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ILL-POSEDNESS OF THE CAUCHY PROBLEM FOR TOTALLY DEGENERATE SYSTEM OF CONSERVATION LAWS

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Abstract. In this paper we answer some open questions concerning totally degenerate systems of conservation laws. We study the augmented Born-Infeld system, which is the Born-Infeld model augmented by two additional conservations laws. This system is a nice example of totally degenerate system of conservation laws and, global smooth solutions are conjectured to exist when the initial-data is smooth. We show that this conjecture is false, for the more natural and general condition of initial-data. In fact, first we show that does not exist global smooth solution for any 2×2 totally degenerated system of conservation laws, which the characteristics speeds do not have singular points. Moreover, we sharpen the conjecture in Majda [20]. Under the same hypothesis of initial-data, we show that the Riemann Problem is not well-posed, which follows for weak solutions of the Cauchy Problem. In the end, we prove some results on the direction of well-posedness for the less physically initial-data.

1. Introduction

In this paper, we are interested in the well-posedness condition of the initial-data problem for the Augmented Born-Infeld equations, also called ABI, as introduced in Brenier [5]. Due to the linear degeneracy of the BI and ABI systems and they very peculiar structure, it seems reasonable to conjecture that both systems admit global in time smooth solutions for any smooth initial data. We show that this conjecture is false, for the most physical important and general initial-data condition for totally linear degenerated systems (see Definition 1.4), which is the ABI and BI case. In fact, this result follows from Theorem 2.1, which says that any 2×2 totally linearly degenerated system of conservation law (which the propagation speeds do not have singular points), does not have global smooth solutions for smooth initial-data of this type. Moreover, the catastrophe appears in the Lipschitz norm for some non-conservative variables and in the sup norm for conservative ones. Therefore, in some sense we sharpen Majda's conjecture. Under the same hypothesis of initial-data, i.e. the more natural one, we show that the Riemann Problem is not well-posed, that is, there does not exist solution. The latter, implies that the Cauchy Problem is not well-posed, at least considering the only existence result for systems of conservation laws (on only one space variable) obtained by the Glimm's scheme, which does not

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work here. In the end, we prove some results of well-posedness and stability of the Cauchy Problem for the less physically condition of initial-data.

Usually in Continuum Physics a truncation process is established in order to simplify the system of balance laws. By dropping some equations and reducing the size of the variables we obtain a simpler system. This process is so good as the entropy structure is not changed and, the number of wave speeds does not increase. It is not the case for the problem of elastodynamics in nonlinear elasticity, obtained by truncation from the thermoelastic one, see Dafermos [12]. So to compensate the lack of convexity in the stored energy function, Dafermos propose an alternative approach, see also [15], which is to embed the original system of elastodynamics into a larger one, augmented by two additional conservation laws. Now, the enlarged system endowed a uniformly convex entropy, so the initial-value problem is locally well-posed for classical solutions, i.e. initial-data in H^s, with $s > 1 + d/2$, where d is the spatial dimension. Further, we have uniqueness and continuously dependence on the initial-data for a broader class of weak solutions, for instance, in the class of Riemann Problems, see [13, 14], also Bressan et al. [6, 7, 8, 9] in the BV case.

In [5], Brenier has done the same procedure for the BI system, which is the most famous model for nonlinear Maxwell's equations, see [4]. He embed the original BI model into a large one (the ABI), augmented by two additional conservation laws by using the stored energy function and the Poynting vector as two additional independent variables. So the ABI model posses a uniformly convex entropy and thus, the comment above takes place here. As the ABI model keeps the linear degeneracy condition of the BI model, Brenier admits the existence of solutions on sufficiently large time intervals and, his goal was in direction of asymptotic analysis, that is, he provided some mathematical confirmation that the BI model establishes a nonlinear transition between wave particle behaviors according to the intensity of the electromagnetic field. In fact, at least for small (smooth)initial data, there exists global smooth solution for the BI model. This result was obtained by Chae and Huh [10].

In this paper, since the Cauchy problem is locally well-posed for classical solutions, the main question is to investigate if the totally linear degeneracy condition implies global existence for any smooth initial-data. Moreover, since we have uniqueness and continuous dependence for the Riemann problem, the fundamental question in this case is to prove existence of solutions. As we shall see, the results obtained here derive from the fact that, spite of the name, totally linear degenerate systems are not simpler in their structure, nor easier to understand, under the pretext that the linear one is less complicated, see [23]. In fact, it is the contrary, for instance linear degenerated fields can lead to solutions which display large oscillations even if of hight frequency. Therefore, it is not natural nor general to assume that, the initial-data has sufficiently small oscillations, indeed, more physically correct is to assume that it really has oscillations.

Finally, we mention the results obtained in [21]. They were able to show that, shocks could appear in the solution of the Riemann problem for BI model. Hence, we prove that the Rankine-Hugoniot condition of the ABI system is not equivalent to the BI one. Moreover, the BI model is not complete by himself and thus, it must be augmented by some selection criteria. Clearly, one of them is the ABI system, see Theorem 3.8, but [24] proposed a different BI enlarged system.

Let us consider the ambient space $\mathbb{R}^3$, so by $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3$ we denote the points in the time-space domain. We consider as independent variables, the electric induction D, the magnetic induction B, the Pointing vector P, all of them taking values in $\mathbb{R}^3$ and, the positive energy function h. Thus the ABI model as proposed in [5] is the following system of equations

$$\partial_t D + \operatorname{curl}_x \left(\frac{-B + D \times P}{h} \right) = 0, \tag{1.1}$$

$$\partial_t B + \operatorname{curl}_x \left(\frac{D + B \times P}{h} \right) = 0, \tag{1.2}$$

$$\operatorname{div}_x D = 0, \quad \operatorname{div}_x B = 0, \tag{1.3}$$

$$\partial_t h + \operatorname{div}_x P = 0, \tag{1.4}$$

$$\partial_t P + \operatorname{div}_x \big(\frac{P \otimes P - D \otimes D - B \otimes B}{h} \big) = \nabla_x \big(\frac{1}{h} \big). \tag{1.5}$$

Equations (1.1), (1.2) come from the Ampere and Faraday's Law respectively, equation (1.3) are compatible constrains, and (1.4), (1.5) are the two additional kinematically induced equations.

Remark 1.1. If the two additional variables, that is, h and P satisfy

$$h = \sqrt{1 + |D|^2 + |B|^2 + |D \times B|^2}, \quad P = D \times B, \tag{1.6}$$

at the initial time, then it remains true for any time when there exists a smooth solution to (1.1)-(1.5). In this sense, we identify the 6 dimensional (algebraic) submanifold of $\mathbb{R}^{10}$, i.e. (1.1)-(1.5) where h, P are given by (1.6), with the original BI model. Moreover, we note that, the smooth solution of the BI model, i.e. (1.1)-(1.3) implies the solution of (1.4), (1.5).

The variables of the ABI model, that is D, B, h and P, satisfy an additional conservation law,

$$\partial_t \eta + \operatorname{div}_x \frac{(\eta h - 1)P + D \times B - (D \otimes D + B \otimes B)P}{h^2} = 0, \tag{1.7}$$

where η is the following uniformly convex function

$$\eta(D,B,h,P) = \frac{1 + |D|^2 + |B|^2 + |P|^2}{2h}.$$

Therefore, η is a convex entropy for the ABI system, see Definition 1.3.

Now, as observed for the Born-Infeld model, see [21], the ABI system itself posses the wave isotropy condition and, we focus on plane waves. Hence, only a single spatial variable is needed. Let us choose $x = x_1$, as a such spatial coordinate. Thus all the fields involved in (1.1)-(1.5) depend on $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$. Moreover, it follows that

$$\begin{gathered} \partial_t D_1 = 0, \quad \partial_x D_1 = 0, \\ \partial_t B_1 = 0, \quad \partial_x B_1 = 0. \end{gathered}$$

Therefore, D_1, B_1 are constant functions and, for simplicity we assume $D_1 = B_1 = 0$. Then, from (1.1)-(1.5), we obtain the following system of conservation laws

$$\partial_t D_2 + \partial_x \big(\frac{B_3 + D_2 P_1}{h} \big) = 0, \tag{1.8}$$

$$\partial_t D_3 + \partial_x\Big(\frac{-B_2 + D_3 P_1}{h}\Big) = 0, \tag{1.9}$$

$$\partial_t B_2 + \partial_x\Big(\frac{-D_3 + B_2 P_1}{h}\Big) = 0, \tag{1.10}$$

$$\partial_t B_3 + \partial_x\Big(\frac{D_2 + B_3 P_1}{h}\Big) = 0, \tag{1.11}$$

$$\partial_t h + \partial_x P_1 = 0, \tag{1.12}$$

$$\partial_t P_1 + \partial_x\Big(\frac{P_1^2 - 1}{h}\Big) = 0, \tag{1.13}$$

$$\partial_t P_2 + \partial_x\Big(\frac{P_1 P_2}{h}\Big) = 0, \tag{1.14}$$

$$\partial_t P_3 + \partial_x\left(\frac{P_1 P_3}{h}\right) = 0. \tag{1.15}$$

Next, we present some mathematical considerations for systems of conservation laws. Let U be an open subset of $\mathbb{R}^n$ and, let $f : U \to \mathbb{R}^n$ be a continuously differentiable map. For some $T \in \mathbb{R}^+$, we consider the following system os conservation laws in one space dimension

$$\operatorname{div}_{t,x}(u, f(u)) \equiv \partial_t u + \partial_x f(u) = 0 \quad (t,x) \in (0,T) \times \mathbb{R}, \tag{1.16}$$

where $u : (0,T) \times \mathbb{R} \to U$ is the unknown and f is given. The set U is called the set of states and the map f the flux-function. We are concerned with initial-value problem, that is, we seek $u(t,x) \in U$ solution of (1.16) and that satisfies an initial-data

$$u(0,x) = u_0 \quad x \in \mathbb{R}, \tag{1.17}$$

where $u_0 : \mathbb{R} \to U$ is a given bounded measurable function. As is well-known, in general for conservation laws there does not exist (global) solutions, even if the data is infinitely differentiable. Consequently, the theory of conservation laws is developed with the concept of weak solutions. The following definition tell us in which sense a bounded mensurable function, is a weak solution of (1.16), (1.17).

Definition 1.2. We say that $u \in L^\infty((0,T) \times \mathbb{R}; U)$ is a weak solution of (1.16), (1.17) if it satisfies

$$\int_0^T\!\!\int_{\mathbb{R}} (u, f(u)) \cdot \nabla_{t,x}\phi(t,x)\, dxdt + \int_{\mathbb{R}} u_0\, \phi(0,x)\, dx = 0, \tag{1.18}$$

for any function $\phi \in C_0^\infty(\mathbb{R}^2)$.

By definition a weak solution is a distributional solution. Moreover, if $u \in L^\infty$ is a C^1 function outside a manifold Γ (with codimension one), across which it has jump discontinuities, then it can be shown using (1.18), see [12, 23], that u must satisfy the so called Rankine-Hugoniot condition

$$n_t[u] + n_x[f(u)] = 0,$$

where $n = (n_t, n_x)$ is the outward unit normal vector along the manifold Γ, $[u] := u^+ - u^-$, $[f(u)] := f(u^+) - f(u^-)$, and

$$u^+ = \lim_{\delta \to 0^+} u((t,x) + \delta n), \quad u^- = \lim_{\delta \to 0^+} u((t,x) - \delta n).$$

Definition 1.3. A real Lipschitz function η is called an entropy for (1.16), with associated entropy flux $q \in W^{1,\infty}(U)$, when for every open set $\Pi \subset (0,T) \times \mathbb{R}$ and for every $u \in C^1$, which solves (1.16) pointwise, we have

$$\partial_t \eta(u) + \partial_x q(u) = 0 \quad \text{in } \mathcal{D}'(\Pi).$$

If, in addition η is a convex function, then we say that (η, q) is a convex entropy pair. Moreover, a weak solution of (1.16), (1.17) is called an entropy solution, when $\partial_t \eta(u) + \partial_x q(u) \leq 0$ in the sense of distributions for every convex entropy pair.

We recall that, see [12, 23], a system of conservation laws is said hyperbolic, when for any $v \in U$, the matrix of entries

$$A_{i,j}(v) := \frac{\partial f_i(v)}{\partial v_j} \quad (i,j = 1, \dots, n),$$

has n real eigenvalues $\lambda_1(v) \leq \lambda_2(v) \leq \dots \leq \lambda_n(v)$ and is diagonalizable. Thus, there exist $r_i(v)$, $(i = 1, \dots, n)$ linearly independent (right) corresponding eigenvectors and

$$A(v)\, r_i(v) = \lambda_i(v)\, r_i(v).$$

Since the ABI system is endowed with a renormalized equation induced by a uniformly convex entropy, from a well-known result, it is symmetrizable and hyperbolic. Moreover, the propagation speeds, i.e. λ's of (1.8)-(1.15) are easily calculated, we have

$$\lambda_i = \frac{P_1 - 1}{h} =: \lambda^- < \lambda_j = \frac{P_1}{h} =: \lambda^o < \lambda_k = \frac{P_1 + 1}{h} =: \lambda^+, \tag{1.19}$$

$(i = 1, 2, 3;\ j = 4, 5;\ k = 6, 7, 8)$.

Definition 1.4. For the system of conservation laws (1.16), a point $v \in U$ is said of linear degeneracy of the i-characteristic family when

$$\nabla_v \lambda_i(v) \cdot r_i(v) = 0, \tag{1.20}$$

otherwise, it is of genuine nonlinearity of the i-characteristic family. If (1.20) holds for every $v \in U$, then the i-characteristic family is called linear degenerated. Moreover, we say that (1.16) is totally linear degenerated, when every i-characteristic is linear degenerated.

Again, the correspondent characteristic fields, i.e. the (right) eigenvectors of (1.8)-(1.15) are easily calculated and, the wave speeds are constant along then, that is, the ABI system is totally linear degenerated. In fact, from (1.19), it is an immediate application of the Boillat's theorem, see [1].

Remark 1.5. Actually, totally degenerated systems are rather common. For instance, besides of course the linear systems, belong to this class, the Einstein equations for vacuum, the von Krmn-Tsien fluid when a and b are constants, see [2, 17], the incompressible relativistic fluid, see [18, 27], the relativistic string, see [3, 22].

Definition 1.6. A smooth function $w : U \to \mathbb{R}$ is called an i-Riemann invariant of (1.16), when for any $v \in U$

$$\nabla_v w(v) \cdot r_i(v) = 0. \tag{1.21}$$

Remark 1.7. We recall the well-known result that, any hyperbolic system of (1.16) with $n = 2$ has a coordinate system of Riemann invariants. Let

$$\begin{aligned} \partial_t u^1 + \partial_x f^1(u^1, u^2) = 0, \\ \partial_t u^2 + \partial_x f^2(u^1, u^2) = 0, \end{aligned} \tag{1.22}$$

be a such 2×2 hyperbolic system. Hence, considering smooth solutions we can rewrite (1.22) in the following form

$$\begin{aligned} \partial_t w^1 + \lambda_1 \, \partial_x w^1 = 0, \\ \partial_t w^2 + \lambda_2 \, \partial_x w^2 = 0, \end{aligned}$$

where w^i and λ_i, $(i = 1, 2)$, are the Riemann invariants and wave speeds respectively. Moreover, if (1.22) is totally linear degenerated and each wave speed does not have singular points, i.e. $\nabla_v \lambda_i(v) \neq 0$, $(i = 1, 2)$, for any $v \in U$, then the linear degeneracy makes λ_i be an i-Riemann invariant. Therefore, we could write (1.22) as

$$\begin{aligned} \partial_t \lambda_1 + \lambda_2 \, \partial_x \lambda_1 = 0, \\ \partial_t \lambda_2 + \lambda_1 \, \partial_x \lambda_2 = 0. \end{aligned} \tag{1.23}$$

2. The Smooth Case

In this section we study the existence of global smooth solutions of the initial-value problem for the ABI system. Regarding the (1.8)-(1.15) system of conservation laws, we observe that it uncouples. In fact, we could resolve first (1.12), (1.13) and, once h, P_1 are obtained, it remains to solve (1.8)-(1.11) and (1.14), (1.15). The later ones, are simple transport equations with constant coefficients. Let $b = P_1/h$, thus P_2 and P_3 are constant functions on the line with the direction $(1, b)$. Hence, without loss of generality, we take $P_2 = P_3 = 0$ and, make $P \equiv P_1$. Then, it remains to solve a 4×4 linear symmetric system of conservation laws with constant coefficients, which is well-known simple to solve, as we shall see at Section 4. Hence, we fix our attention in the following initial-value problem

$$\partial_t h + \partial_x P = 0 \quad \text{in } (0, T) \times \mathbb{R}, \tag{2.1}$$

$$\partial_t P + \partial_x \big(\frac{P^2 - 1}{h} \big) = 0 \quad \text{in } (0, T) \times \mathbb{R}, \tag{2.2}$$

$$(h, P) = (h_0, P_0) \quad \text{in } \{0\} \times \mathbb{R}, \tag{2.3}$$

where h_0 and P_0 are given bounded smooth scalar functions. As mentioned at the introduction, we are not assuming that h_0, P_0 have small sup norm, nor have sufficiently small oscillations. The system (2.1), (2.2) has wave speeds

$$\lambda^- = \frac{P - 1}{h} < \lambda^+ = \frac{P + 1}{h}, \tag{2.4}$$

and an easy computation shows that, it is totally linear degenerated. Therefore, since for any (h, P)

$$\begin{aligned} \nabla \lambda^- = \big(-\frac{P - 1}{h^2}, -\frac{1}{h} \big) \neq 0, \\ \nabla \lambda^+ = \big(-\frac{P + 1}{h^2}, +\frac{1}{h} \big) \neq 0, \end{aligned}$$

from Remark 1.7, we can rewrite (2.1)-(2.3) in the following form

$$\begin{aligned} \partial_t \lambda^- + \lambda^+ \, \partial_x \lambda^- = 0 \quad \text{in } (0,T) \times \mathbb{R}, \\ \partial_t \lambda^+ + \lambda^- \, \partial_x \lambda^+ = 0 \quad \text{in } (0,T) \times \mathbb{R}, \end{aligned} \tag{2.5}$$

$$(\lambda^-, \lambda^+) = (\lambda_0^-, \lambda_0^+), \quad \text{in } \{0\} \times \mathbb{R}, \tag{2.6}$$

where

$$\lambda_0^-(x) := \frac{P_0(x) - 1}{h_0(x)} \quad \text{and} \quad \lambda_0^+(x) := \frac{P_0(x) + 1}{h_0(x)}.$$

Now, we assume the more natural and general condition, that is

$$\lambda_M^- - \lambda_m^+ > 0, \tag{2.7}$$

where

$$\lambda_m^+ := \inf_{x \in \mathbb{R}} \lambda_0^+(x), \quad \lambda_M^- := \sup_{x \in \mathbb{R}} \lambda_0^-(x).$$

In fact, the important point is that, when the initial energy is not small and moreover the initial-data has oscillations, we can always take $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$, such that

$$\lambda_0^-(x_1) > \lambda_0^+(x_2). \tag{2.8}$$

Indeed, once h_0 is not small, for each $x \in \mathbb{R}$, $\lambda_0^+(x)$ is not so distant of $\lambda_0^-(x)$, since

$$\lambda_0^+(x) - \lambda_0^-(x) = \frac{2}{h_0(x)}.$$

Moreover, since h_0, P_0 have oscillations instead of global, we could take a local condition, that is, there exists an interval $I \subset \mathbb{R}$, such that

$$\lambda_M^- := \sup_{x \in I} \lambda_0^-(x) > \inf_{x \in I} \lambda_0^+(x) =: \lambda_m^+.$$

Considering this more physically correct condition of initial-data for linear degenerated systems, we have the following theorem.

Theorem 2.1. *Let λ_0^-, λ_0^+ be two bounded smooth functions, which satisfy condition* (2.7)*. Then, there does not exist globally smooth solution of the initial-value problem* (2.5), (2.6) *with initial-data* $(\lambda_0^-, \lambda_0^+)$*. Moreover, there exists a finite maximal time T^*, such that*

$$\limsup_{t \to T^*} (\|\partial_t u(t)\|_{L^\infty} + \|\partial_x u(t)\|_{L^\infty}) = +\infty \quad (u = (\lambda^-, \lambda^+)). \tag{2.9}$$

That is, the catastrophe appears in the Lipschitz norm for (λ^-, λ^+)*.*

Proof. 1. The first part of the proof follows by contradiction, i.e. the assumption of (2.7) and the existence of global smooth solution, implies a contradiction. By (2.7), there exist $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$ that satisfies (2.8), i.e.

$$\lambda_0^-(x_1) > \lambda_0^+(x_2).$$

Let (λ^-, λ^+) be the global smooth solution of (2.5) with initial-data $(\lambda_0^-, \lambda_0^+)$. We recall that, by the equivalence between (2.1), (2.2) with (2.5) and from (2.4), we have for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$

$$\lambda^-(t,x) < \lambda^+(t,x).$$

Set $\gamma^+(t) = (t, x(t))$, $\gamma^-(t) = (t, x(t))$ the characteristics curves, solutions respectively of the differential equations

$$\frac{dx}{dt} = \lambda^+(t, x(t)), \quad \frac{dx}{dt} = \lambda^-(t, x(t)).$$

Hence, from (2.5) we obtain that, λ^- is constant over $\gamma^+(t)$ and, λ^+ is constant over $\gamma^-(t)$, which means that λ^-, λ^+ are bounded functions. Indeed, since

$$\lambda_m^- \le \lambda_0^-(x) \le \lambda_M^-, \quad \lambda_m^+ \le \lambda_0^+(x) \le \lambda_M^+,$$

and the solution is globally, it follows that

$$\lambda_m^- \le \lambda^-(t,x) \le \lambda_M^-, \quad \lambda_m^+ \le \lambda^+(t,x) \le \lambda_M^+,$$

for any $t \ge 0$. For convenience we denote by γ_i^+, γ_i^-, $(i = 1, 2)$, when respectively

$$\begin{aligned} \lambda^-(t,x) &= \lambda_0^-(x_i) \quad (\text{over } \gamma_i^+), \\ \lambda^+(t,x) &= \lambda_0^+(x_i) \quad (\text{over } \gamma_i^-), \end{aligned}$$

for some point $x_i \in \mathbb{R}$. Thus, over $\gamma_1^+(t)$,

$$\frac{dx}{dt} = \lambda^+(t, x(t)) > \lambda^-(t, x(t)) \equiv \lambda_0^-(x_1).$$

Analogously, over $\gamma_2^-(t)$,

$$\frac{dx}{dt} = \lambda^-(t, x(t)) < \lambda^+(t, x(t)) \equiv \lambda_0^+(x_2).$$

Now, we use the comparison principle for ordinary differential equations, see [19], applied to the characteristics curves $\gamma_1^+(t)$, $\gamma_2^-(t)$, which implies respectively

$$x(t) > x_1 + \lambda_0^-(x_1)t, \tag{2.10}$$

$$x(t) < x_2 + \lambda_0^+(x_2)t. \tag{2.11}$$

It follows from (2.10), (2.11) that, for every $t > 0$, the characteristics $\gamma_1^+(t), \gamma_2^-(t)$ are respectively in the right and left sides of the lines

$$y_1(t) = x_1 + \lambda_0^-(x_1)t, \quad y_2(t) = x_2 + \lambda_0^+(x_2)t.$$

Moreover, from (2.8) the lines $y_1(t)$ and $y_2(t)$ intersect. Therefore, there must exists a point $(\tau, \xi) \in \mathbb{R}^+ \times \mathbb{R}$, such that

$$\gamma_1^+(\tau) = \gamma_2^-(\tau),$$

where we have used that (λ^-, λ^+) is bounded. Consequently, given $\varepsilon > 0$, we have

$$\lambda^-(\gamma_1^+(\tau - \varepsilon)) \equiv \lambda_0^-(x_1) > \lambda_0^+(x_2) \equiv \lambda^+(\gamma_2^-(\tau - \varepsilon)).$$

Letting $\varepsilon \to 0^+$, we obtain a contradiction.
2. The second part of the proof is an application of the Continuation Principle for classical solutions of conservation laws, see [20]. Let $[0, T^*)$ be the maximal interval of smooth existence, then for any $0 \le t < T^*$

$$\lambda_m^- \le \lambda^-(t,x) \le \lambda_M^-, \quad \lambda_m^+ \le \lambda^+(t,x) \le \lambda_M^+.$$

Therefore, the Continuation Principle implies (2.9). □

Observe that, we proved the catastrophe in the Lipschitz norm for (λ^-, λ^+), which are non-conservative variables of the (2.5), (2.6) initial-value problem. Although, if we stick to the original conservatives variables of the (2.1), (2.2) system, h in particular, we have blow up in sup norm. It means that, the blow up in sup norm of the conservative variables may coincide with the blow up in Lipschitz norm of some non conservative variables. In fact, we have shown that, for the more natural and general condition of initial-data, any 2×2 totally degenerated system of conservation laws, which the characteristics speeds do not have singular points, does not have global smooth solution. The important fact was the existence of oscillations in linear degenerated fields.

On the other hand, if sufficiently small oscillations are assumed, then we could have

$$\lambda_m^+ - \lambda_M^- > 0, \tag{2.12}$$

which is a sufficient condition for existence of global smooth solutions in this case of 2×2 systems, see [20]. Therefore, in some sense we have proved that (2.12) is also a necessary condition. Moreover, we sharpen Majda's conjecture [20] which says that, for totally linear degenerated system of conservation laws the blow up in the Lipschitz norm never happens for any smooth initial-data.

Now, we return our attention to the ABI system. From Remark 1.1 once (1.6) is satisfied at the initial time, it remains true for any time in the maximal interval of the existence of smooth solution. Furthermore, we could choose an initial-data for the ABI system, which satisfies (1.6) and the condition (2.7). Therefore, if we suppose the existence of global solution for the ABI system, then from Theorem 2.1 we obtain a contradiction. Finally:

Corollary 2.2. *Let D_0, B_0 be two bounded smooth fields. Let*

$$h_0 = \sqrt{1 + |D_0|^2 + |B_0|^2 + |D_0 \times B_0|^2}, \quad P_0 = D_0 \times B_0,$$

such that, for some interval $I \subset \mathbb{R}$

$$\sup_{x \in I} \frac{P_0(x) - 1}{h_0(x)} > \inf_{x \in I} \frac{P_0(x) + 1}{h_0(x)}.$$

Then, there does not exist globally smooth solution of the initial-value problem (1.8)-(1.15) *with initial-data* (D_0, B_0, h_0, P_0).

Therefore, the conjecture which says that the ABI system or the BI itself has global in time smooth solutions for any smooth initial-data is false. In fact, for the more natural and general condition of initial-data, we do not have global smooth solution.

3. The Riemann Problem

The aim of this section is to study the existence of solution for the ABI system in the class of Riemann Problem. Thus, we consider initial-data, see equation (1.17), of the following form

$$u_0(x) = \begin{cases} u_0^\ell & \text{if } x < 0, \\ u_0^r & \text{if } x > 0, \end{cases} \tag{3.1}$$

where u_0^ℓ, u_0^r are given constants. We recall that, since there exist results of weak-strong uniqueness, once we obtain the existence of solution for the Riemann Problem, we have well-posedness. Hence, we seek for self-similar solutions of (1.16), (3.1), that is

$$v(\xi) = u(t,x) \quad (\xi = \frac{x}{t})$$

in $BV_{\rm loc}(\mathbb{R};\mathbb{R}^n) \cap L^\infty(\mathbb{R};\mathbb{R}^n)$ that satisfies in the sense of distributions the ordinary differential equation

$$[\xi\, v(\xi) - f(v(\xi))]' = v(\xi) \quad ('\equiv \frac{d}{d\xi}) \tag{3.2}$$

obtained from (1.16) and, the boundary conditions

$$v(-\infty) = u_0^\ell, \quad v(+\infty) = u_0^r. \tag{3.3}$$

In fact, v is a Lipschitz function and thus by the Rademacher's Theorem it is differentiable $\mathcal{L}^1$-a.e., see [16]. Hence, (3.2) is satisfied by:
i) Constant states; for each Lebesgue point ξ, where $v'(\xi) = 0$.
ii) Jump discontinuities; for each discontinuity point ξ, where the Rankine-Hugoniot jump condition must hold, i.e.

$$\xi\,[v^+ - v^-] = f(v^+) - f(v^-) \quad (v^+ := v(\xi^+),\, v^- := v(\xi^-)).$$

iii) Centered simple waves; for each Lebesgue point ξ, where $v'(\xi) \neq 0$. From (3.2), i.e. $[Df(v(\xi)) - \xi\, I_d]\, v'(\xi) = 0$, we must have

$$\xi = \lambda_i(v(\xi)) \quad \text{and} \quad v'(\xi) = c(\xi) r_i(v(\xi)) \quad (i = 1, \dots, n). \tag{3.4}$$

Moreover, if we set

$$\mathcal{C} := \{\xi \in \mathbb{R}\,;\ v'(\xi) = 0\},$$
$$\mathcal{J} := \{\xi \in \mathbb{R}\,; \text{the Rankine-Hugoniot condition holds}\},$$
$$\mathcal{W} := \{\xi \in \mathbb{R}\,;\ v'(\xi) \neq 0\}$$

then, $\mathbb{R}$ is the union of these pairwise disjoint sets. Therefore, the solutions $v(\xi)$ of (3.2), (3.3) are given by a combination of (i)-(iii).

Remark 3.1. By differentiating the first relation in (3.4) and, utilizing the second, we obtain

$$[D\lambda_i(v(\xi)) \cdot r_i(v(\xi)] c(v(\xi)) = 1.$$

Since v' is a locally finite Radon measure, we observe that, the centered simple waves are points of genuine nonlinearity of the i-characteristic family. Moreover, from the above expression, we determine the scalar function c. Therefore, for totally linear degenerated systems of conservation laws, we have $\mathcal{W} = \emptyset$.

Usually, the jump $v^+ - v^-$ is called the amplitude and its size $|v^+ - v^-|$ is the strength of the jump discontinuity. Moreover, when the strength of the jump discontinuity is less than a positive (sufficiently) small δ, we say that the the jump discontinuity is weak.

Definition 3.2. We say that the jump discontinuity $(v^-, v^+; \xi)$ is a i-classical shock (or i-Lax shock, or i-compressive shock), when there exists an index i, $(1 \leq i \leq n)$ such that

$$\begin{gathered} \lambda_i(v^+) < \xi < \lambda_i(v^-), \\ \lambda_{i-1}(v^-) < \xi < \lambda_{i+1}(v^+). \end{gathered} \tag{3.5}$$

It implies that at the point of discontinuity, there are $n+1$ incoming characteristics, of which the speeds are the eigenvalues

$$\lambda_1(v^+), \dots, \lambda_i(v^+), \lambda_i(v^-), \dots, \lambda_n(v^-).$$

Moreover, (3.5) is called the Lax shock admissibility criterion. When the left or the right part of $(3.5)_1$ is satisfied as equality, the jump discontinuity is called a left or a right i-contact discontinuity and, if both parts holds as equalities, then we have a i-contact discontinuity. When there are at least $n+2$ incoming characteristics, the jump discontinuity $(v^-, v^+; \xi)$ is called a i-overcompressive shock, that is, there exists a index i such that

$$\lambda_{i+1}(v^+) < \xi < \lambda_i(v^-). \tag{3.6}$$

When there are n incoming characteristics, the jump discontinuity $(v^-, v^+; \xi)$ is called a i-undercompressive shock (or i-transitional shock), that is, there exists a index i such that

$$\lambda_i(v^\pm) < \xi < \lambda_{i+1}(v^\pm). \tag{3.7}$$

When there are $n-1$ incoming characteristics, the jump discontinuity $(v^-, v^+; \xi)$ is called a i-rarefaction shock (or i-counter Lax shock), that is, there exists a index i such that

$$\begin{gathered} \lambda_i(v^-) < \xi < \lambda_i(v^+), \\ \lambda_{i-1}(v^+) < \xi < \lambda_{i+1}(v^-). \end{gathered} \tag{3.8}$$

In any (3.6)-(3.8) case, we say that the jump discontinuity is a non-classical shock.

To solve the Riemann Problem for the ABI system, we start studying when given two constant states

$$\begin{aligned} u^\ell &= (D_2^\ell, D_3^\ell, B_2^\ell, B_3^\ell, h^\ell, P_1^\ell, P_2^\ell, P_3^\ell), \\ u^r &= (D_2^r, D_3^r, B_2^r, B_3^r, h^r, P_1^r, P_2^r, P_3^r), \end{aligned} \tag{3.9}$$

not necessarily close, nor small, how they could be connected. Since the ABI system of equations is totally linear degenerated, from Remark 3.1, we are not allowed to use centered simple waves. So, it rest to connect $u^\ell \equiv u^-$ and $u^r \equiv u^+$ by jump discontinuities. Therefore, for any $s := \xi \in \mathcal{J}$, we regard the Rankine-Hugoniot jump condition given from (1.8)-(1.15), that is

$$s(D_2^+ - D_2^-) = \frac{B_3^+ + D_2^+ P_1^+}{h^+} - \frac{B_3^- + D_2^- P_1^-}{h^-}, \tag{3.10}$$

$$s(D_3^+ - D_3^-) = \frac{-B_2^+ + D_3^+ P_1^+}{h^+} - \frac{-B_2^- + D_3^- P_1^-}{h^-}, \tag{3.11}$$

$$s(B_2^+ - B_2^-) = \frac{-D_3^+ + B_2^+ P_1^+}{h^+} - \frac{-D_3^- + B_2^- P_1^-}{h^-}, \tag{3.12}$$

$$s(B_3^+ - B_3^-) = \frac{D_2^+ + B_3^+ P_1^+}{h^+} - \frac{D_2^- + B_3^- P_1^-}{h^-}, \tag{3.13}$$

$$s(h^+ - h^-) = P_1^+ - P_1^-, \tag{3.14}$$

$$s(P_1^+ - P_1^-) = \frac{(P_1^+)^2 - 1}{h^+} - \frac{(P_1^-)^2 - 1}{h^-}, \tag{3.15}$$

$$s(P_2^+ - P_2^-) = \frac{P_1^+ P_2^+}{h^+} - \frac{P_1^- P_2^-}{h^-}, \tag{3.16}$$

$$s(P_3^+ - P_3^-) = \frac{P_1^+ P_3^+}{h^+} - \frac{P_1^- P_3^-}{h^-}. \tag{3.17}$$

First, we study (3.14), (3.15), which is the Rankine-Hugoniot condition for the 2×2 totally linear degenerated system given by (2.1), (2.2), with $P \equiv P_1$. From

(3.14) and (3.15), we have

$$\frac{(P_1^+ - P_1^-)^2}{h^+ - h^-} = \frac{(P_1^+)^2 - 1}{h^+} - \frac{(P_1^-)^2 - 1}{h^-}.$$

Hence, after some algebra we obtain

$$\Big(P_1^+ \big(\frac{h^-}{h^+}\big)^{1/2} - P_1^- \big(\frac{h^+}{h^-}\big)^{1/2}\Big)^2 = \frac{(h^+ - h^-)^2}{h^+ \, h^-}.$$

Therefore, it follows that

$$P_1^+ \big(\frac{h^-}{h^+}\big)^{1/2} - P_1^- \big(\frac{h^+}{h^-}\big)^{1/2} = -\frac{h^+ - h^-}{h^+ \, h^-}$$

or,

$$P_1^+ \big(\frac{h^-}{h^+}\big)^{1/2} - P_1^- \big(\frac{h^+}{h^-}\big)^{1/2} = \frac{h^+ - h^-}{h^+ \, h^-}.$$

From the former and the second, we obtain respectively

$$\frac{P_1^- - 1}{h^-} = \frac{P_1^+ - 1}{h^+}, \tag{3.18}$$

$$\frac{P_1^- + 1}{h^-} = \frac{P_1^+ + 1}{h^+}. \tag{3.19}$$

If we explicit P_1^- in (3.18), and analogously in (3.19), then from (3.14) we calculate the value of s. Thus we have respectively

$$s = \frac{P_1^+ - 1}{h^+} \quad \text{and} \quad s = \frac{P_1^+ + 1}{h^+}.$$

Therefore, considering the 2×2 system of conservation laws given by (2.1), (2.2), with $P \equiv P_1$, we have the following statement.

Lemma 3.3. *Let $u^- = (h^-, P^-)$, $u^+ = (h^+, P^+)$ be two given constant states. Let (2.1), (2.2) be the system of conservation laws for $u = (h, P)$. Then u^-, u^+ could be connected only by contact discontinuities in the following form:*
i) When (u^-, u^+) satisfies (3.18), by a contact discontinuity of speed

$$s = \lambda^-(u^-) = \lambda^-(u^+).$$

ii) When (u^-, u^+) satisfies (3.19), by a contact discontinuity of speed

$$s = \lambda^+(u^-) = \lambda^+(u^+).$$

Since the Rankine-Hugoniot condition does not depend on contact discontinuities, nor the conservative form of the system, so the following theorem gives general features and explicit solutions of the Riemann problem for any 2×2 totally degenerated system of conservation laws, which satisfies the conditions in Remark 1.7.

Theorem 3.4. *A self-similar weak solution (h, P) of (2.1), (2.2) in $\mathbb{R}^+ \times \mathbb{R}$, with initial-data*

$$(h(0,x), P(0,x)) = \begin{cases} (h_0^\ell, P_0^\ell) & \text{if } x < 0, \\ (h_0^r, P_0^r) & \text{if } x > 0, \end{cases} \tag{3.20}$$

is given at most by:
i) One contact discontinuity; with speed $s = \lambda^-(u_0^\ell) = \lambda^-(u_0^r)$ *or* $s = \lambda^+(u_0^r) = \lambda^+(u_0^\ell)$*, when respectively*

$$\frac{P_0^\ell - 1}{h_0^\ell} = \frac{P_0^r - 1}{h_0^r} \quad \textit{or} \quad \frac{P_0^\ell + 1}{h_0^\ell} = \frac{P_0^r + 1}{h_0^r}.$$

In any case, the solution is given by

$$(h, P)(t, x) = \begin{cases} (h_0^\ell, P_0^\ell) & \textit{if } x < s\,t, \\ (h_0^r, P_0^r) & \textit{if } x > s\,t. \end{cases}$$

ii) Two contact discontinuities; one of speed $s_1 = \lambda^-(u_0^\ell) = \lambda^-(\bar{u})$*, and another with speed* $s_2 = \lambda^+(\bar{u}) = \lambda^+(u_0^r)$*, with* $s_1 < s_2$*, when*

$$\bar{h} = \frac{2}{\lambda^+(u_0^r) - \lambda^-(u_0^\ell)} \quad \textit{and} \quad \bar{P} = \frac{\lambda^+(u_0^r) + \lambda^-(u_0^\ell)}{\lambda^+(u_0^r) - \lambda^-(u_0^\ell)}.$$

The solution is given by

$$(h, P)(t, x) = \begin{cases} (h_0^\ell, P_0^\ell) & \textit{if } x < s_1\,t, \\ (\bar{h}, \bar{P}) & \textit{if } s_1\,t < x < s_2\,t, \\ (h_0^r, P_0^r) & \textit{if } x > s_2\,t. \end{cases}$$

Proof. 1. The (i) and (ii) type solutions follow easy from Lemma 3.3. Further, the $(\bar{h}, \bar{P})$ intermediate state in (ii) is implied from the condition that

$$\frac{P_0^\ell - 1}{h_0^\ell} = \frac{\bar{P} - 1}{\bar{h}} \quad \text{and} \quad \frac{\bar{P} + 1}{\bar{h}} = \frac{P_0^r + 1}{h_0^r}.$$

2. From Lemma 3.3 we only have contact discontinuities, which follows that we are not allowed to have two or more intermediate distinct states. For instance, let us suppose two, that is

$$\bar{u} = (\bar{h}, \bar{P}) \quad \text{and} \quad \bar{\bar{u}} = (\bar{\bar{h}}, \bar{\bar{P}}).$$

So, the solution would be given by

$$(h, P)(t, x) = \begin{cases} (h_0^\ell, P_0^\ell) & \text{if } x < s_1\,t, \\ (\bar{h}, \bar{P}) & \text{if } s_1\,t < x < s_2\,t, \\ (\bar{\bar{h}}, \bar{\bar{P}}) & \text{if } s_2\,t < x < s_3\,t, \\ (h_0^r, P_0^r) & \text{if } x > s_3\,t. \end{cases} \tag{3.21}$$

Since for any two states u^-, u^+, we have

$$s = \lambda^\pm(u^-) = \lambda^\pm(u^+),$$

we are not allowed to have two distinct contact discontinuities of the same kind side by side. Indeed, suppose that

$$s_1 = \lambda^-(u_0^\ell), \quad s_2 = \lambda^-(\bar{u}), \quad s_3 = \lambda^+(u_0^r).$$

Hence, we have $s_1 = \lambda^-(u_0^\ell) = \lambda^-(\bar{u}) = s_2$, which implies a contraction. Analogously,

$$s_1 = \lambda^-(u_0^\ell), \quad s_2 = \lambda^+(\bar{u}), \quad s_3 = \lambda^+(u_0^r).$$

Now, we observe that it is not possible to intercalate two different kinds of contact discontinuities. For instance, if $s_1 = \lambda^-(u_0^\ell)$, $s_2 = \lambda^+(\bar{u})$ and $s_3 = \lambda^-(u_0^r)$, then

$$s_2 = \lambda^+(\bar{u}) = \lambda^+(\bar{\bar{u}}) > \lambda^-(\bar{\bar{u}}) = \lambda^-(u_0^r) = s_3.$$

Consequently, the solution is ill-defined, that is, (h, P) given by (3.21) is a multi-valued function. Therefore, we must have only one intermediate state. The another cases are similar. □

Now, if we assume the more natural and general condition of initial-data for degenerated fields, i.e. (2.7) and moreover (2.8), where in this case of constant states means that

$$\lambda^-(u_0^\ell) - \lambda^+(u_0^r) > 0, \tag{3.22}$$

then it follows from Theorem 3.4:

Corollary 3.5. *There does not exist solution in $BV_{\rm loc}\cap L^\infty$ of the Riemann problem for* (2.1), (2.2), *when the initial-data* (3.20) *satisfies the condition* (3.22), *that is*

$$\frac{P_0^\ell - 1}{h_0^\ell} > \frac{P_0^r + 1}{h_0^r}.$$

Remark 3.6. We recall that, when (h_0^ℓ, P_0^ℓ) and (h_0^r, P_0^r) are sufficiently close, they can be always connected to each other. That is, any weak jump discontinuity associated a linear degenerated characteristic family, could be always connected by a contact discontinuity. In fact, there is not a contradiction with Corollary 3.5, since in this case we do not have (3.22) condition of initial-data satisfied.

We return to the ABI system, that is, the Rankine-Hugoniot condition (3.10)-(3.17). Now, if we set

$$d_i^\pm := \frac{D_i^\pm}{h^\pm}, \quad b_i^\pm := \frac{B_i^\pm}{h^\pm}, \quad p_i^\pm := \frac{P_i^\pm}{h^\pm} \quad (i = 2, 3),$$

then we can rewrite (3.10)-(3.17) as

$$d_2^+\zeta^+ - d_2^-\zeta^- - b_3^+ + b_3^- = 0, \tag{3.23}$$
$$d_3^+\zeta^+ - d_3^-\zeta^- + b_2^+ - b_2^- = 0, \tag{3.24}$$
$$b_2^+\zeta^+ - b_2^-\zeta^- + d_3^+ - d_3^- = 0, \tag{3.25}$$
$$b_3^+\zeta^+ - b_3^-\zeta^- - d_2^+ + d_2^- = 0, \tag{3.26}$$
$$\zeta^+ - \zeta^- = 0, \tag{3.27}$$
$$(P_1^+\zeta^+ + 1)/(h^+) - (P_1^-\zeta^- + 1)/h^- = 0, \tag{3.28}$$
$$p_2^+\zeta^+ - p_2^-\zeta^- = 0, \tag{3.29}$$
$$p_3^+\zeta^+ - p_3^-\zeta^- = 0, \tag{3.30}$$

where $\zeta^\pm := s\,h^\pm - P_1^\pm$. So instead of s, we have two unknowns, i.e. $\zeta^\pm$. Hence, we obtain one more equation to be satisfied

$$\phi(\zeta^+, \zeta^-) := \frac{P_1^+ + \zeta^+}{h^+} - \frac{P_1^- + \zeta^-}{h^-} = 0, \tag{3.31}$$

which implies that s must have the same value given by

$$\zeta^+ = s\,h^+ - P_1^+ \quad \text{or} \quad \zeta^- = s\,h^- - P_1^-.$$

Once we obtain $\zeta^\pm$ that satisfies (3.23)-(3.31), the Rankine-Hugoniot condition is satisfied. Clearly from (3.27), we must have

$$\zeta^- = \zeta^+ =: \zeta.$$

Therefore, it follows from (3.23)-(3.31) that:
i) When $\zeta = 0$, we must have $d_i^- = d_i^+, b_i^- = b_i^+$, $(i = 2, 3)$, $h^- = h^+$, and $P_1^- = P_1^+$, that is

$$(D_2^-, D_3^-, B_2^-, B_3^-, h^-, P_1^-) = (D_2^+, D_3^+, B_2^+, B_3^+, h^+, P_1^+), \tag{3.32}$$

and we could have a jump in P_i, $(i = 2, 3)$. Moreover, we have

$$s = \frac{P_1^+}{h^+} = \frac{P_1^-}{h^-}.$$

ii) When $\zeta \neq 0$, we must have $p_i^- = p_i^+$, $(i = 2, 3)$,

$$\begin{gathered}
(\zeta^2 - 1)(d_i^+ - d_i^-) = 0 \quad (i = 2, 3),\\
(\zeta^2 - 1)(b_i^+ - b_i^-) = 0 \quad (i = 2, 3),\\
\frac{P_1^+\zeta + 1}{P_1^-\zeta + 1} = \frac{h^+}{h^-} = \frac{P_1^+ + \zeta}{P_1^- + \zeta}.
\end{gathered}$$

The latter, implies that

$$(\zeta^2 - 1)(P_1^+ - P_1^-) = 0.$$

If $(\zeta^2 - 1) \neq 0$, then u^- must be equal to u^+, which is not the case. Consequently, we must have

$$\zeta = \pm 1.$$

Hence, we could have a jump in (D_i, B_i, h, P_1, P_i), $(i = 2, 3)$. Moreover, we have

$$s = \frac{P_1^+ \pm 1}{h^+} = \frac{P_1^- \pm 1}{h^-}.$$

Therefore, considering the ABI system of conservation laws given by (1.8), (1.15), we have the following:

Lemma 3.7. *Let $u^\pm = (D_2^\pm, D_3^\pm, B_2^\pm, B_3^\pm, h^\pm, P_1^\pm, P_2^\pm, P_3^\pm)$ be two given constant states. Let* (1.8)-(1.15) *be the ABI system of conservation laws for $u = (D, B, h, P)$. Then u^-, u^+ could be connected only by contact discontinuities in the following form:*
i) When (u^-, u^+) satisfies (3.32)*, by a contact discontinuity of speed*

$$s = \lambda^o(u^-) = \lambda^o(u^+).$$

ii) When (u^-, u^+) satisfies

$$\begin{gathered}
\frac{D_2^+ + B_3^+}{h^+} = \frac{D_2^- + B_3^-}{h^-}, \quad \frac{D_3^+ - B_2^+}{h^+} = \frac{D_3^- - B_2^-}{h^-},\\
\frac{P_1^+ - 1}{h^+} = \frac{P_1^- - 1}{h^-}, \quad \frac{P_i^+}{h^+} = \frac{P_i^-}{h^-} \quad (i = 2, 3),
\end{gathered}$$

by a contact discontinuity of speed

$$s = \lambda^-(u^-) = \lambda^-(u^+).$$

iii) When (u^-, u^+) satisfies

$$\begin{gathered}
\frac{D_2^+ - B_3^+}{h^+} = \frac{D_2^- - B_3^-}{h^-}, \quad \frac{D_3^+ + B_2^+}{h^+} = \frac{D_3^- + B_2^-}{h^-},\\
\frac{P_1^+ + 1}{h^+} = \frac{P_1^- + 1}{h^-}, \quad \frac{P_i^+}{h^+} = \frac{P_i^-}{h^-} \quad (i = 2, 3),
\end{gathered}$$

by a contact discontinuity of speed $s = \lambda^+(u^-) = \lambda^+(u^+)$.

Obviously, if (3.9) satisfies (3.10)-(3.17) with $h^\pm$ and $P^\pm$ given by

$$h^\pm = \sqrt{1 + |D^\pm|^2 + |B^\pm|^2 + |D^\pm \times B^\pm|^2}, \quad P^\pm = D^\pm \times B^\pm, \tag{3.33}$$

then $D^\pm$, $B^\pm$ satisfy the Rankine-Hugoniot condition for the BI system. However, it is not clear that we have the converse. In fact, it is not true. Indeed, as showed in [21], we could have jump discontinuities in the BI model, which are not contact discontinuities. On the other hand, from Lemma 3.7, we have only contact discontinuities in the case of the ABI system. Consequently, the Rankine-Hugoniot condition of these systems are not equivalent. For instance, there exist states $(D^\pm, B^\pm)$ which satisfy the Rankine-Hugoniot condition for the BI system, but do not satisfy the ABI one, with $h^\pm$ and $P^\pm$ given by (3.33). Next we show that, if a field (D, B, h, P) with h, P given by (1.6) is a piecewise smooth solution of the ABI system, then it does not have dissipative shocks.

Theorem 3.8. *Let (D^+, B^+), (D^-, B^-) be two given constant states, and $h^\pm$, $P^\pm$ given by* (3.33)*. If $(D^\pm, B^\pm, h^\pm, P^\pm)$ satisfy the Rankine-Hugoniot condition for the ABI system* (1.8)-(1.15)*, then $(D^\pm, B^\pm)$ also satisfy the Rankine-Hugoniot condition for the entropy equation* (1.7)*.*

Proof. First, let us note that, since $D_1 = B_1 = 0$ and $P = D \times B$

$$P_2 = P_3 = 0,$$
$$P \equiv P_1 = D_2 B_3 - D_3 B_2.$$

Furthermore, we have

$$\frac{(D \otimes D)P}{h^2} = \frac{(D \cdot P)\, D}{h^2} = 0,$$
$$\frac{(B \otimes B)P}{h^2} = \frac{(B \cdot P)\, B}{h^2} = 0.$$

Hence, the Rankine-Hugoniot condition for the entropy equation (1.7), is given by

$$s(\eta^+ - \eta^-) = \frac{\eta^+ P_1^+}{h^+} - \frac{\eta^- P_1^-}{h^-},$$

where $\eta^\pm = \eta(D^\pm, B^\pm, h^\pm, P^\pm)$. Therefore, for $\zeta^\pm = s h^\pm - P_1^\pm$, we must have

$$\frac{\eta^+ \zeta^+}{h^+} - \frac{\eta^- \zeta^-}{h^-} = 0. \tag{3.34}$$

Since $(D^\pm, B^\pm, h^\pm, P^\pm)$ satisfy the Rankine-Hugoniot condition for the ABI system (1.8)-(1.15), we have $\zeta^+ = \zeta^-$. Moreover, from (3.33) and the definition of η, we get

$$\eta^\pm = \frac{h^\pm}{2}.$$

Consequently, the right-hand side of (3.34) is zero, which completes the proof. $\square$

The following theorem gives general features and explicit solutions of the Riemann problem for the ABI system of conservation laws.

Theorem 3.9. *A self-similar weak solution (D, B, h, P) of* (1.8)-(1.15) *in $\mathbb{R}^+ \times \mathbb{R}$, with initial-data*

$$(D, B, h, P)(0, x) = \begin{cases} (D_0^\ell, B_0^\ell, h_0^\ell, P_0^\ell) & \text{if } x < 0, \\ (D_0^r, B_0^r, h_0^r, P_0^r) & \text{if } x > 0, \end{cases} \tag{3.35}$$

is given at most by:
i) One contact discontinuity; with speed $s=\lambda^-(u_0^\ell)=\lambda^-(u_0^r)$, *when*

$$\frac{D_{2_0}^\ell+B_{3_0}^\ell}{h_0^\ell}=\frac{D_{2_0}^r+B_{3_0}^r}{h_0^r},\quad \frac{D_{3_0}^\ell-B_{2_0}^\ell}{h_0^\ell}=\frac{D_{3_0}^r-B_{2_0}^r}{h_0^r},$$
$$\frac{P_{1_0}^\ell-1}{h_0^\ell}=\frac{P_{1_0}^r-1}{h_0^r},\quad \frac{P_{i_0}^\ell}{h_0^\ell}=\frac{P_{i_0}^r}{h_0^r}\quad (i=2,3);$$

or, with speed $s=\lambda^o(u_0^\ell)=\lambda^o(u_0^r)$ *when*

$$(D_{2_0}^\ell,D_{3_0}^\ell,B_{2_0}^\ell,B_{3_0}^\ell,h_0^\ell,P_{1_0}^\ell)=(D_{2_0}^r,D_{3_0}^r,B_{2_0}^r,B_{3_0}^r,h_0^r,P_{1_0}^r);$$

or, with speed $s=\lambda^+(u_0^\ell)=\lambda^+(u_0^r)$ *when*

$$\frac{D_{2_0}^\ell-B_{3_0}^\ell}{h_0^\ell}=\frac{D_{2_0}^r-B_{3_0}^r}{h_0^r},\quad \frac{D_{3_0}^\ell+B_{2_0}^\ell}{h_0^\ell}=\frac{D_{3_0}^r+B_{2_0}^r}{h_0^r},$$
$$\frac{P_{1_0}^\ell+1}{h_0^\ell}=\frac{P_{1_0}^r+1}{h_0^r},\quad \frac{P_{i_0}^\ell}{h_0^\ell}=\frac{P_{i_0}^r}{h_0^r}\quad (i=2,3).$$

In any case, the solution is given by

$$(D,B,h,P)(t,x)=\begin{cases}(D_0^\ell,B_0^\ell,h_0^\ell,P_0^\ell) & \text{if } x<s\,t,\\ (D_0^r,B_0^r,h_0^r,P_0^r) & \text{if } x>s\,t.\end{cases}$$

ii) Two contact discontinuities; one of speed $s_1=\lambda^-(u_0^\ell)=\lambda^-(\bar u)$, *and another with speed* $s_2=\lambda^o(\bar u)=\lambda^o(u_0^r)$, *with* $s_1<s_2$, *when*

$$\frac{D_{2_0}^\ell+B_{3_0}^\ell}{h^\ell}=\frac{\bar D_2+\bar B_3}{\bar h},\quad \frac{D_{3_0}^\ell-B_{2_0}^\ell}{h^\ell}=\frac{\bar D_3-\bar B_2}{\bar h},\quad \frac{P_{1_0}^\ell-1}{h_0^\ell}=\frac{\bar P_1-1}{\bar h},$$

where $\bar u=(\bar D,\bar B,\bar h,\bar P)$ *is given by*

$$\bar D=D_0^r,\quad \bar B=B_0^r,\quad \bar h=h_0^r,\quad \bar P_1=P_{1_0}^r,$$
$$\bar P_i=P_{i_0}^\ell\,\frac{h_0^r}{h_0^\ell}\quad (i=2,3);$$

or, one of speed $s_1=\lambda^o(u_0^\ell)=\lambda^o(\bar u)$, *and another with speed* $s_2=\lambda^+(\bar u)=\lambda^+(u_0^r)$, *with* $s_1<s_2$, *when*

$$\frac{D_{2_0}^r-B_{3_0}^r}{h_0^r}=\frac{\bar D_2-\bar B_3}{\bar h},\quad \frac{D_{3_0}^r+B_{2_0}^r}{h_0^r}=\frac{\bar D_3+\bar B_2}{\bar h},\quad \frac{P_{1_0}^r+1}{h_0^r}=\frac{\bar P_1+1}{\bar h},$$

where $\bar u=(\bar D,\bar B,\bar h,\bar P)$ *is given by*

$$\bar D=D_0^\ell,\quad \bar B=B_0^\ell,\quad \bar h=h_0^\ell,\quad \bar P_1=P_{1_0}^\ell,$$
$$\bar P_i=P_{i_0}^r\,\frac{h_0^\ell}{h_0^r}\quad (i=2,3);$$

or, one of speed $s_1=\lambda^-(u_0^\ell)=\lambda^-(\bar u)$, *and another with speed* $s_2=\lambda^+(\bar u)=\lambda^+(u_0^r)$, *with* $s_1<s_2$, *when*

$$\bar D_2=\Big(\frac{D_{2_0}^r-B_{3_0}^r}{h_0^r}+\frac{D_{2_0}^\ell+B_{3_0}^\ell}{h_0^\ell}\Big)/(\lambda^+(u_0^r)-\lambda^-(u_0^\ell)),$$
$$\bar D_3=\Big(\frac{D_{3_0}^\ell-B_{2_0}^\ell}{h_0^\ell}+\frac{D_{3_0}^r+B_{2_0}^r}{h_0^r}\Big)/(\lambda^+(u_0^r)-\lambda^-(u_0^\ell)),$$

$$\begin{aligned}
\bar{B}_2 &= \Big(\frac{D^r_{3_0}+B^r_{2_0}}{h^r_0} - \frac{D^\ell_{3_0}-B^\ell_{2_0}}{h^\ell_0}\Big)/(\lambda^+(u^r_0)-\lambda^-(u^\ell_0)),\\
\bar{B}_3 &= \Big(\frac{D^\ell_{2_0}+B^\ell_{3_0}}{h^\ell_0} - \frac{D^r_{2_0}-B^r_{3_0}}{h^r_0}\Big)/(\lambda^+(u^r_0)-\lambda^-(u^\ell_0)),\\
\bar{h} &= \frac{2}{\lambda^+(u^r_0)-\lambda^-(u^\ell_0)}, \quad \bar{P}_1 = \frac{\lambda^+(u^r_0)+\lambda^-(u^\ell_0)}{\lambda^+(u^r_0)-\lambda^-(u^\ell_0)},\\
\bar{P}_i &= \frac{2P^\ell_{i_0}/h^\ell_0}{(\lambda^+(u^r_0)-\lambda^-(u^\ell_0))} = \frac{2P^r_{i_0}/h^r_0}{(\lambda^+(u^r_0)-\lambda^-(u^\ell_0))} \quad (i=2,3).
\end{aligned}$$

In any case, the solution is given by

$$(D,B,h,P)(t,x) = \begin{cases} (D^\ell_0, B^\ell_0, h^\ell_0, P^\ell_0) & \text{if } x < s_1\,t,\\ (\bar{D},\bar{B},\bar{h},\bar{P}) & \text{if } s_1\,t < x < s_2\,t,\\ (D^r_0, B^r_0, h^r_0, P^r_0) & \text{if } x > s_2\,t. \end{cases}$$

iii) Three contact discontinuities; the first of speed $s_1 = \lambda^-(u^\ell_0) = \lambda^-(\bar{u})$*, the second of speed* $s_2 = \lambda^o(\bar{u}) = \lambda^o(\tilde{u})$*, and the third of speed* $\lambda^+(\tilde{u}) = \lambda^+(u^r_0)$*, with* $s_1 < s_2 < s_3$*, when*

$$\begin{aligned}
\tilde{D}_2 = \bar{D}_2 &= \Big(\frac{D^r_{2_0}-B^r_{3_0}}{h^r_0} + \frac{D^\ell_{2_0}+B^\ell_{3_0}}{h^\ell_0}\Big)/(\lambda^+(u^r_0)-\lambda^-(u^\ell_0)),\\
\tilde{D}_3 = \bar{D}_3 &= \Big(\frac{D^\ell_{3_0}-B^\ell_{2_0}}{h^\ell_0} + \frac{D^r_{3_0}+B^r_{2_0}}{h^r_0}\Big)/(\lambda^+(u^r_0)-\lambda^-(u^\ell_0)),\\
\tilde{B}_2 = \bar{B}_2 &= \Big(\frac{D^r_{3_0}+B^r_{2_0}}{h^r_0} - \frac{D^\ell_{3_0}-B^\ell_{2_0}}{h^\ell_0}\Big)/(\lambda^+(u^r_0)-\lambda^-(u^\ell_0)),\\
\tilde{B}_3 = \bar{B}_3 &= \Big(\frac{D^\ell_{2_0}+B^\ell_{3_0}}{h^\ell_0} - \frac{D^r_{2_0}-B^r_{3_0}}{h^r_0}\Big)/(\lambda^+(u^r_0)-\lambda^-(u^\ell_0)),\\
\tilde{h} = \bar{h} &= \frac{2}{\lambda^+(u^r_0)-\lambda^-(u^\ell_0)}, \quad \tilde{P}_1 = \bar{P}_1 = \frac{\lambda^+(u^r_0)+\lambda^-(u^\ell_0)}{\lambda^+(u^r_0)-\lambda^-(u^\ell_0)},\\
\bar{P}_i &= \frac{2P^\ell_{i_0}/h^\ell_0}{(\lambda^+(u^r_0)-\lambda^-(u^\ell_0))}, \quad \tilde{P}_i = \frac{2P^r_{i_0}/h^r_0}{(\lambda^+(u^r_0)-\lambda^-(u^\ell_0))} \quad (i=2,3).
\end{aligned}$$

The solution is given by

$$(D,B,h,P)(t,x) = \begin{cases} (D^\ell_0, B^\ell_0, h^\ell_0, P^\ell_0) & \text{if } x < s_1\,t,\\ (\bar{D},\bar{B},\bar{h},\bar{P}) & \text{if } s_1\,t < x < s_2\,t,\\ (\tilde{D},\tilde{B},\tilde{h},\tilde{P}) & \text{if } s_2\,t < x < s_3\,t,\\ (D^r_0, B^r_0, h^r_0, P^r_0) & \text{if } x > s_3\,t. \end{cases}$$

The proof follows from Lemma 3.7 in a similar way given at Theorem 3.4.

Again, if we assume the more natural and general condition of initial-data for degenerated fields, which here implies that we have either

$$\begin{aligned}
&\lambda^-(u^\ell_0) > \lambda^o(u^r_0),\\
&\lambda^o(u^\ell_0) > \lambda^+(u^r_0),\\
&\lambda^-(u^\ell_0) > \lambda^+(u^r_0),
\end{aligned} \tag{3.36}$$

then it follows from Theorem 3.8 that we do not have solution of the Riemann problem for the ABI system.

Corollary 3.10. *There does not exist solution of the Riemann problem for* (1.8), (1.15)*, when the initial-data* (3.35) *satisfies* (3.36)*.*

4. The well-posed case

In this section we focus on the well-posedness case. So we have to assume the less natural initial-data condition for degenerated fields. As we have noted in Section 2, under the (2.12) initial-data condition the system (2.1), (2.2) has smooth global solution for smooth initial-data. Next, we recall a result in Serre [23], which gives a global solution of (2.5), (2.6) for non-smooth initial-data.

Proposition 4.1. *Let λ_0^-, λ_0^+ be two $BV \cap L^\infty$ scalar functions that satisfy* (2.12)*, that is*

$$\sup_{x\in\mathbb{R}} \lambda_0^-(x) < \inf_{x\in\mathbb{R}} \lambda_0^+(x).$$

Then, for every $T > 0$, there exists a weak solution (λ^-, λ^+) of the Cauchy problem (2.5), (2.6) *with initial-data $(\lambda_0^-, \lambda_0^+)$, such that, for any entropy pair $F(u,v) = (\eta(u,v), q(u,v))$, $u \neq v \in \mathbb{R}$,*

$$\eta(u,v) := \frac{\alpha(u)+\beta(v)}{v-u}, \quad q(u,v) := \frac{v\,\alpha(u)+u\,\beta(v)}{v-u} \quad (\alpha,\beta \in C(\mathbb{R})),$$

(λ^-, λ^+) satisfies

$$\begin{aligned} &\int_0^T\int_{\mathbb{R}} F(\lambda^-(t,x),\lambda^+(t,x))\cdot\nabla_{t,x}\phi(t,x)\,dx\,dt \\ &+\int_{\mathbb{R}} \eta(\lambda_0^-(x),\lambda_0^+(x))\,\phi(0,x)dx = 0, \end{aligned} \tag{4.1}$$

for any function $\phi \in C_0^\infty((-\infty,T)\times\mathbb{R})$. Moreover, for almost every $t \in (0,T)$,

$$\begin{aligned} \inf_{x\in\mathbb{R}} \lambda_0^-(x) \le \lambda^-(t,x) \le \sup_{x\in\mathbb{R}} \lambda_0^-(x), \\ \inf_{x\in\mathbb{R}} \lambda_0^+(x) \le \lambda^+(t,x) \le \sup_{x\in\mathbb{R}} \lambda_0^+(x), \end{aligned} \tag{4.2}$$

and

$$TV(\lambda^-(t)) \le TV(\lambda_0^-), \quad TV(\lambda^+(t)) \le TV(\lambda_0^+).$$

The proof is obtained via Glimm's scheme and the compensated compactness theory. Furthermore, we have well-posedness from the results given by Bressan et al. [6, 7, 8, 9].

Now, we look at the ABI system. In the smooth case, that is, when we seek a solution of the Cauchy problem (1.8)-(1.15) for smooth initial-data, once h, P_1 are known it remains to solve a 6×6 linear system of conservation laws with smooth coefficients. Further, this system is symmetric, which follows easy if we denote $u = (D_2, D_3, B_2, B_3, P_2, P_3)$ and

$$f(u) = \Big(\frac{B_3+D_2P}{h}, \frac{-B_2+D_3P}{h}, \frac{-D_3+B_2P}{h}, \frac{D_2+B_3P}{h}, \frac{P_1P_2}{h}, \frac{P_1P_3}{h}\Big)$$

then the differential of f with respect to u is

$$A := \begin{bmatrix} \frac{P_1}{h} & 0 & 0 & \frac{1}{h} & 0 & 0 \\ 0 & \frac{P_1}{h} & -\frac{1}{h} & 0 & 0 & 0 \\ 0 & -\frac{1}{h} & \frac{P_1}{h} & 0 & 0 & 0 \\ \frac{1}{h} & 0 & 0 & \frac{P_1}{h} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{P_1}{h} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{P_1}{h} \end{bmatrix}.$$

Consequently, this 6×6 system is hyperbolic and since we are in the case of a single space dimension, we have an extremely elementary solution of the Cauchy problem. Indeed,

$$u(t,x) = \sum_{i=1}^{6} \phi_i(x - \lambda_i t) r_i,$$

where λ_i, $(i = 1, \dots, 6)$, are the eigenvalues of A with correspondent r_i eigenvectors. Moreover, we have $u(0,x) = u_0(x)$ since

$$u_0(x) = \sum_{i=1}^{6} \phi_i(x) r_i,$$

which means that, ϕ_i is the component of u_0 along r_i, $(i = 1, \dots, 6)$. Hence, we have the following corollary.

Corollary 4.2. *Let (D_0, B_0, h_0, P_0) be bounded smooth functions, such that*

$$\sup_{x\in\mathbb{R}} \frac{P_0(x) - 1}{h_0(x)} < \inf_{x\in\mathbb{R}} \frac{P_0(x) + 1}{h_0(x)}. \tag{4.3}$$

Then there exists a global smooth solution $u = (D, B, h, P)$ of the Cauchy problem for (1.8)-(1.15) *with $u_0 = (D_0, B_0, h_0, P_0)$. Moreover, for every $t \in \mathbb{R}^+$*

$$\sup_{x\in\mathbb{R}} |u(t,x)| \leq \sup_{x\in\mathbb{R}} |u_0(x)|. \tag{4.4}$$

Finally, as in [5] we prove a stability result for the ABI system.

Theorem 4.3. *Let $(D^\ell, B^\ell, h^\ell, P^\ell)$ be a sequence of uniformly bounded smooth functions that satisfy the ABI system* (1.8)-(1.15)*, such that, at the initial time*

$$\sup_{x\in\mathbb{R}} \frac{P^\ell(0,x) - 1}{h^\ell(0,x)} < \inf_{x\in\mathbb{R}} \frac{P^\ell(0,x) + 1}{h^\ell(0,x)},$$
$$h^\ell(0,x) = \sqrt{1 + |D^\ell(0,x)|^2 + |B^\ell(0,x)|^2 + |D^\ell(0,x) \times B^\ell(0,x)|^2},$$
$$P^\ell(0,x) = D^\ell(0,x) \times B^\ell(0,x).$$

Then, there exists a weak limit (D, B, h, P) which satisfies the ABI system.

Proof. 1. First, let us note that, since for each ℓ, $P^\ell(0) = D^\ell(0) \times B^\ell(0)$, we have for every $t \geq 0$, and $\ell \in \mathbb{Z}^+$

$$P_2^\ell(t) = P_3^\ell(t) = 0, \quad P_1^\ell(t) = D_2^\ell(t) B_3^\ell(t) - D_3^\ell(t) B_2^\ell(t).$$

Hence, from (1.8)-(1.15), we have

$$\partial_t D_2^\ell + \partial_x \Big(\frac{B_3^\ell + D_2^\ell P^\ell}{h^\ell}\Big) = 0, \tag{4.5}$$

$$\partial_t D_3^\ell + \partial_x(\frac{-B_2^\ell + D_3^\ell P^\ell}{h^\ell}) = 0, \tag{4.6}$$

$$\partial_t B_2^\ell + \partial_x(\frac{-D_3^\ell + B_2^\ell P^\ell}{h^\ell}) = 0, \tag{4.7}$$

$$\partial_t B_3^\ell + \partial_x(\frac{D_2^\ell + B_3^\ell P^\ell}{h^\ell}) = 0, \tag{4.8}$$

$$\partial_t h^\ell + \partial_x P^\ell = 0, \tag{4.9}$$

$$\partial_t P^\ell + \partial_x\big(\frac{(P^\ell)^2 - 1}{h^\ell}\big) = 0. \tag{4.10}$$

2. The uniformly bound of $u^\ell := (D_2^\ell, D_3^\ell, B_2^\ell, B_3^\ell, h^\ell, P^\ell)$, that is

$$\sup_\ell |u^\ell| \le C,$$

implies the existence of $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R}^6)$, such that, for a subsequence u^{ℓ_k} of u^ℓ, we have

$$u^{\ell_k} \overset{ast}{\rightharpoonup} u \quad \text{in } L^\infty(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R}^6).$$

Clearly, we have that

$$\{\partial_t u^k + \partial_x f(u^k)\} \text{ is pre-compact in } W^{-1,2}_{\text{loc}}.$$

Now, let the entropy pair $F(u,v) = (\eta(u,v), q(u,v))$ as given in Proposition 4.1. As standard in the theory of compensated compactness, for each k, we set μ^k the following Radon measure

$$\mu^k := \partial_t \eta(\lambda_k^-, \lambda_k^+) + \partial_x q(\lambda_k^-, \lambda_k^+)$$

where

$$\lambda_k^\pm := \frac{P^k \pm 1}{h^k}.$$

Since $\lambda_k^\pm$ is bounded, it follows that $\mu^k \in W^{-1,\infty}$. Moreover, by the Compactness Theorem for Radon measures, see [16], we have

$$\{\mu^k\} \text{ is pre-compact in } W^{-1,q}_{\text{loc}}, \quad 1 \le q < 2.$$

Hence, from a well-known interpolation result, we also have

$$\{\partial_t \eta(\lambda_k^-, \lambda_k^+) + \partial_x q(\lambda_k^-, \lambda_k^+)\} \text{ is pre-compact in } W^{-1,2}_{\text{loc}}.$$

By to Young measures theory, see [23, 25, 26], associated with the subsequence $\{(\lambda_k^-, \lambda_k^+)\}_{k=1}^\infty$ there exists a measurable family of Young measures $\nu_{(.)} : \mathbb{R}^+ \times \mathbb{R} \to \text{Prob}(\mathbb{R}^2)$, such that

$$\operatorname{supp} \nu_{(t,x)} \subset K \quad \text{for a.e. } (t,x) \in \mathbb{R}^+ \times \mathbb{R},$$

where K is a compact set of $\mathbb{R}^2$ and, $\text{Prob}(\mathbb{R}^2)$ is the space of nonnegative Radon measures over $\mathbb{R}$. Moreover, for any $g \in C(\mathbb{R}^2)$ the weak star limit

$$g(\lambda_k^-, \lambda_k^+) \overset{ast}{\rightharpoonup} \bar{g} \quad \text{in } L^\infty(\mathbb{R}^2),$$

as $k \to \infty$ exists, and

$$\bar{g}(t,x) = \int_{\mathbb{R}^2} g(\xi)\, d\nu_{(t,x)}(\xi) = \langle \nu_{(t,x)}, g \rangle \quad \text{for a.e. } (t,x) \in \mathbb{R}^+ \times \mathbb{R}.$$

Therefore, we are in the exact conditions to apply the Murat-Tartar div-curl Lemma, which we shall do five times in the following simple form. Let

$$a^k \rightharpoonup a, \quad b^k \rightharpoonup b, \quad c^k \rightharpoonup c, \quad d^k \rightharpoonup d,$$

weakly in $L^2_{\rm loc}$ as $k \to \infty$. Suppose that

$$\{\partial_t a^k + \partial_x b^k,\ \partial_t c^k + \partial_x d^k\} \subset U,$$

where U is a compact set of $W^{-1,2}_{\rm loc}$. Then, for a subsequence

$$a^k d^k - b^k c^k \rightharpoonup ad - bc \quad \text{as } k \to \infty$$

in the weak topology of measures. Hence, we have the following:

i) $a^k = P^k$, $b^k = \frac{(P^k)^2-1}{h^k}$, $c^k = D^k_2$, $d^k = \frac{B^k_3 + D^k_2 P^k}{h^k}$,

$$\frac{D^k_2}{h^k} + \frac{B^k_3 P^k}{h^k} \rightharpoonup \frac{D_2}{h} + \frac{B_3 P}{h};$$

ii) $a^k = P^k$, $b^k = \frac{(P^k)^2-1}{h^k}$, $c^k = B^k_3$, $d^k = \frac{D^k_2 + B^k_3 P^k}{h^k}$,

$$\frac{B^k_3}{h^k} + \frac{D^k_2 P^k}{h^k} \rightharpoonup \frac{B_3}{h} + \frac{D_2 P}{h};$$

iii) $a^k = P^k$, $b^k = \frac{(P^k)^2-1}{h^k}$, $c^k = -B^k_2$, $d^k = -\big(\frac{-D^k_3 + B^k_2 P^k}{h^k}\big)$,

$$\frac{-B^k_2}{h^k} + \frac{D^k_3 P^k}{h^k} \rightharpoonup \frac{-B_2}{h} + \frac{D_3 P}{h};$$

iv) $a^k = P^k$, $b^k = \frac{(P^k)^2-1}{h^k}$, $c^k = -D^k_3$, $d^k = -\big(\frac{B^k_2 + D^k_3 P^k}{h^k}\big)$,

$$\frac{-D^k_3}{h^k} + \frac{B^k_2 P^k}{h^k} \rightharpoonup \frac{-D_3}{h} + \frac{B_2 P}{h};$$

v) $a^k = \eta^k_1$, $b^k = q^k_2$, $c^k = \eta^k_2$, $d^k = q^k_1$,

where

$$\eta^k_i := \frac{\alpha_i(\lambda^-_k) + \beta_i(\lambda^+_k)}{\lambda^+_k - \lambda^-_k}, \quad q^k_i := \frac{\lambda^+_k\, \alpha_i(\lambda^-_k) + \lambda^-_k\, \beta_i(\lambda^+_k)}{\lambda^+_k - \lambda^-_k},$$

$$\eta^k_1\, q^k_2 - \eta^k_2\, q^k_1 \rightharpoonup \bar\eta_1\, \bar q_2 - \bar\eta_2\, \bar q_1 = \langle \nu, \eta_1 \rangle \langle \nu, q_2 \rangle - \langle \nu, \eta_2 \rangle \langle \nu, q_1 \rangle.$$

Here, for brevity $\nu \equiv \nu_{(t,x)}$.

3. The items (i)-(iv) are enough to pass to the limit as $k \to \infty$ in (4.5)-(4.8). Moreover, (4.9) is trivial since it is linear. Now, we show that (v) is also enough to pass to the limit in (4.10). Indeed, from (v)

$$\eta^k_1\, q^k_2 - \eta^k_2\, q^k_1 \rightharpoonup \langle \nu, \eta_1 \langle \rangle \nu, q_2 \rangle - \langle \nu, \eta_2 \rangle \langle \nu, q_1 \rangle.$$

On the other hand,

$$g(\lambda^-_k, \lambda^+_k) = \eta^k_1\, q^k_2 - \eta^k_2\, q^k_1 \rightharpoonup \bar g = \langle \nu, \eta_1\, q_2 - \eta_2\, q_1 \rangle.$$

Consequently, we obtain

$$\langle \nu, \eta_1\, q_2 - \eta_2\, q_1 \rangle = \langle \nu, \eta_1 \rangle \langle \nu, q_2 \rangle - \langle \nu, \eta_2 \rangle \langle \nu, q_1 \rangle. \tag{4.11}$$

Now, we note that

$$(\eta_1\, q_2 - \eta_2\, q_1)(\lambda^-, \lambda^+) = \frac{\alpha_2(\lambda^-)\, \beta_1(\lambda^+) - \alpha_1(\lambda^-)\, \beta_2(\lambda^+)}{\lambda^+ - \lambda^-},$$

and set the following positive Radon measure

$$\mu_{(t,x)} := \frac{\nu_{(t,x)}}{\lambda^+ - \lambda^-}.$$

Thus, from (4.11) it follows that

$$\begin{aligned}\langle \mu, \alpha_2\,\beta_1 - \alpha_1\,\beta_2\rangle &= \langle \mu, \alpha_1 + \beta_1\rangle\langle \mu, \lambda^+\alpha_2 + \lambda^-\beta_2\rangle \\ &\quad - \langle \mu, \alpha_2 + \beta_2\rangle\langle \mu, \lambda^+\alpha_1 + \lambda^-\beta_1\rangle.\end{aligned}$$

Hence, for $\beta_1 = \beta_2 = 0$, $\alpha_1 = \alpha_2 = 0$ and $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = -\beta_2 = \beta$, we obtain respectively

$$0 = \langle \mu, \alpha_1\rangle\langle \mu, \lambda^+\alpha_2\rangle - \langle \mu, \alpha_2\rangle\langle \mu, \lambda^+\alpha_1\rangle, \tag{4.12}$$

$$0 = \langle \mu, \beta_1\rangle\langle \mu, \lambda^-\beta_2\rangle - \langle \mu, \beta_2\rangle\langle \mu, \lambda^-\beta_1\rangle, \tag{4.13}$$

$$\langle \mu, \alpha\,\beta\rangle = \langle \mu, \beta\rangle\langle \mu, \lambda^+\alpha\rangle - \langle \mu, \alpha\rangle\langle \mu, \lambda^-\beta\rangle. \tag{4.14}$$

From (4.12), there exists a constant C^+ such that, for any continuous function α

$$\langle \mu, \lambda^+\,\alpha\rangle = C^+\langle \mu, \alpha\rangle.$$

Analogously, from (4.13) there exists a constant C^- such that, for any continuous function β

$$\langle \mu, \lambda^-\,\beta\rangle = C^-\langle \mu, \beta\rangle.$$

Therefore, from (4.14) we have

$$\langle \mu, \alpha\,\beta\rangle = (C^+ - C^-)\langle \mu, \alpha\rangle\langle \mu, \beta\rangle, \tag{4.15}$$

which means that, μ is the tensor product of two positive measures with supports contained respectively in $\mathbb{R}_{\lambda^-}$ and $\mathbb{R}_{\lambda^+}$, that is

$$\mu_{(t,x)} = \sigma^{\lambda^-}_{(t,x)} \otimes \theta^{\lambda^+}_{(t,x)}.$$

Moreover, since $\langle \mu, 1\rangle$ is positive it follows from (4.15)

$$1 = (C^+ - C^-)\langle \mu, 1\rangle = (C^+ - C^-)\langle \sigma, 1\rangle\langle \theta, 1\rangle. \tag{4.16}$$

Further, for all α, β

$$\begin{gathered}\langle \nu, \eta\rangle = \langle \mu, \alpha + \beta\rangle = \langle \sigma, \alpha\rangle\langle \theta, 1\rangle + \langle \sigma, 1\rangle\langle \theta, \beta\rangle, \\ \langle \nu, q\rangle = \langle \mu, \lambda^+\,\alpha + \lambda^-\,\beta\rangle = \langle \sigma, \alpha\rangle\langle \theta, \lambda^+\rangle + \langle \sigma, \lambda^-\rangle\langle \theta, \beta\rangle.\end{gathered}$$

Now, we recall that

$$h^k = \frac{2}{\lambda_k^+ - \lambda_k^-}, \quad P^k = \frac{\lambda_k^- + \lambda_k^+}{\lambda_k^+ - \lambda_k^-}.$$

Furthermore, we have

$$\frac{(P^k)^2 - 1}{h^k} = 2\frac{\lambda_k^-\,\lambda_k^+}{\lambda_k^+ - \lambda_k^-}.$$

Then, passing to the limit as $k \to \infty$, we get

$$h = \operatorname*{w-lim}_{k\to\infty} h^k = \langle \nu, \frac{2}{\lambda^+ - \lambda^-}\rangle = 2\langle \mu, 1\rangle = 2\langle \sigma, 1\rangle\langle \theta, 1\rangle,$$

$$\begin{aligned}P = \operatorname*{w-lim}_{k\to\infty} P^k &= \langle \nu, \frac{\lambda^- + \lambda^+}{\lambda^+ - \lambda^-}\rangle = \langle \mu, \lambda^- + \lambda^+\rangle \\ &= \langle \sigma, 1\rangle\langle \theta, \lambda^+\rangle + \langle \sigma, \lambda^-\rangle\langle \theta, 1\rangle,\end{aligned}$$

$$\operatorname*{w-lim}_{k\to\infty} \frac{(P^k)^2-1}{h^k} = 2\langle\nu, \frac{\lambda^-\lambda^+}{\lambda^+-\lambda^-}\rangle = 2\langle\mu, \lambda^-\lambda^+\rangle = 2\langle\sigma, \lambda^-\rangle\langle\theta, \lambda^+\rangle.$$

It remains to show that

$$\frac{P^2-1}{h} = 2\langle\mu, \lambda^-\lambda^+\rangle$$

or equivalently, $2h\langle\mu, \lambda^-\lambda^+\rangle = \langle\mu, \lambda^- + \lambda^+\rangle^2 - 1$. However,

$$\begin{aligned} 2h\langle\mu, \lambda^-\lambda^+\rangle &= 4\langle\mu, 1\rangle\langle\mu, \lambda^-\lambda^+\rangle \\ &= 4\langle\mu, 1\rangle C^+\langle\mu, \lambda^-\rangle \\ &= 4C^+C^-\langle\mu, 1\rangle^2 \\ &= [(C^+ + C^-)^2 - (C^+ - C^-)^2]\langle\mu, 1\rangle^2 \\ &= [(C^+ + C^-)\langle\mu, 1\rangle]^2 - 1 \\ &= \langle\mu, \lambda^- + \lambda^+\langle^2 - 1, \end{aligned}$$

where we have used (4.16). Hence, the proof is complete. □

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