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CARATHÉODORY PERTURBATION OF A SECOND-ORDER DIFFERENTIAL INCLUSION WITH CONSTRAINTS

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Abstract. We prove the existence of local solutions for the second-order viability problem

$$\ddot{x}(t) \in f(t, x(t), \dot{x}(t)) + F(x(t), \dot{x}(t)), \quad x(t) \in K.$$

Here K is a closed subset of $\mathbb{R}^n$, F is an upper semicontinuous multifunction with compact values contained in the subdifferential of a convex proper lower semicontinuous function V, and f is a Carathéodory function.

1. Introduction

This paper concerns the second-order nonconvex viability problem

$$\begin{gathered} \ddot{x}(t) \in f(t, x(t), \dot{x}(t)) + F(x(t), \dot{x}(t)) \\ (x(0), \dot{x}(0)) = (x_0, v_0) \\ x(t) \in K. \end{gathered} \tag{1.1}$$

Where F is a globally upper semicontinuous multifunction, cyclically monotone, i.e. $F(x, y) \subset \partial V(y)$, defined from $K \times U$ into the subset of all nonempty compact subset of $\mathbb{R}^n$; f is a Carathéodory function from $\mathbb{R} \times K \times U$ to $\mathbb{R}^n$, where K is a closed subset of $\mathbb{R}^n$, U is an open subset of $\mathbb{R}^n$ and ∂V is the subdifferential of a lower semicontinuous and convex function from U into $\mathbb{R}^n$.

Existence of solutions of differential inclusions with upper semicontinuous and cyclically monotone right-hand side was first established by Bressan, Cellina and Colombo [6]. It has been proved the existence of local solutions of a first order problem without constraints: $\dot{x}(t) \in F(x(t))$. The approach is based on some technics related to the subdifferential properties applied to approximate solutions. To avoid the difficulty of the weak convergence of the derivatives of such approximate solutions, authors rely on the basic relation

$$\frac{d}{dt}(V(x(t))) = \|\dot{x}(t)\|^2.$$

Regarding the existence of viable solutions of second-order upper semicontinuous differential inclusions without convexity, we refer to Lupulescu [10, 11], where two

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results are given in this subject: the first deals with the problem (1.1) where $f \equiv 0$; while the second studies problem (1.1), but without constraint, i.e. $K = \mathbb{R}^n$.

The first work in the second-order viability problem, was done by Cornet and Haddad [8], the authors were subject concerned by a problem with upper semicontinuous right-hand side, not cyclically monotone but convex. This research program was pursued by some works, see [4, 12]. For the nonconvex case, existence results may be found in [1, 9, 13].

It is known that viability problems require tangential conditions. So far as we know, for a second-order viability problem, most of the time, authors use the second-order contingent set

$$A_K(x,y) = \big\{z \in \mathbb{R}^n : \liminf_{h\to 0^+} \frac{d_K(x+hy+\frac{1}{2}h^2 z)}{\frac{h^2}{2}} = 0\big\}$$

introduced by Ben-Tal. In the present paper we prove the existence solutions to (1.1) assuming the tangential condition: For all $(t,x,y) \in I \times K \times U$, there exists $w \in F(x,y)$ such that

$$\liminf_{h\to 0^+} \frac{1}{h^2} d_K(x+hy+\frac{h^2}{2}w + \int_t^{t+h} f(\tau,x,y)d\tau) = 0.$$

This condition is also used by Lupulescu [10] with $f \equiv 0$.

2. Notation and statement of the main result

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with scalar product $\langle\, ;\, \rangle$ and norm $\|\ \|$. Let K be a closed subset of $\mathbb{R}^n$, U be a nonempty open subset of $\mathbb{R}^n$ and denote $\Omega = K \times U$. For each $x \in \mathbb{R}^n$ we denote by $d_K(x)$ the distance from x to K. For $r > 0$, $B(x,r)$ stands for the ball centered at x with radius r and $\overline{B}(x,r)$ its closure, B is the unit ball of $\mathbb{R}^n$.

Let F be a multifunction from Ω into the set of all nonempty compact subsets of $\mathbb{R}^n$. Let f be a function from $\mathbb{R} \times \Omega$ into $\mathbb{R}^n$. Assume that F and f satisfy the following conditions:

- (A1) F is upper semicontinuous, i.e. for all (x,y) and for every $\varepsilon > 0$ there exists $\delta > 0$ such that $F(x',y') \subseteq F(x,y) + \varepsilon B$, whenever $\|(x,y)-(x',y')\| \le \delta$;
- (A2) There exists a convex proper and lower semicontinuous function $V : \mathbb{R}^n \to \mathbb{R}^n$ such that $F(x,y) \subset \partial V(y)$, where ∂V denotes the subdifferential of the function V;
- (A3) $f : \mathbb{R} \times \Omega \to \mathbb{R}^n$ is a Carathéodory function, i.e. for each $(x,y) \in \Omega$, $t \to f(t,x,y)$ is measurable and for all $t \in \mathbb{R}$, $(x,y) \to f(t,x,y)$ is continuous;
- (A4) There exists $m \in L^2(\mathbb{R})$ such that $\|f(t,x,y)\| \le m(t)$ for all $(t,x,y) \in \mathbb{R}\times\Omega$;
- (A5) (Tangential condition) For all $(t,x,v) \in \mathbb{R} \times \Omega$, there exists $w \in F(x,v)$ such that
$$\liminf_{h\to 0^+} \frac{1}{h^2} d_K\Big(x+hv+\frac{h^2}{2}w + \int_t^{t+h} f(\tau,x,v)d\tau\Big) = 0.$$

Let $(x_0,y_0) \in \Omega$. Assuming that F and f satisfy (A1)–(A5), we shall prove the following result.

Theorem 2.1. *There exist $T > 0$ and an absolutely continuous $x : [0,T] \to \mathbb{R}^n$ for which $\dot{x}$ is also absolutely continuous such that*

$$\ddot{x}(t) \in f(t,x(t),\dot{x}(t)) + F(x(t),\dot{x}(t)) \quad a.e.\ t \in [0,T]$$

$$
\begin{gathered}
(x(0), \dot{x}(0)) = (x_0, y_0) \\
x(t) \in K \quad \forall t \in [0, T].
\end{gathered}
$$

3. Proof of the main result

We begin by recalling the following result which was proved in [6], and will be used in the second step of the proof of the main result.

Lemma 3.1. *Let V be a convex proper lower semicontinuous function such that $F(x, y) \subset \partial V(y)$, for any $(x, y) \in \Omega$. Then there exist $r = r_{x,y} > 0$ and $M = M_{x,y} > 0$ such that*

$$
\|F(x, y)\| = \sup_{z \in F(x,y)} \|z\| \le M \quad \text{on } B((x, y), r)
$$

and V is Lipschitz continuous on $B(y, r)$ with constant M.

Let r and M be the real numbers defined in the Lemma above, and such that $B(y_0, r) \subset U$. Choose $T_1 > 0$ such that

$$
\int_0^{T_1} (m(s) + M + 1)\, ds < \frac{r}{3}. \tag{3.1}
$$

Set

$$
T_2 = \min\big\{\frac{r}{3(M+1)}, \frac{2r}{3(\|y_0\| + r)}\big\}. \tag{3.2}
$$

In the sequel, we denote by Ω_0 the compact set $(K \times \overline{B}(y_0, r)) \cap \overline{B}((x_0, y_0), r)$ and choose T such that

$$
T \in]0, \min\{T_1, T_2\}]\,. \tag{3.3}
$$

The following result will be used for proving the viability property of the solutions to (1.1).

Lemma 3.2. *Let F and f satisfy assumptions (A1)–(A5). Then for each $\varepsilon > 0$ there exists $\eta \in]0, \varepsilon[$ such that for each (t, x, v) in $[0, T] \times \Omega_0$, there exist w in $F(x, v) + \frac{\varepsilon}{T} B$ and h in $[\eta, \varepsilon]$; that is,*

$$
\big(x + hv + \frac{h^2}{2} w + \int_t^{t+h} f(\tau, x, v) d\tau\big) \in K.
$$

Proof. Let $(t, x, v) \in [0, T] \times \Omega_0$, let $\varepsilon > 0$. Since F is upper semicontinuous, then there exists $\delta_{(x,v)} > 0$ such that

$$
F(y, u) \subset F(x, v) + \frac{\varepsilon}{T} B \quad \forall (y, u) \in B((x, v), \delta_{(x,v)}). \tag{3.4}
$$

On the other hand, for all $(s, y, u) \in [0, T] \times \Omega_0$, by the tangential condition, there exist $h_{(s,y,u)} \in]0, \varepsilon]$ and $c \in F(y, u)$ such that

$$
d_K\Big(y + h_{(s,y,u)} u + \frac{h^2_{(s,y,u)}}{2} c + \int_s^{s + h_{(s,y,u)}} f(\tau, y, u) d\tau\Big) < h^2_{(s,y,u)} \frac{\varepsilon}{4T}.
$$

Consider the subset

$$
\begin{aligned}
N(s, y, u) = &\big\{(l, a, b) \in \mathbb{R} \times (\mathbb{R}^n)^2 : d_K\big(a + h_{(s,y,u)} b + \frac{h^2_{(s,y,u)}}{2} c \\
&+ \int_l^{l + h_{(s,y,u)}} f(\tau, a, b) d\tau\big) < h^2_{(s,y,u)} \frac{\varepsilon}{4T}\big\}.
\end{aligned}
$$

Since $\|f(l,a,b)\| \leq m(l)$ for all $(l,a,b) \in \mathbb{R}\times\Omega$, the dominated convergence theorem shows that the function

$$(l,a,b) \to a + h_{(s,y,u)}b + \frac{h^2_{(s,y,u)}}{2}c + \int_l^{l+h_{(s,y,u)}} f(\tau,a,b)d\tau$$

is continuous. So that, the function

$$(l,a,b) \to d_K(a + h_{(s,y,u)}b + \frac{h^2_{(s,y,u)}}{2}c + \int_l^{l+h_{(s,y,u)}} f(\tau,a,b)d\tau)$$

is continuous and consequently the subset $N(s,y,u)$ is open. Moreover, since (s,y,u) belongs to $N(s,y,u)$, there exists a ball $B((s,y,u),\eta_{(s,y,u)})$ with radius $\eta_{(s,y,u)} < \delta_{(x,v)}$ and contained in $N(s,y,u)$. Therefore, the compactness of $[0,T]\times\Omega_0$ implies that it can be covered by q such balls $B((s_i,y_i,u_i),\eta_{(s_i,y_i,u_i)})$. For simplicity, put

$$h_{(s_i,y_i,u_i)} := h_i, \quad \eta_{(s_i,y_i,u_i)} := \eta_i, \quad \eta := \min_{i=1,\dots,q} h_i > 0.$$

Let $(t,x,v) \in [0,T]\times\Omega_0$. Since (t,x,v) belongs to one of the balls $B((s_i,y_i,u_i),\eta_i)$, there exist $x_i \in K$ and $c_i \in F(y_i,u_i)$ such that

$$\begin{aligned}
&\big\|c_i - \frac{2}{h_i^2}(x_i - x - h_i v - \int_t^{t+h_i} f(\tau,x,v)\,d\tau)\big\| \\
&\leq \frac{1}{h_i^2} d_K(x + h_i v + \frac{h_i^2}{2}c_i + \int_t^{t+h_i} f(\tau,x,v)\,d\tau) + \frac{\varepsilon}{4T} \leq \frac{\varepsilon}{2T}.
\end{aligned}$$

Let us set

$$w = \frac{2}{h_i^2}(x_i - x - h_i v - \int_t^{t+h_i} f(\tau,x,v)\,d\tau),$$

then

$$(x + h_i v + \frac{h_i^2}{2}w + \int_t^{t+h_i} f(s\tau,x,v)d\tau) \in K \quad \text{and} \quad \|c_i - w\| \leq \frac{\varepsilon}{2T}.$$

Since $(t,x,v) \in B((s_i,y_i,u_i),\eta_i)$ and $\eta_i < \delta_{(x,v)}$, relation (3.4) implies

$$F(y_i,u_i) \subset F(x,v) + \frac{\varepsilon}{2T}B;$$

so that $w \in F(x,v) + \frac{\varepsilon}{T}B$. Hence the Lemma is proved. $\square$

Now, we are able to prove the main result. Our approach consists of constructing, in a first step, a sequence of approximate solutions and deduce, in a second step, from available estimates that a subsequence converges to a solution of (1.1).

Step 1. Construction of approximate solutions. Let $(x_0,y_0) \in \Omega_0$ and $\varepsilon > 0$. By Lemma 3.2, there exist $\eta > 0$, h_0 in $[\eta,\varepsilon]$ and w_0 in $F(x_0,y_0) + \frac{\varepsilon}{T}B$ such that

$$\big(x_0 + h_0 y_0 + \frac{h_0^2}{2}w_0 + \int_0^{h_0} f(\tau,x_0,y_0)d\tau\big) \in K.$$

Put

$$x_1 = x_0 + h_0 y_0 + \frac{h_0^2}{2}w_0 + \int_0^{h_0} f(\tau,x_0,y_0)d\tau \quad \text{and} \quad y_1 = y_0 + h_0 w_0.$$

Since $w_0 \in F(x_0, y_0) \subset B(0, M+1)$, $\|f(t, x_0, y_0)\| \leq m(t)$, by (3.1), (3.2), we obtain

$$\begin{aligned}
\|x_1 - x_0\| &= \big\|h_0 y_0 + \frac{h_0^2}{2} w_0 + \int_0^{h_0} f(\tau, x_0, y_0)\, d\tau\big\| \\
&\leq T\|y_0\| + \frac{T}{2}\|w_0\| + \|\int_0^{h_0} f(\tau, x_0, y_0)\, d\tau\| \\
&\leq T\|y_0\| + \int_0^T (M + 1 + m(\tau)) d\tau \\
&\leq T\|y_0\| + \frac{r}{3} \leq r,
\end{aligned}$$

and

$$\|y_1 - y_0\| = \|h_0 w_0\| \leq T\|w_0\| < \frac{r}{3} < r$$

and thus $(x_1, y_1) \in \Omega_0$. By induction, for $p \geq 2$ and for every $i = 1, \dots, p-1$, we construct $(h_i, (x_i, y_i), w_i)$ in $[\eta, \varepsilon] \times \Omega_0 \times \mathbb{R}^n$ such that $\sum_{i=0}^{p-1} h_i \leq T$ and

$$\begin{gathered}
x_i = (x_{i-1} + h_{i-1} y_{i-1} + \frac{h_{i-1}^2}{2} w_{i-1} + \int_{h_{i-2}}^{h_{i-2}+h_{i-1}} f(\tau, x_{i-1}, y_{i-1})\, d\tau) \in K; \\
y_i = y_{i-1} + h_{i-1} w_{i-1}; \\
w_i \in F(x_i, y_i) + \frac{\varepsilon}{T} B.
\end{gathered}$$

Since $h_i \in]\eta, \varepsilon[$ there exists an integer s, such that

$$\sum_{i=0}^{s-1} h_i < T \leq \sum_{i=0}^{s} h_i.$$

In what follows, choose ε small such that

$$\sum_{i=0}^{s-1} \frac{h_i^2}{2} \leq \sum_{i=0}^{s-1} h_i < T.$$

For all $p = 1, \dots, s-1$ define $(h_p)_p \subset [\eta, \varepsilon]$, $(x_p, y_p)_p \subset \Omega_0$, and $(w_p)_p$ as follows

$$\begin{gathered}
x_p = (x_{p-1} + h_{p-1} y_{p-1} + \frac{h_{p-1}^2}{2} w_{p-1} + \int_{h_{p-2}}^{h_{p-2}+h_{p-1}} f(\tau, x_{p-1}, y_{p-1})\, d\tau) \in K; \\
y_p = y_{p-1} + h_{p-1} w_{p-1}; \\
w_p \in F(x_p, y_p) + \frac{\varepsilon}{T} B.
\end{gathered}$$

Claim: For $p = 1, \dots, s-1$, the points (x_p, y_p) are in Ω_0.

Indeed, by definition of (x_p, y_p), we have

$$\begin{gathered}
x_p = x_0 + \sum_{i=0}^{p-1} h_i y_i + \sum_{i=0}^{p-1} \frac{h_i^2}{2} w_i + \int_0^{h_0} f(\tau, x_0, y_0)\, d\tau + \sum_{i=1}^{p-1} \int_{\sum_{j=0}^{i-1} h_j}^{\sum_{j=0}^{i} h_j} f(\tau, x_i, y_i)\, d\tau; \\
y_p = y_{p-1} + h_{p-1} w_{p-1}; \\
w_p \in F(x_p, y_p) + \frac{\varepsilon}{T} B.
\end{gathered}$$

Hence

$$\|y_p - y_0\| \leq \|\sum_{i=0}^{p-1} h_i w_i\| \leq T(M+1) \leq \frac{r}{3} \leq r, \tag{3.5}$$

and

$$
\begin{aligned}
&\|x_p - x_0\| \\
&= \Big\| \sum_{i=0}^{p-1} h_i y_i + \sum_{i=0}^{p-1} \frac{h_i^2}{2} w_i + \int_0^{h_0} f(\tau, x_0, y_0)\, d\tau + \sum_{i=1}^{p-1} \int_{\sum_{j=0}^{i-1} h_j}^{\sum_{j=0}^{i} h_j} f(\tau, x_i, y_i)\, d\tau \Big\| \\
&\le (\|y_0\| + \frac{r}{3}) \sum_{i=0}^{p-1} h_i + (M+1) \sum_{i=0}^{p-1} \frac{h_i^2}{2} + \int_0^T m(\tau)\, d\tau.
\end{aligned}
$$

Since

$$
\sum_{i=0}^{p-1} h_i \le T \quad \text{and} \quad \sum_{i=0}^{p-1} \frac{h_i^2}{2} \le T,
$$

by (3.1)–(3.3), we have

$$
\begin{gathered}
\|x_p - x_0\| \le (\|y_0\| + \frac{r}{3})T + \int_0^T (M + 1 + m(\tau))\, d\tau, \\
\|x_p - x_0\| \le T(\|y_0\| + \frac{r}{3}) + \frac{r}{3} \le r;
\end{gathered} \tag{3.6}
$$

hence $(x_p, y_p)_p \subset \Omega_0$ which proves the claim.

For any nonzero integer k and for $q = 1, \dots, s$ denote by h_q^k a real associated to $\varepsilon = \frac{1}{k}$ and $(t, x, y) = (h_{q-1}^k, x_q, y_q)$ given by Lemma 3.2. Let the sequence $(\tau_k^q)_k$ defined by

$$
\begin{gathered}
\tau_k^0 = 0, \quad \tau_k^s = T, \\
\tau_k^q = h_0^k + \dots + h_{q-1}^k;
\end{gathered}
$$

and consider the sequence of functions $(x_k(.))_k$ defined on each interval $[\tau_k^{q-1}, \tau_k^q[$ by

$$
\begin{gathered}
x_k(t) = x_{q-1} + (t - \tau_k^{q-1}) y_{q-1} + \frac{(t - \tau_k^{q-1})^2}{2} w_{q-1} \\
+ \int_{\tau_k^{q-1}}^t (t - \tau) f(\tau, x_{q-1}, y_{q-1}) d\tau; \\
x_k(0) = x_0.
\end{gathered}
$$

Step 2. Convergence of approximate solutions. By the definition of x_k, for all $t \in [\tau_k^{q-1}, \tau_k^q[$ we have

$$
\begin{gathered}
\dot{x}_k(t) = y_{q-1} + (t - \tau_k^{q-1}) w_{q-1} + \int_{\tau_k^{q-1}}^t f(\tau, x_{q-1}, y_{q-1}) d\tau; \\
\ddot{x}_k(t) = w_{q-1} + f(t, x_{q-1}, y_{q-1}).
\end{gathered}
$$

Hence by (3.5) and (3.6) we have the estimates

$$
\|\ddot{x}_k(t)\| \le \|w_{q-1}\| + \|f(t, x_{q-1}, y_{q-1})\| \le M + 1 + m(t); \tag{3.7}
$$

$$
\begin{gathered}
\|\dot{x}_k(t)\| = \|\dot{x}_k(\tau_k^{q-1}) + \int_{\tau_k^{q-1}}^t \ddot{x}_k(\tau)) d\tau\|; \\
\|\dot{x}_k(t)\| \le \|y_{q-1}\| + \Big\| \int_0^T (M + 1 + m(\tau)) d\tau \Big\|
\end{gathered}
$$

$$\leq \|y_{q-1}\| + \frac{r}{3} \leq \|y_0\| + \frac{2r}{3};$$

and

$$\begin{aligned}\|x_k(t)\| &= \|x_k(\tau_k^{q-1}) + \int_{\tau_k^{q-1}}^t \dot{x}_k(\tau))d\tau\| \\ &\leq \|x_{q-1}\| + \int_0^T (\|y_0\| + \frac{2r}{3})d\tau \\ &\leq \|x_0\| + T\|y_0\| + (1 + \frac{2T}{3})r.\end{aligned}$$

By (3.7) one has

$$\int_0^T \|\ddot{x}_k(t)\|^2 dt \leq \int_0^T (M + 1 + m(t))^2 dt.$$

Then the sequence $(\ddot{x}_k(.))_k$ is bounded in $L^2([0,T],\mathbb{R}^n)$ and $(\dot{x}_k(.))_k$ is equiuniformly continuous. Moreover, we see that $(x_k(.))_k$ is equi-Lipschitzian, hence equiuniformly continuous. Therefore, the sequence $(\ddot{x}_k(.))_k$ is bounded in $L^2([0,T],\mathbb{R}^n)$, $(\dot{x}_k(.))_k$ and $(x_k(.))_k$ are bounded in $C([0,T],\mathbb{R}^n)$ and equiuniformly continuous, hence, by [3, Theorem 0.3.4] there exist a subsequence, still denoted by $(x_k(.))_k$ and an absolutely continuous function $x : [0,T] \to \mathbb{R}^n$ such that

(i) x_k converges uniformly to x;
(ii) $\dot{x}_k$ converges uniformly to $\dot{x}$;
(iii) $\ddot{x}_k$ converges weakly in $L^2([0,T],\mathbb{R}^n)$ to $\ddot{x}$.

The family of approximate solutions x_k satisfies the following property.

Proposition 3.3. *For every $t \in [0,T[$ there exits $q \in \{1,\dots,s\}$ such that*

$$\lim_{k\to\infty} d_{grF}\big(x_k(t), \dot{x}_k(t); \ddot{x}_k(t) - f(t, x_k(\tau_k^{q-1}), \dot{x}_k(\tau_k^{q-1}))\big) = 0$$

Proof. Let $t \in [0,T]$. By construction of τ_k^q, there exists $q \in 1,\dots s$ such that $t \in [\tau_k^{q-1}, \tau_k^q[$ and $(\tau_k^q)_k$ converges to t. Moreover, for $q = 1,\dots s$

$$\ddot{x}_k(t) - f(t, x_k(\tau_k^{q-1}), \dot{x}_k(\tau_k^{q-1})) = w_{q-1} \in F(x_k(\tau_k^{q-1}), \dot{x}_k(\tau_k^{q-1})) + \frac{1}{kT}B,$$

then

$$\begin{aligned}&\lim_{k\to\infty} d_{grF}((x_k(t), \dot{x}_k(t)); \ddot{x}_k(t) - f(t, x_k(\tau_k^{q-1}), \dot{x}_k(\tau_k^{q-1}))) \\ &\leq \lim_{k\to\infty}(\|x_k(t) - x_k(\tau_k^{q-1})\| + \|\dot{x}_k(t) - \dot{x}_k(\tau_k^{q-1})\| + \frac{1}{kT}).\end{aligned}$$

Since $\|\ddot{x}_k(t)\| \leq M + 1 + m(t)$, $\|\dot{x}_k(t)\| \leq \|y_0\| + \frac{2r}{3}$ and $(\tau_k^q)_k$ converges to t, it follows that

$$\lim_{k\to\infty} \|x_k(t) - x_k(\tau_k^{q-1})\| = \lim_{k\to\infty} \|\dot{x}_k(t) - \dot{x}_k(\tau_k^{q-1})\| = 0,$$

hence

$$\lim_{k\to\infty} d_{grF}\big((x_k(t), \dot{x}_k(t)); \ddot{x}_k(t) - f(t, x_k(\tau_k^{q-1}), \dot{x}_k(\tau_k^{q-1}))\big) = 0.$$

This completes the proof. □

Since $x_k \to x$ uniformly, $\dot{x}_k \to \dot{x}$ uniformly, $\ddot{x}_k \to \ddot{x}$ weakly in $L^2([0,T],\mathbb{R}^n)$ and $(f(.,x_k(\tau_k^{q-1}),\dot{x}_k(\tau_k^{q-1})))_k$ converges to $f(.,x(.),\dot{x}(.))$ in $L^2([0,T];\mathbb{R}^n)$ and F is upper semicontinuous, then by [3, Theorem 1.4.1], x is a solution of the convexified problem

$$\begin{gathered}\ddot{x}(t) \in f(t,x(t),\dot{x}(t)) + \operatorname{co}(F(x(t),\dot{x}(t))) \text{ a.e. on } [0,T];\\ x(0) = x_0, \quad \dot{x}(0) = y_0.\end{gathered}$$

Consequently for all $t \in [0,T]$ we have

$$\ddot{x}(t) - f(t,x(t),\dot{x}(t)) \in \partial V(\dot{x}(t)) \tag{3.8}$$

Proposition 3.4. *The application x is a solution of* (1.1).

Proof. By (3.8) and [7, Lemma 3.3], we obtain

$$\frac{d}{dt}(V(\dot{x}(t))) = \langle \ddot{x}(t), \ddot{x}(t) - f(t,x(t),\dot{x}(t))\rangle \quad \text{a.e in } [0,T];$$

therefore,

$$V(\dot{x}(T)) - V(y_0) = \int_0^T \|\ddot{x}(\tau)\|^2 d\tau - \int_0^T \langle \ddot{x}(\tau); f(\tau,x(\tau),\dot{x}(\tau))\rangle d\tau. \tag{3.9}$$

On the other hand, for $q = 1,\dots,s$ and $t \in [\tau_k^{q-1},\tau_k^q[$,

$$\Big(\ddot{x}_k(t) - f(t,x_k(\tau_k^{q-1}),\dot{x}_k(\tau_k^{q-1}))\Big) \in F(x_k(\tau_k^{q-1}),\dot{x}_k(\tau_k^{q-1})) + \frac{1}{kT}B.$$

Then

$$\Big(\ddot{x}_k(t) - f(t,x_k(\tau_k^{q-1}),\dot{x}_k(\tau_k^{q-1}))\Big) \in \partial V(\dot{x}_k(\tau_k^{q-1}) + \frac{1}{kT}B,$$

hence, there exists $b_q \in B$ such that

$$\Big(\ddot{x}_k(t) - f(t,x_k(\tau_k^{q-1}),\dot{x}_k(\tau_k^{q-1})) + \frac{1}{kT}b_q\Big) \in \partial V(\dot{x}_k(\tau_k^{q-1}). \tag{3.10}$$

Properties of the subdifferential of a convex function imply that for every z in $\partial V(\dot{x}_k(\tau_k^{q-1})$, we have

$$V(\dot{x}_k(\tau_k^q)) - V(\dot{x}_k(\tau_k^{q-1})) \geq \langle \dot{x}_k(\tau_k^q) - \dot{x}_k(\tau_k^{q-1}); z\rangle. \tag{3.11}$$

Then by (3.10)

$$\begin{aligned}&V(\dot{x}_k(\tau_k^q)) - V(\dot{x}_k(\tau_k^{q-1}))\\ &\geq \langle \dot{x}_k(\tau_k^q) - \dot{x}_k(\tau_k^{q-1}); \ddot{x}_k(t) - f(t,x_k(\tau_k^{q-1}),\dot{x}_k(\tau_k^{q-1})) + \frac{1}{kT}b_q\rangle;\end{aligned}$$

thus

$$\begin{aligned}&V(\dot{x}_k(\tau_k^q)) - V(\dot{x}_k(\tau_k^{q-1}))\\ &\geq \langle \int_{\tau_k^{q-1}}^{\tau_k^q} \ddot{x}_k(\tau)d\tau; \ddot{x}_k(t) - f(t,x_k(\tau_k^{q-1}),\dot{x}_k(\tau_k^{q-1})) + \frac{1}{kT}b_q\rangle\,.\end{aligned}$$

Since $\ddot{x}_k$ is constant in $[\tau_k^{q-1}, \tau_k^q[$, it follows that

$$\begin{aligned} V(\dot{x}_k(\tau_k^q)) - V(\dot{x}_k(\tau_k^{q-1})) \geq & \int_{\tau_k^{q-1}}^{\tau_k^q} \langle \ddot{x}_k(\tau); \ddot{x}_k(\tau) \rangle d\tau \\ & - \int_{\tau_k^{q-1}}^{\tau_k^q} \langle \ddot{x}_k(\tau); f(t, x_k(\tau_k^{q-1}), \dot{x}_k(\tau_k^{q-1})) \rangle d\tau \\ & + \int_{\tau_k^{q-1}}^{\tau_k^q} \langle \ddot{x}_k(\tau); \frac{1}{kT} b_q \rangle d\tau; \end{aligned}$$

hence we have

$$\begin{aligned} & V(\dot{x}_k(T)) - V(y_0) \\ & \geq \int_0^T \|\ddot{x}_k(\tau)\|^2 d\tau - \sum_{q=1}^{s} \int_{\tau_k^{q-1}}^{\tau_k^q} \langle \ddot{x}_k(\tau); f(\tau, x_k(\tau_k^{q-1}), \dot{x}_k(\tau_k^{q-1})) \rangle d\tau \\ & \quad + \sum_{q=1}^{s} \frac{1}{kT} \int_{\tau_k^{q-1}}^{\tau_k^q} \langle \ddot{x}_k(\tau); b_q \rangle d\tau. \end{aligned} \tag{3.12}$$

Claim: The sequence $\big(\sum_{q=1}^s \int_{\tau_k^{q-1}}^{\tau_k^q} \langle \ddot{x}_k(\tau); f(\tau, x_k(\tau_k^{q-1}), \dot{x}_k(\tau_k^{q-1})) \rangle d\tau)_k$ converges to $\int_0^T \langle \ddot{x}(\tau); f(\tau, x(\tau), \dot{x}(\tau)) \rangle d\tau$.

Proof. Since $[0, T] = \bigcup_{q=1}^s [\tau_k^{q-1}, \tau_k^q]$, we have

$$\begin{aligned} & \| \sum_{q=1}^{s} \int_{\tau_k^{q-1}}^{\tau_k^q} \langle \ddot{x}_k(\tau); f(\tau, x_k(\tau_k^{q-1}), \dot{x}_k(\tau_k^{q-1})) \rangle d\tau - \int_0^T \langle \ddot{x}(\tau); f(\tau, x(\tau), \dot{x}(\tau)) \rangle d\tau \| \\ & = \| \sum_{q=1}^{s} \int_{\tau_k^{q-1}}^{\tau_k^q} (\langle \ddot{x}_k(\tau); f(\tau, x_k(\tau_k^{q-1}), \dot{x}_k(\tau_k^{q-1})) \rangle - \langle \ddot{x}(\tau); f(\tau, x(\tau), \dot{x}(\tau)) \rangle) d\tau \| \\ & \leq \sum_{q=1}^{s} \int_{\tau_k^{q-1}}^{\tau_k^q} \| \langle \ddot{x}_k(\tau); f(\tau, x_k(\tau_k^{q-1}), \dot{x}_k(\tau_k^{q-1})) \rangle - \langle \ddot{x}(\tau); f(\tau, x(\tau), \dot{x}(\tau)) \rangle \| d\tau \,. \end{aligned}$$

Since

$$\begin{aligned} & \sum_{q=1}^{s} \int_{\tau_k^{q-1}}^{\tau_k^q} \| \langle \ddot{x}_k(\tau); f(\tau, x_k(\tau_k^{q-1}), \dot{x}_k(\tau_k^{q-1})) \rangle - \langle \ddot{x}(\tau); f(\tau, x(\tau), \dot{x}(\tau)) \rangle \| d\tau \\ & \leq \sum_{q=1}^{s} \int_{\tau_k^{q-1}}^{\tau_k^q} \| \langle \ddot{x}_k(\tau); f(\tau, x_k(\tau_k^{q-1}), \dot{x}_k(\tau_k^{q-1})) \rangle - \langle \ddot{x}_k(\tau); f(\tau, x_k(\tau), \dot{x}_k(\tau)) \rangle \| d\tau \\ & \quad + \sum_{q=1}^{s} \int_{\tau_k^{q-1}}^{\tau_k^q} \| \langle \ddot{x}_k(\tau); f(\tau, x_k(\tau), \dot{x}_k(\tau)) \rangle - \langle \ddot{x}_k(\tau); f(\tau, x(\tau), \dot{x}(\tau)) \rangle \| d\tau \\ & \quad + \sum_{q=1}^{s} \int_{\tau_k^{q-1}}^{\tau_k^q} \| \langle \ddot{x}_k(\tau); f(\tau, x(\tau), \dot{x}(\tau)) \rangle - \langle \ddot{x}(\tau); f(\tau, x(\tau), \dot{x}(\tau)) \rangle \| d\tau \\ & = \sum_{q=1}^{s} \int_{\tau_k^{q-1}}^{\tau_k^q} \| \langle \ddot{x}_k(\tau); f(\tau, x_k(\tau_k^{q-1}), \dot{x}_k(\tau_k^{q-1})) \rangle - \langle \ddot{x}_k(\tau); f(\tau, x_k(\tau), \dot{x}_k(\tau)) \rangle \| d\tau \end{aligned}$$

$$+ \int_0^T \|\langle \ddot{x}_k(\tau); f(\tau, x_k(\tau), \dot{x}_k(\tau))\rangle - \langle \ddot{x}_k(\tau); f(\tau, x(\tau), \dot{x}(\tau))\rangle\| d\tau$$
$$+ \int_0^T \|\langle \ddot{x}_k(\tau); f(\tau, x(\tau), \dot{x}(\tau))\rangle - \langle \ddot{x}(\tau); f(\tau, x(\tau), \dot{x}(\tau))\rangle\| d\tau,$$

it follows that

$$\begin{aligned}
&\Big\| \sum_{q=1}^{s} \int_{\tau_k^{q-1}}^{\tau_k^q} \langle \ddot{x}_k(\tau); f(\tau, x_k(\tau_k^{q-1}), \dot{x}_k(\tau_k^{q-1}))\rangle d\tau - \int_0^T \langle \ddot{x}(\tau); f(\tau, x(\tau), \dot{x}(\tau))\rangle d\tau \Big\| \\
&= \sum_{q=1}^{s} \int_{\tau_k^{q-1}}^{\tau_k^q} \|\langle \ddot{x}_k(\tau); f(\tau, x_k(\tau_k^{q-1}), \dot{x}_k(\tau_k^{q-1}))\rangle - \langle \ddot{x}_k(\tau); f(\tau, x_k(\tau), \dot{x}_k(\tau))\rangle\| d\tau \\
&\quad + \int_0^T \|\langle \ddot{x}_k(\tau); f(\tau, x_k(\tau), \dot{x}_k(\tau))\rangle - \langle \ddot{x}_k(\tau); f(\tau, x(\tau), \dot{x}(\tau))\rangle\| d\tau \\
&\quad + \int_0^T \|\langle \ddot{x}_k(\tau); f(\tau, x(\tau), \dot{x}(\tau))\rangle - \langle \ddot{x}(\tau); f(\tau, x(\tau), \dot{x}(\tau))\rangle\| d\tau\,.
\end{aligned}$$

Since f is a Carathéodory function, x_k and $\dot{x}_k$ are uniformly lipschitz continuous, $\|\ddot{x}_k(s)\| \leq M + 1 + m(s)$, $m \in L^2([0,T], \mathbb{R}^n)$, $x_k \to x$, $\dot{x}_k \to \dot{x}$ uniformly and $\ddot{x}_k \to \ddot{x}$ weakly in $L^2([0,T], \mathbb{R}^n)$ then the last term converges to 0. Hence the claim is proved. □

Since

$$\lim_{k\to\infty} \sum_{q=1}^{s} \frac{1}{k} \int_{\tau_k^{q-1}}^{\tau_k^q} \langle \ddot{x}_k(\tau); b_q\rangle d\tau = 0,$$

by passing to the limit as $k \to \infty$ in (3.12) and using the continuity of the function V on the ball $B(y_0, r)$, we obtain the estimate

$$V(\dot{x}(T)) - V(y_0) \geq \lim_{k\to\infty} \sup \int_0^T \|\ddot{x}_k(\tau)\|^2 d\tau - \int_0^T < \ddot{x}(\tau); f(\tau, x(\tau), \dot{x}(\tau) > d\tau.$$

Moreover, by (3.8), we have

$$\|\ddot{x}\|_2^2 \geq \lim_{k\to\infty} \sup \|\ddot{x}_k\|_2^2,$$

and by the weak lower semicontinuity of the norm, it follows that

$$\|\ddot{x}\|_2^2 \leq \lim_{k\to\infty} \inf \|\ddot{x}_k\|_2^2.$$

Hence $\lim_{k\to\infty} \|\ddot{x}_k\|_2^2 = \|\ddot{x}\|_2^2$, i.e. $((\ddot{x}_k))_k$ converges to $\ddot{x}$ strongly in $L^2([0,T], \mathbb{R}^n)$. So that there exists a subsequence $\ddot{x}_k$ which converges pointwisely almost every where to $\ddot{x}$. In view of Proposition 3.3, we conclude that

$$d_{grF}(x(t), \dot{x}(t), \ddot{x}(t) - f(t, x(t), \dot{x}(t))) = 0 \quad \text{a.e. } t \in [0,T].$$

Since the graph of F is closed, we have

$$\ddot{x}(t) \in f(t, x(t), \dot{x}(t)) + F(x(t), \dot{x}(t)) \text{ a.e. } t \in [0,T].$$

Finally, let $t \in [0,T]$. Recall that there exits $(\tau_k^q)_k$ such that $\lim_{k\to\infty} \tau_k^q = t$ for all $t \in [0,T]$. Since $\lim\limits_{k\to\infty} \|x(t) - x_k(\tau_k^q)\| = 0$, $x_k(\tau_k^q) \in K$ and K is closed, by passing to the limit for $k \to \infty$ we obtain $x(t) \in K$. This completes the proof. □

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