same arguments give $-U \leq G(w)$ a.e. in M. Note that, a.e. in M, we have

$$|G(w)| \leq U \leq \max_{B_R(x_0,\mathbb{R}^m)} U = \lambda.$$

By assumption of this Theorem, the set $f(W)$ is bounded in $L^\infty(M)$; i.e., $|f(W)| \leq 2m\lambda/R^2$. Consequently the set $\Delta^{-1} f(W)$ is bounded in $\widetilde{H}^{2,r}(M)$. $\square$

Note that Lemma 3.2 and formula (3.1) imply the set $G(W)$ being precompact in $C_0^1(\overline{M})$. Observing that W is a convex set, we apply Schauder's fixed point theorem to the mapping $G : W \to W$ and obtain desired fixed point $u = G(u) \in \widetilde{H}^{2,r}(M)$. This completes the proof of the main Theorem.

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Oleg Zubelevich
Department of Differential Equations and Mathematical Physics, Peoples Friendship University of Russia, Ordzhonikidze st., 3, 117198, Moscow, Russia
E-mail address: ozubel@yandex.ru
Current address: 2-nd Krestovskii Pereulok 12-179, 129110, Moscow, Russia