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EXISTENCE, UNIQUENESS AND CONSTRUCTIVE RESULTS FOR DELAY DIFFERENTIAL EQUATIONS

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Abstract. Here, we investigate boundary-value problems (BVPs) for systems of second-order, ordinary, delay-differential equations. We introduce some differential inequalities such that all solutions (and their derivatives) to a certain family of BVPs satisfy some a priori bounds. The results are then applied, in conjunction with topological arguments, to prove the existence of solutions. We then apply earlier abstract theory of Petryshyn to formulate some constructive results under which solutions to BVPs for systems of second-order, ordinary, delay-differential equations are A-solvable and may be approximated via a Galerkin method. Finally, we provide some differential inequalities such that solutions to our equations are unique.

1. Introduction

This paper considers the so-called system of delay-differential equations

$$x''(t) = f(t, x(t), x(h(t)), x'(t)), \quad t \in [0, T], \tag{1.1}$$

subject to the boundary conditions

$$x(t) = \phi(t), \quad t \in [-V, 0], \tag{1.2}$$

$$x(T) = B, \tag{1.3}$$

where $T > 0$, $f : [0, T] \times \mathbb{R}^{3d} \to \mathbb{R}^d$, $V \geq 0$, $h : [0, T] \to [-V, T]$, $t - V \leq h(t) \leq t$, $\phi : [-V, 0] \to \mathbb{R}^d$ and $B \in \mathbb{R}^d$. We call (1.1)–(1.3) a boundary-value problem (BVP) for delay-differential equations. The given functions f, h and ϕ are continuous and we use (1.2) to solve (1.1) forward in time. A solution $x = x(t)$ to (1.1)–(1.3) is a function $x : [-V, T] \to \mathbb{R}^d$ satisfying (1.1) for all $t \in [0, T]$, (1.2) for all $t \in [-V, 0]$ and (1.3) for $t = T$ with

$$x \in S := C\big([-V, T]; \mathbb{R}^d\big) \cap C^2\big([0, T]; \mathbb{R}^d\big).$$

If p is a vector then $\|p\|$ denotes the Euclidean norm of p and $\langle \cdot, \cdot \rangle$ denotes the usual scalar product.

If $\mathcal{I}$ is an interval in $\mathbb{R}$, then we define the notation $\|x\|_{\mathcal{I}}$ by

$$\|x\|_{\mathcal{I}} := \sup_{t \in \mathcal{I}} \|x(t)\|.$$

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Finally, we define the special norm

$$\|x\|_1 := \max\{\|x\|_{[-V,T]}, \|x'\|_{[0,T]}\}.$$

BVPs for delay-differential equations arise, for example, in control theory where variational problems are complicated by the effect of time delays in signal transmission and the lead to a BVP of the type (1.1)–(1.3) (see [11] and references therein).

In the literature, existence, uniqueness and numerical results concerning solutions to BVPs for delay-differential equations appear in [6, 12, 2, 3, 11, 10, 1, 7] and references therein.

The existence results in [6] and [12] use topological transversality involving a priori bounds on solutions. The a priori bound theory involves a Lyapunov-type function, maximum principles and Nagumo-type conditions. The existence results in [2, 11, 3] employ the Schauder-Tychonoff fixed point theorem involving a modification method and a priori bounds on solutions. The a priori bound theory involves upper and lower solutions and Nagumo-type conditions.

The uniqueness results of [3] involve Lipschitz conditions. The numerical results in [10, 1, 7] employ the finite difference method and the shooting method.

In comparison with the above works, the contribution that this paper makes to the field of delay-differential equations is two-fold: The methods used herein are new in the delay-differential equation setting (adapted from [4]); and the results contained herein are quite general.

For example, concerning the existence of solutions, our methods for a priori bounds on solutions do not involve maximum principles, rather they rely on new inequalities in the delay-differential setting of the type

$$\|f(t, x, x_h, p)\| \le 2\alpha[\langle x, f(t, x, x_h, p\rangle + \|p\|^2] + K.$$

In addition, concerning uniqueness of solutions, our methods employ maximum principles, rather than Lipschitz conditions.

Furthermore, our new constructive methods involve A-proper mappings and the Galerkin method applied to delay-differential equations, rather than a finite difference or shooting method.

Additionally, our results apply to systems of equations and therefore are quite general. In fact, it appears that even in the scalar case the results contained herein are new (excluding Theorem 2.1).

For more on the theory of delay-differential equations we refer the reader to [4].

2. A General Existence Theorem

This section contains the general existence theorem that we will rely on throughout the remainder of the paper. The theorem can be found in papers such as [6], [12].

Consider the family of delay-differential equations

$$x''(t) = \lambda f(t, x(t), x(h(t)), x'(t)), \quad t \in [0, T], \tag{2.1}$$

subject to the family of boundary conditions

$$x(t) = \lambda\phi(t), \quad t \in [-V, 0], \tag{2.2}$$

$$x(T) = \lambda B, \tag{2.3}$$

where $\lambda \in [0, 1]$.

Theorem 2.1. *Let f, h and ϕ be continuous functions. If all possible solutions x of (2.1)–(2.3) satisfy $\|x\|_1 < R$ for some positive constant R with R independent of λ, then for each $\lambda \in [0,1]$, the BVP (2.1)–(2.3) has a solution x. In particular, the BVP (1.1)-(1.3) has a solution, for $\lambda = 1$.*

Proof. We provide a proof for two reasons: First, for the convenience of the reader; and second, to introduce some concepts that will be used later in the paper. It can be easily checked that the BVP (2.1)–(2.3) is equivalent to the problem of finding $x \in S$ such that

$$x(t) = \begin{cases} \int_0^T \big[-G(t,s)\lambda f(s,x(s),x(h(s)),x'(s))\big]ds + \lambda k(t), & t \in [0,T], \\ \lambda\phi(t), & t \in [-V,0], \end{cases}$$

where $\lambda \in [0,1]$,

$$G(t,s) := \begin{cases} \frac{(T-t)s}{T}, & 0 \le s \le t \le T, \\ \frac{t(T-s)}{T}, & 0 \le t \le s \le T, \end{cases}$$

and

$$k(t) := \frac{T\phi(0) + (B-\phi(0))t}{T}, \quad t \in [0,T].$$

Define an operator $J : C([-V,T];\mathbb{R}^d) \cap C^1([0,T];\mathbb{R}^d) \to C\big([-V,T];\mathbb{R}^d\big)$ by

$$(Jx)(t) := \begin{cases} \int_0^T \big[-G(t,s) f(s,x(s),x(h(s)),x'(s))\big]ds + k(t), & t \in [0,T], \\ \phi(t), & t \in [-V,0], \end{cases}$$

and define a set

$$\Omega := \Big\{x \in C([-V,T];\mathbb{R}^d) \cap C^1([0,T];\mathbb{R}^d) : \ \|x\|_1 < R\Big\}.$$

Since h, f and ϕ are continuous, we see that for each $\lambda \in [0,1]$, J is continuous and completely continuous (by Arzela-Ascoli Theorem). Thus, $J : \overline{\Omega} \to C\big([-V,0];\mathbb{R}^d\big)$ is a compact map since J is restricted to the closure of a bounded, open set Ω.

Consider the family of problems

$$(I - \lambda J)(x) = 0, \quad \lambda \in [0,1],$$

which is equivalent to (2.1)–(2.3). Since J is compact and $\|x\|_1 < R$ (so $x \notin \partial\Omega$) with R independent of λ, the following Leray-Schauder degrees are defined and a homotopy principle applies

$$\begin{aligned} d(I - \lambda J), \Omega, 0) &= d(I - J, \Omega, 0) \\ &= d(I, \Omega, 0) \\ &= 1 \ne 0, \end{aligned}$$

since $0 \in \Omega$. Therefore,

$$(I - \lambda J)(x) = 0$$

has a solution in $x \in C([-V,T];\mathbb{R}^d) \cap C^1([0,T];\mathbb{R}^d)$ for each $\lambda \in [0,1]$. By elementary methods the solution is also in $C^2([0,T])$. This concludes the proof. $\square$

3. On the equation $x''(t) = f(t, x(t), x(h(t)))$

Theorem 2.1 shows that bounds on solutions to families of BVPs for delay-differential equations play an important role in developing existence results. This section introduces some new inequalities for delay-differential equations such that all solutions x to the family of equations

$$x''(t) = \lambda f(t, x(t), x(h(t))), \quad t \in [0, T], \tag{3.1}$$

subject to the family of boundary conditions

$$x(t) = \lambda\phi(t), \quad t \in [-V, 0], \tag{3.2}$$

$$x(T) = \lambda B, \tag{3.3}$$

where $\lambda \in [0, 1]$, satisfy $\|x\|_1 < R$ with R independent of λ. The result is then applied, in conjunction with Theorem 2.1, to give some existence results for the system of equations

$$x''(t) = f(t, x(t), x(h(t))), \quad t \in [0, T], \tag{3.4}$$

subject to the boundary conditions

$$x(t) = \phi(t), \quad t \in [-V, 0], \tag{3.5}$$

$$x(T) = B. \tag{3.6}$$

Since (3.1) is independent of x', once a priori bounds on x are obtained then a priori bounds on x' naturally follow, with these bounds also independent of λ. Define

$$\beta := \max\big\{\|\phi(0)\|, \|B\|\big\}.$$

Theorem 3.1. *Let f, h and ϕ be continuous. Assume there exist scalar constants $\alpha \geq 0$, $K \geq 0$, such that*

$$\|f(t, w, y)\| \leq 2\alpha\langle w, f(t, w, y)\rangle + K, \quad \text{for all } t \in [0, T],\ (w, y) \in \mathbb{R}^{2d}. \tag{3.7}$$

Then all solutions x of (3.1)–(3.3) satisfy

$$\|x\|_{[-V,T]} \leq \max\big\{\alpha\beta^2 + \beta + \frac{KT^2}{8}, \|\phi\|_{[-V,0]}\big\} := M.$$

Proof. Let $0 \leq \lambda \leq 1$. See that if (3.7) holds, then multiplying both sides by λ, we obtain

$$\|\lambda f(t, w, y)\| \leq 2\alpha\langle w, \lambda f(t, w, y)\rangle + \lambda K \leq 2\alpha\langle w, \lambda f(t, w, y)\rangle + K. \tag{3.8}$$

Now consider the family of BVPs (3.1)-(3.3) and its equivalent integral representation

$$x(t) = \lambda(Jx)(t) := \begin{cases} \int_0^T \big[-G(t, s)\lambda f(s, x(s), x(h(s)))\big]ds + \lambda k(t), & t \in [0, T], \\ \lambda\phi(t), & t \in [-V, 0], \end{cases} \tag{3.9}$$

where G and k are given in the proof of Theorem 2.1. Since $\lambda \in [0, 1]$, it is easy to see that

$$\|x\|_{[-V,0]} \leq \|\phi\|_{[-V,0]}.$$

Now since $\lambda \in [0,1]$ and $G \geq 0$, taking norms in (3.9) and using (3.8) we obtain

$$\begin{aligned}
\|x(t)\| &\leq \int_0^T G(t,s)\|\lambda f(s,x(s),x(h(s)))\|ds + \|\lambda k(t)\|, \quad t \in [0,T], \\
&\leq \int_0^T G(t,s)\big[2\alpha\langle x(s), \lambda f(s,x(s),x(h(s)))\rangle + K\big]ds + \beta, \\
&\leq \int_0^T G(t,s)\big[2\alpha\langle x(s), \lambda f(s,x(s),x(h(s)))\rangle + 2\alpha\|x'(s)\|^2 + K\big]ds + \beta, \\
&= \int_0^T \big(G(t,s)\big[\alpha[\|x(s)\|^2\big]'' + K\big)ds + \beta, \\
&\leq \alpha \int_0^T G(t,s)[\|x(s)\|^2]''ds + \frac{KT^2}{8} + \beta,
\end{aligned}$$

where we have used the identity

$$[\|x(t)\|^2]'' = 2\langle x(t), x''(t)\rangle + 2\|x'(t)\|^2, \tag{3.10}$$

and the inequality

$$\int_0^T G(t,s)ds \leq \frac{T^2}{8}, \quad t \in [0,T].$$

The above inequality is readily obtained since the explicit form of G is known. Continuing to employ the explicit form of G,

$$\begin{aligned}
\|x(t)\| &\leq \frac{(T-t)\alpha}{T} \int_0^t s[\|x(s)\|^2]''ds, \\
&\quad + \frac{t\alpha}{T}\int_t^T (T-s)[\|x(s)\|^2]''ds + \frac{KT^2}{8} + \beta, \\
&= I_1 + I_2 + \frac{KT^2}{8} + \beta,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &:= \frac{(T-t)\alpha}{T} \int_0^t s[\|x(s)\|^2]''ds, \\
I_2 &:= \frac{t\alpha}{T}\int_t^T (T-s)[\|x(s)\|^2]''ds.
\end{aligned}$$

A simple integration by parts on I_1 and I_2 gives

$$\begin{aligned}
I_1 &= \big(\frac{T-t}{T}\big)\alpha[t(\|x(t)\|^2)' - \|x(t)\|^2 + \|x(0)\|^2], \\
I_2 &= \frac{t\alpha}{T}[-(T-t)(\|x(t)\|^2)' + \|x(T)\|^2 - \|x(t)\|^2].
\end{aligned}$$

Therefore, adding I_1 and I_2 and noting some cancellation of terms and the non-negativity of $\|x(t)\|^2$ we obtain,

$$\begin{aligned}
I_1 + I_2 &\leq \alpha\big[\big(\frac{T-t}{T}\big)\|x(0)\|^2 + \frac{t}{T}\|x(T)\|^2\big], \\
&\leq \alpha\big[\big(\frac{T-t}{T}\big)\beta^2 + \frac{t}{T}\beta^2\big] = \alpha\beta^2.
\end{aligned}$$

Hence

$$\|x\|_{[-V,T]} \leq \max\Big\{\alpha\beta^2 + \beta + \frac{KT^2}{8}, \|\phi\|_{[-V,0]}\Big\} = M.$$

This concludes the proof. □

Theorem 3.2. *Let the conditions of Theorem 3.1 hold. Then the BVP* (3.1)–(3.3) *has a solution for each* $\lambda \in [0,1]$*. In particular, the BVP* (3.4)–(3.6) *has a solution.*

Proof. From Theorem 3.1 there exists an $M \geq 0$ such that $\|x\|_{[-V,T]} \leq M$ with M independent of λ. The a priori bound on x' now naturally follows by differentiating the first line of (3.9) and taking norms. This bound $P \geq 0$ on x' will depend on M and is independent of λ. Hence there is a constant $R > 0$ such that

$$\|x\|_1 < \max\{M, P\} + 1 =: R,$$

and by Theorem 2.1 the result follows. This concludes the proof. □

The proof of the next result is immediate from Theorem 3.2 and will be needed to develop constructive theory in Section 5.

Corollary 3.3. *Consider the BVP* (3.1)–(3.3) *with* $\phi(0) = 0 = B$*. If the conditions of Theorem 3.2 hold then the BVP* (3.1)–(3.3) *has a solution* x*, for each* $\lambda \in [0,1]$ *and the bound on* x *is given by*

$$\|x\|_{[-V,T]} \leq \max\big\{\frac{KT^2}{8},\ \|\phi\|_{[-V,0]}\big\}.$$

Corollary 3.4. *Let* h*,* f *and* ϕ *be continuous, scalar-valued* ($d = 1$) *functions. Assume there exist constants* $\alpha \geq 0, K \geq 0$ *such that*

$$|f(t,w,y)| \leq 2\alpha w f(t,w,y) + K, \quad \text{for all } t \in [0,T], (w,y) \in \mathbb{R}^2. \tag{3.11}$$

Then the BVP (3.1)–(3.3) *has a solution* x *for each* $\lambda \in [0,1]$*. In particular,* (3.4)–(3.6) *has a solution.*

The result in the above corollary is a special case of Theorem 3.2.

We now present some examples to illustrate the theory of this section.

Example 3.5. Consider the scalar BVP

$$x''(t) = t[x(t)]^3 \exp\{-[x(h(t))]^2\}, \quad t \in [0,T],$$
$$x(t) = 1, \quad t \in [-V,0],\ x(T) = 0.$$

See that $|w|^3 \leq w^4 + 1$ for all w, so multiplying both sides of this inequality by $t\exp\{-[y]^2\}$ we obtain for $\alpha = 1/2$ and $K = T$,

$$\begin{aligned} |f(t,w,y)| &\leq t\exp\{-[y]^2\}(w^4+1) \\ &= wf(t,w,y) + t\exp\{-[y]^2\} \\ &\leq wf(t,w,y) + T. \end{aligned}$$

Hence (3.11) will hold for the choices $\alpha = 1/2$ and $K = T$. Therefore, by Corollary 3.4, the BVP will have a solution.

Example 3.6. Consider the BVP

$$x''(t) = t^2\cos(x(t))\exp\{-x(h(t))\}, \quad t \in [0,1],$$
$$x(t) = \phi(t), \quad t \in [-2,0],\ x(1) = 0.$$

See that for $\alpha = 0$ and $K = 1$,

$$|f(t,w,y)| = |t^2\cos(w)\exp\{-[y]^2\}|, \leq 1 = 2\alpha w f(t,w,y) + K.$$

Hence (3.11) will hold for the choices $\alpha = 0$ and $K = 1$. Therefore, by Corollary 3.4, the BVP will have a solution.

Note that in both examples, h continuous and $t-T \le h(t) \le t$ is sufficient. Also note that (3.7) is useful in the case that f is a polynomial in w and bounded in y.

4. On the equation $x''(t) = f(t, x(t), x(h(t)), x'(t))$

In this section we consider the class of BVPs for delay-differential equations (1.1)–(1.3) and its corresponding family of problems (2.1)–(2.3). Since f depends on x' the a priori bounds on x of Section 3 may not directly imply bounds on x'. Therefore we need to impose additional assumptions to obtain these bounds on x'.

Theorem 4.1. *Let h, f and ϕ be continuous. Assume there exist constants $\alpha \ge 0, K \ge 0$ such that*

$$\|f(t,w,y,z)\| \le 2\alpha\Big[\langle w, f(t,w,y,z)\rangle + \|z\|^2\Big] + K, \tag{4.1}$$

for all $t \in [0,T]$, $(w,y,z) \in \mathbb{R}^{3d}$. Then all solutions x of (2.1)–(2.3) satisfy $\|x\|_{[-V,T]} \le M$ where M is independent of $\lambda \in [0,1]$ and is defined in the proof of Theorem 3.1.

The proof of the above theorem is almost identical to that of Theorem 3.1 and so is omitted.

Theorem 4.2. *Let the conditions of Theorem 4.1 hold. If, in addition, $2\alpha M < 1$ then, for all solutions x of (2.1)–(2.3),*

$$\|x'\|_{[0,T]} \le \frac{M[2\alpha M + KT^2/8]}{T(1-2\alpha M)/2} := N,$$

with N independent of $\lambda \in [0,1]$.

Proof. Let x be a solution of (2.1)–(2.3). For $t \in [0, \frac{T}{2}]$ apply Taylor's formula to obtain

$$x(t+\frac{T}{2}) - x(t) - \frac{T}{2}x'(t) = \int_t^{t+\frac{T}{2}} (t+\frac{T}{2}-s)x''(s)ds.$$

Thus,

$$-\frac{T}{2}x'(t) = x(t) - x(t+\frac{T}{2}) + \int_t^{t+\frac{T}{2}} (t+\frac{T}{2}-s)\lambda f(s,x(s),x(h(s)),x'(s))ds.$$

Now taking norms and using: $\lambda \in [0,1], \|x\|_{[0,T]} \le M, t+\frac{T}{2}-s \ge 0$, (3.10) and (4.1) we obtain

$$\begin{aligned}\frac{T}{2}\|x'(t)\| &\le 2M + \int_t^{t+\frac{T}{2}} (t+\frac{T}{2}-s)\{\alpha[\|x(s)\|^2]'' + K\}ds, \quad t \in [0,\frac{T}{2}]\\ &= 2M + \alpha\int_t^{t+\frac{T}{2}} (t+\frac{T}{2}-s)[\|x(s)\|^2]''ds + K\int_t^{t+\frac{T}{2}} (t+\frac{T}{2}-s)ds.\end{aligned}$$

Using Taylor's formula once more for $[\|x(s)\|^2]''$ we obtain

$$\begin{aligned}\frac{T}{2}\|x'(t)\| &\le 2M + \alpha\Big[\|x(t+\frac{T}{2})\|^2 - \|x(t)\|^2 - \frac{T}{2}[\|x(t)\|^2]'\Big] + \frac{KT^2}{8}, \quad t \in [0,\frac{T}{2}],\\ &\le 2M + \alpha\Big[M^2 - T\langle x(t), x'(t)\rangle\Big] + \frac{KT^2}{8}\\ &\le 2M + \alpha\Big[M^2 + MT\|x'(t)\|\Big] + \frac{KT^2}{8}.\end{aligned}$$

So rearranging we have

$$\|x'\|_{[0,\frac{T}{2}]} \le \frac{M(2+\alpha M)+KT^2/8}{T(1-2\alpha M)/2} := N,$$

with N independent of $\lambda \in [0,1]$. For $t \in [\frac{T}{2}, T]$ we use the Taylor formula

$$x(t) - x(t-\frac{T}{2}) - \frac{T}{2}x'(t) = -\int_{t-\frac{T}{2}}^{t} (t-\frac{T}{2}-s)x''(s)ds.$$

By arguing in a similar fashion to the case $t \in [0, \frac{T}{2}]$ we obtain $\|x'\|_{[\frac{T}{2},\, T]} \le N$ with N defined above and independent of $\lambda \in [0,1]$. This concludes the proof. □

Theorem 4.3. *Let the conditions of Theorem 4.1 hold. If $2\alpha M < 1$ (with M and α defined in Theorem 3.1) then (2.1)–(2.3) has a solution x for all $\lambda \in [0,1]$. In particular, (1.1)–(1.3) has a solution.*

Proof. The conditions of Theorem 4.1 guarantee the existence of constants M, N such that

$$\|x\|_1 < \max\{M, N, \|\phi\|_{[-V,0]}\} + 1 := R.$$

with R independent of $\lambda \in [0,1]$. Now applying Theorem 2.1 the result follows and this concludes the proof. □

The proof of the next result is immediate from Theorem 4.2.

Corollary 4.4. *Consider the BVP (2.1)–(2.3) with $\phi(0) = 0 = B$. If the conditions of Theorem 3.2 hold then the BVP (3.1)-(3.3) has a solution for each $\lambda \in [0,1]$ and the bound on x is given by*

$$\|x\|_{[-V,T]} \le \max\big\{\frac{KT^2}{8}, \|\phi\|_{[-V,0]}\big\}.$$

Corollary 4.5. *Let h, f and ϕ be continuous, scalar-valued (d=1) functions such that*

$$|f(t,w,y,z)| \le 2\alpha\Big[wf(t,w,y,z) + |z|^2\Big] + K, \text{ for all } t \in [0,T],\ (w,y,z) \in \mathbb{R}^3,$$

and for some constants $\alpha \ge 0, K \ge 0$. Then the scalar BVP (2.1)–(2.3) has a solution x for each $\lambda \in [0,1]$. In particular, (1.1)–(1.3) has a solution.

The above corollary is a special case of Theorem 4.3.

5. Some Constructive Results

In this section we develop some constructive results for the BVP (1.1)–(1.3). The ideas rely on the a priori bounds on solutions of previous sections and an abstract result due to Petryshyn [9]. To apply these results, we consider homogeneous boundary conditions $\phi(0) = 0$, $B = 0$.

We introduce the following notation so we can readily apply Petryshyn's abstract result. Let X and Y denote real Banach spaces. let $L : D(L) \subset X \to Y$ denote a Fredholm map of index 0; in particular, L is linear. Let Null(L) denote be the null space of L and let Rank(L) denote the rank of L. Let $P : D(P) \subset X \to Y$ denote a nonlinear map.

Let $\{X_n\} \subset X, \{Y_n\} \subset Y$, be sequences of finite-dimensional spaces and for each $n \in \mathbb{Z}^+$ let $Q_n : Y \to Y_n$ denote a linear projection. Define the scheme $\Gamma = \{X_n, Y_n, Q_n\}$ as **admissible** for maps $X \to Y$ provided:

$\dim X_n = \dim Y_n$, for each n, and
$\operatorname{dist}(x, X_n) := \inf\{\|x - v\|_X : v \in X_n\} \to 0$ as $n \to \infty$, for each $x \in X_n$.

For given maps L and P, the equation

$$Lx = Px, \quad x \in D(L) \cap D(P),$$

is said to be strongly (feebly) **A-solvable** with respect to Γ if there exists an $N_0 \in \mathbb{Z}^+$ such that the finite dimensional equation

$$Q_n Lx = Q_n Px, \quad x \in (D(L) \cap D(P)) \cap X_n,$$

has a solution $x \in D(L) \cap D(P)) \cap X_n$ for each $n \geq N_0$ such that $x_n \to x \in X$, $(x_{n_j} \to x \in X)$ and $Lx = Px$.

If the equation $Lx = Px$ is strongly A-solvable then we follow the lead of Petryshyn and say that the Galerkin method applies.

The mapping $L - P : D(L) \cap D(P) \subset X \to Y$ is said to be **A-proper** with respect to Γ if

$$Q_n L - Q_n P : (D(L) \cap D(P)) \cap X_n \subset X_n \to Y_n$$

is continuous for each $n \in \mathbb{Z}^+$ and if $\{x_{n_j} : x_{n_j} \in (D(L) \cap D(P)) \cap X_n\}$ is any bounded sequence in X such that

$$Q_{n_j} Lx_{n_j} - Q_{n_j} Px_{n_j} \to 0 \quad \text{in} Y,$$

then there is a subsequence $\{x_{n_k}\}$ of $\{x_{n_j}\}$ and $x \in D(L) \cap D(P)$ such that

$$x_{n_k} \to x \quad \text{in } X \quad \text{and} \quad Lx = Px.$$

Since L is a Fredholm map of index zero, there exists closed subspaces $X_1 \subset X$ and $Y_2 \subset Y$ such that $X = \operatorname{Null}(L) \oplus X_1$ and $Y = Y_2 \oplus \operatorname{Ran}(L)$. Let Q be the linear projection of Y onto Y_2 and assume there exists a continuous bilinear form $[\cdot, \cdot]$ on $Y \times X$ mapping (y, x) into $[y, x]$ such that

$$y \in \operatorname{Ran}(L) \text{ iff } [y, x] = 0, \quad \text{for all } x \in \operatorname{Null}(L).$$

We first need the following result of Petryshyn [9, Theorem A], .

Theorem 5.1. *Let L be a Fredholm map of index zero. Assume* $\operatorname{Null}(L) = \{0\}$. *Assume there exists a bounded open ball $G \subset X$ with $0 \in G$ such that*

- (a) *$P(\bar{G})$ is bounded,*
- (b) *$L - \lambda P : \overline{G} \to Y$ is A-proper w.r.t. Γ for each $\lambda \in [0, 1]$,*
- (b) *$Lx \neq \lambda Px$ for $x \in \partial G$ and $\lambda \in (0, 1]$.*

Then $L - P$ is feebly A-solvable with respect to Γ. In particular the BVP (2.1)–(2.3) *has a solution x. If that solution x is unique in G, then $L - P$ is strongly A-solvable with respect to Γ and the Galerkin method is applicable to the BVP* (2.1)–(2.3).

Remark 5.2. Petryshyn's Theorem A is more general than Theorem 5.1 (see [9, Remark 1.2,]). We assume $\operatorname{Null}(L) = \{0\}$ both for simplicity of statement and for the specific application with $\phi(0) = 0, B = 0$.

Theorem 5.3. *Let the conditions of Theorem 4.3 hold. Then the BVP* (1.1)–(1.3) *(with $\phi(0) = 0$) is feebly $A-$solvable with respect to Γ. If x is the unique solution then the BVP* (1.1)–(1.3) *(with $\phi(0) = 0$) is strongly $A-$solvable with respect to Γ. That is, the Galerkin Method is applicable.*

Proof. Let

$$Lx := \begin{cases} x''(t), & t \in [0,T] \\ x(t), & t \in [-V,0]; \end{cases}$$

$$Px := \begin{cases} f(t,x(t),x(h(t)),x'(t)), & t \in [0,T] \\ \phi(t), \quad t \in [-V,0]; \end{cases}$$

$$X := \Big\{ x \in C([-V,T];\mathbb{R}^d) \cap C^1([0,T];\mathbb{R}^d) : x(T) = x(0) = 0,\ x = \phi \text{ on } [-V,0] \Big\};$$

$$Y := C([-V,T];\mathbb{R}^d).$$

Set

$$G = \{x \in X : \|x\|_1 < \max\{M, N, \|\phi\|_{[-V,0]}\} + 1\},$$

where M and N are given in the proof of Theorem 4.3.

It is easy to see that $P(\bar{G})$ is bounded. Now L is A-proper with respect to Γ by [8]. Since $P : X \to Y$ is continuous see that N is also completely continuous because X is compactly embedded into $C^1([0,T];\mathbb{R}^d)$ and therefore P is A-proper with respect to Γ (see [9]). Hence $L - \lambda P : \overline{G} \to Y$ is A-proper with respect to Γ for each $\lambda \in [0,1]$.

Finally, we see that

$$Lx \neq \lambda Px \quad \text{for all } x \in \partial\Omega \text{ and all } \lambda \in [0,1],$$

since the a priori bound theory of Theorems 4.1 and 4.2 is applicable. Therefore, all of the conditions of Theorem 5.1 are satisfied and the result follows. This concludes the proof. □

6. On Uniqueness of Solutions

This brief section provides some results which guarantee the uniqueness of solutions to the BVPs for delay differential equations. Our interest here is twofold. Firstly, the constructive results of Section 5 rely on uniqueness of solutions. Secondly, BVPs with deviating arguments can introduce solutions which do not appear for the "associated" non-deviating BVP (see [3]).

Theorem 6.1. *If f satisfies*

$$\langle u - v, f(t,u,z,u') - f(t,v,w,v') \rangle > 0, \quad \textit{for all } t \in [0,T], \tag{6.1}$$

and $u, z, u', v, w, v' \in \mathbb{R}^d$ with $u \neq v$, and $\langle u - v, u' - v' \rangle = 0$, then (1.1) *has, at most, one solution satisfying* (1.2)-(1.3)*.*

Proof. Assume u and v are solutions to the BVP (1.1)–(1.3). Then $u - v$ satisfies the BVP

$$u''(t) - v''(t) = f(t,u(t),u(h(t)),u'(t)) - f(t,v(t),v(h(t)),v'(t)), \quad t \in [0,T],$$

$$u(t) - v(t) = 0, \quad t \in [-V,0],$$

$$u(T) - v(T) = 0.$$

Consider $r(t) := \|u(t) - v(t)\|^2$, $t \in [-V,T]$. Now r must have a positive maximum at some point $c \in [-V,T]$. From the boundary conditions, $c \in (0,T)$. Therefore, by a maximum principle we must have

$$r'(c) = 0, \quad r''(c) \leq 0. \tag{6.2}$$

So using the product rule on r we have

$$r''(c) \geq 2\big\langle u(c) - v(c), f(c, u(c), u(h(c)), u'(c)) - f(c, v(c), v(h(c)), v'(c))\big\rangle > 0,$$

which contradicts (6.2). Therefore $r(t) = \|u(t) - v(t)\|^2 = 0$ for all $t \in [-V, T]$, and solutions of the BVP (1.1)–(1.3) must be unique. $\square$

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