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DECAY OF SOLUTIONS TO EQUATIONS MODELLING INCOMPRESSIBLE BIPOLAR NON-NEWTONIAN FLUIDS

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Abstract. This article concerns systems of equations that model incompressible bipolar non-Newtonian fluid motion in the whole space $\mathbb{R}^n$. Using the improved Fourier splitting method, we prove that a weak solution decays in the L^2 norm at the same rate as $(1+t)^{-n/4}$ as the time t approaches infinity. Also we obtain optimal L^2 error-estimates for Newtonian and Non-Newtonian flows.

1. Introduction

Consider the viscous incompressible fluid motion governed by the momentum and continuity equations

$$\partial_t u + (u\cdot\nabla)u - \nabla\cdot\tau^v + \nabla\pi = 0 \quad \text{in } \mathbb{R}^n\times(0,\infty), \tag{1.1}$$

$$\nabla\cdot u = 0 \quad \text{in } \mathbb{R}^n\times(0,\infty), \tag{1.2}$$

with the initial condition

$$u(x,0) = u_0 \quad \text{in } \mathbb{R}^n. \tag{1.3}$$

Here $n\ge 2$, the gradient $\nabla = (\partial_{x_1},\dots,\partial_{x_n})$, $u=(u_1,\dots,u_n)$ and π denote the unknown velocity and pressure of the fluid motion. $\tau^v = (\tau^v_{ij})$ is the stress tensor specified in the form

$$\tau^v_{ij} = 2\ (\mu_0 + \mu_1|e(u)|^{p-2})\ e_{ij}(u) - 2\mu_2\Delta e_{ij}(u) \tag{1.4}$$

with the constant viscosities $\mu_0>0$, μ_1, $\mu_2\ge 0$ and the symmetric deformation velocity tensor $e(u) = (e_{ij}(u))$,

$$e_{ij}(u) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \quad |e(u)| = (e_{ij}(u)e_{ij}(u))^{1/2}. \tag{1.5}$$

When $\mu_1=\mu_2=0$, the Stokes Law

$$\tau^v_{ij} = 2\mu_0 e_{ij}(u) \tag{1.6}$$

holds. The fluids, such as water and alcohol, satisfying the linear equation expressed by (1.6) is said to be Newtonian, and (1.1) turns out to be the famous Navier-Stokes equations (refer to [10, 16]). However the nonlinear constitutive equation expressed by (1.4) with $\mu_1,\mu_2>0$ is related to other non-Newtonian fluids such as the molten

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plastics, dyes, adhesives, paints and greases. When $\mu_2 = 0$, system (1.1)-(1.4) was first proposed by Ladyzhenskaya [11] and is known as the Ladyzhenskaya equations. The fluid is said to be monopolar because the only first order derivative of the velocity field is involved in the stress tensor (see (1.4) for $\mu_2 = 0$), whereas the fluid is bipolar if the second order derivative arises in τ^v (see (1.4) for $\mu_1, \mu_2 > 0$). The theory of bipolar fluids is compatible with the basic principles of thermodynamics, including the Clausius-Duhem inequality and the material frame indifference (See [1] for a detailed description of multipolar fluids). Moreover, the fluid is shear thinning if $p < 2$ and shear thickening if $p > 2$ (when $p = 2$, the system turns out to be Navier-Stokes equations).

There is an extensive literature on the solutions of the incompressible non-Newtonian fluids. Ladyzhenskaya [11] and Lions [12] first discussed the existence and uniqueness of weak solutions of the sort monopolar model (see (1.4) for $\mu_2 = 0$), and more recently, Du and Gunzguiger [5] studied the somewhat more general existence and uniqueness results in bounded domains. Pokorny [14] investigated the Cauchy problem for both monopolar and bipolar fluids in whole spaces. As for the decay properties of solutions, on the one hand, Necăsová and Penel [13] recently examined the logarithmic decay in $\mathbb{R}^2$ and algebraic decay in $\mathbb{R}^3$ with respect to the monopolar shear thickening fluids ($p \geq 3$) by the Schonbek's Fourier splitting method [15]. With the aid of Wiegner's method [17], Guo and Zhu [6] improved the algebraic decay rates. Higher decay rates were recently proved by Dong [3] based on the arguments of Kajikiya and Miyakawa [8]. In particular, by improving Schonbek's Fourier splitting method, the optimal algebraic decay rate in $\mathbb{R}^2$ of this monopolar model was obtained by Dong and Li [2] in the following form

$$\|u(t)\|_{L^2} \leq C(1+t)^{-1/2}, \quad \|u(t) - e^{t\Delta}u_0\|_{L^2} \leq C(1+t)^{-3/4}, \quad \forall\ t > 0. \tag{1.7}$$

On the other hand, Guo and Zhu [7] considered also the decay of the weak solution of the bipolar fluids (see (1.4) for $\mu_1, \mu_2 > 0$). Based on the fourth order linear parabolic equation and the Wiegner's method [17], the decay rate of the L^2 norm they obtained is only one-half of the decay rate to the linear heat equation, that is

$$\begin{aligned} &\|u(t)\|_{L^2} \leq C(1+t)^{-\frac{n}{4r}-\frac{n}{8}}, \forall\ t > 0;\\ &\|u(t) - e^{t\Delta}u_0\|_{L^2} \to 0, \quad \text{as } t \to \infty. \end{aligned} \tag{1.8}$$

assuming $L^r \cap L^2$ integrability of the initial data. Furthermore, as for the time decay of Navier-Stokes equations in whole spaces, the sharp decay rates were obtained by Schonbek[15], Kajikiya and Miyakawa [8], Wiegner[17] and references cited therein.

The aim of this paper is investigate the optimal rate of decay of global solutions to the Cauchy problem of the bipolar shear thinning fluids (1.1)-(1.4) ($p \geq 3,\ \mu_1, \mu_2 > 0$ in (1.4)). We use the improved Fourier splitting methods developed by Dong et al [2, 4] and Zhang [18], and the rigorous analysis of the lower frequency effect of the lower dissipative term Δu which determines mainly the time decay rates of the solutions. We obtain the optimal L^2-decay rate, which is the same as that of the linear heat equation

$$\|u(t)\|_{L^2} \leq C(1+t)^{-n/4}, \quad \forall\ t > 0, \tag{1.9}$$

assuming $L^1 \cap L^2$ integrability of the initial data. Furthermore, since the bipolar non-Newtonian flow is modified from the Newtonian flow, we examine the L^2-decay estimates of the error $u(t) - \tilde{u}(t)$. Here $u(t)$ denotes weak solution of the non-Newtonian system (1.1)-(1.4) with $\mu_1, \mu_2 > 0$, whereas $\tilde{u}$ denotes the weak solution of the Newtonian system of (1.1)-(1.4) with $\mu_1 = \mu_2 = 0$. The optimal error estimates we obtained are the following

$$\|u(t) - \tilde{u}(t)\| = o\left((1+t)^{-n/4}\right), \quad \text{as } t \to \infty. \tag{1.10}$$

This paper is organized as follows. In Sections 2 we define weak solutions and state some preliminary lemmas. Decay estimates of the non-Newtonian flow are described in Section 3. Decay estimates of the error between the non-Newtonian and Newtonian flows $u(t) - \tilde{u}(t)$ are derived in Section 4.

2. Preliminaries

Let $\|\cdot\|_q = \|\cdot\|_{L^q}$ ($\|\cdot\| = \|\cdot\|_2$) be the norm of the usual scalar and vector Lebesgue space $L^q(\mathbb{R}^n)$ and $\|\cdot\|_{m,p} = \|\cdot\|_{W^{m,p}}$ be the norm of the Sobolev space $W^{m,p}(\mathbb{R}^n)$. The space $\mathbf{H}$ denotes the L^2-closure of $C_{0,\sigma}^\infty(\mathbb{R}^n)$, which is the set of smooth divergence-free vector fields with compact supports in $\mathbb{R}^n$. The space $W_{0,\sigma}^{1,q}(\mathbb{R}^n)$ denotes the closure of $C_{0,\sigma}^\infty(\mathbb{R}^n)$ in $W^{1,q}(\mathbb{R}^n)$. The Fourier transformation of a function f is denoted by $\hat{f}$ or $F[f]$. $C > 0$, independent of the quantities t, x, ρ, u and $\tilde{u}$, is a generic constant, which may depend on the initial data u_0.

Without loss of generality, we assume that $\mu_0 = \mu_1 = \mu_2 = 1$ in (1.4). Substitution of (1.4) into (1.1) produces

$$u_t - \triangle u + \Delta^2 u + (u\cdot\nabla)u - \nabla\cdot(|\nabla u|^{p-2}\nabla u) + \nabla\pi = 0, \tag{2.1}$$

in $\mathbb{R}^n \times (0,\infty)$.

By a weak solution of the initial value problem (1.2)-(1.4) and (2.1) for $n \geq 2$ and $p \geq 1 + \frac{2n}{n+2}$ (see [11, 12, 14]), we mean a vector field

$$u \in L^p(0,T;W_{0,\sigma}^{1,p}(\mathbb{R}^n)) \cap L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;W_{0,\sigma}^{2,2}(\mathbb{R}^n)), \quad \forall T > 0$$

satisfying

$$\begin{aligned} &\int_{\mathbb{R}^n} u(t)\cdot\varphi(t)\,dx - \int_0^t\int_{\mathbb{R}^n} u\cdot\frac{\partial\varphi}{\partial s}\,dxds \\ &+ \int_0^t\int_{\mathbb{R}^n} u_j\frac{\partial u_i}{\partial x_j}\varphi_i\,dxds + \int_0^t\int_{\mathbb{R}^n}\tau_{ij}(e(u))\cdot e_{ij}(\varphi)dxds = \int_{\mathbb{R}^n} u_0\cdot\varphi(0)\,dx \end{aligned} \tag{2.2}$$

a. e. $t \in (0,T)$ for every $\varphi \in C^1([0,T),\mathbf{H}) \cap C([0,T), W_{0,\sigma}^{2,2}(\mathbb{R}^n) \cap W_{0,\sigma}^{1,p}(\mathbb{R}^n))$ and $\varphi(x,T) = 0$. Moreover, we assume that the weak solution also satisfies the following energy inequality

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}|u|^2dx + \int_{\mathbb{R}^n}|\nabla u|^2\,dx + \int_{\mathbb{R}^n}|\Delta u|^2\,dx + \int_{\mathbb{R}^n}|\nabla u|^p\,dx \leq 0. \tag{2.3}$$

It should be noted that a weak solution can be specified as a limit of a sequence of smooth approximate solutions in a local L^2-norm due to the standard Faedo-Galerkin argument. Thus the decay estimates with respect to the weak solution become limits of those of the smooth approximate solutions (see, for example, Kajikiya and Miyakawa [8]). Since we only consider the decay estimates of the weak solution in L^2-norm, without loss of generality, we may suppose that the weak solution admit enough regularity so that we can work on the weak solution directly rather than on the sequence of smooth approximate solutions.

Let us now recall some preliminary lemmas.

lemma 2.1 (Gronwall Inequality). *Let $f(t), g(t), h(t)$ be nonnegative continuous functions and satisfying the inequality*

$$g(t) \le f(t) + \int_0^t g(s)h(s)ds, \quad \forall\ t > 0,$$

where $f'(t) \ge 0$. Then

$$g(t) \le f(t) \exp\Big(\int_0^t h(s)ds \Big), \quad \forall\ t > 0. \tag{2.4}$$

lemma 2.2. *Assume that $u_0 \in \mathbf{H} \cap L^1(\mathbb{R}^n)$ and u is a weak solution of (1.2)-(1.4) and (2.1). Then*

$$\sup_{0 \le t \le \infty} \|u(t)\| \le \|u_0\|, \tag{2.5}$$

and (i) $2 < p < 3, n = 2$,

$$|\hat{u}(\xi, t)| \le C + C|\xi| \int_0^t \|u(s)\|^2 ds + C|\xi| \Big(\int_0^t \|u(s)\|^{\frac{2}{4-p}} ds \Big)^{\frac{4-p}{2}}, \tag{2.6}$$

(ii) $1 + \frac{2n}{n+2} \le p < 3, n \ge 3$,

$$|\hat{u}(\xi, t)| \le C + C|\xi| \int_0^t \|u(s)\|^2 ds + C|\xi| \left(\int_0^t \|u(s)\|^{\frac{2\alpha}{2-\beta}} ds \right)^{\frac{2-\beta}{2}}, \tag{2.7}$$

where $\alpha = \frac{2n-(n-2)(p-1)}{4}$, $\beta = \frac{(n+2)(p-1)-2n}{4}$,
(iii) $p \ge 3, n \ge 2$,

$$|\hat{u}(\xi, t)| \le C + C|\xi| \int_0^t \|u(s)\|^2 ds + C|\xi|. \tag{2.8}$$

Proof. From the energy inequality (2.3), it is easy to get the first inequality (2.5). We now prove (2.6)-(2.8), first, applying the Fourier transformation of (2.1) we have

$$\hat{u}_t + (|\xi|^2 + |\xi|^4)\hat{u} = F[\nabla \cdot (|e(u)|^{p-2}e(u)) - (u \cdot \nabla)u - \nabla\pi] =: G(\xi, t). \tag{2.9}$$

Now we estimate $G(\xi, t)$. Taking divergence in (2.1) to get,

$$\Delta\pi = \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} [-u_i u_j + |e(u)|^{p-2} e_{ij}(u)].$$

The Fourier transformation yields

$$|\xi|^2 F[\pi] = \sum_{i,j} \xi_i \xi_j F[-u_i u_j + |e(u)|^{p-2} e_{ij}(u)],$$

and thus

$$|F[\nabla\pi]| = |\xi| F[\pi] \le |F[\nabla \cdot (|e(u)|^{p-2}e(u))]| + |F[(u \cdot \nabla)u]|. \tag{2.10}$$

Furthermore,

$$|F[(u \cdot \nabla)u]| = |F[\mathrm{div}(u \otimes u)]| \le \sum_{i,j} \int_{\mathbb{R}^n} |u_i u_j|\, |\xi_j| dx \le |\xi|\, \|u\|^2, \tag{2.11}$$

$$|F[\nabla \cdot (|e(u)|^{p-2}e(u))]| \le |\xi| |F[|e(u)|^{p-2}e(u)]| \le |\xi| \|\nabla u\|_{p-1}^{p-1}. \tag{2.12}$$

So inserting (2.10)-(2.12) into $G(\xi, t)$, we have

$$|G(\xi, t)| \le C|\xi| \|u\|^2 + C|\xi| \|\nabla u\|_{p-1}^{p-1}. \tag{2.13}$$

From (2.9), it follows easily that,

$$\frac{d}{dt}\left(\hat{u}e^{(|\xi|^2+|\xi|^4)t}\right) \le G(\xi,t)e^{(|\xi|^2+|\xi|^4)t}.$$

Integrating in time gives,

$$\begin{aligned}
|\hat{u}(\xi,t)| &\le \left|e^{-(|\xi|^2+|\xi|^4)t}\hat{u}_0(\xi) + \int_0^t G(\xi,t)e^{-(|\xi|^2+|\xi|^4)(t-s)}ds\right| \\
&\le |\hat{u}_0(\xi)| + \int_0^t |G(\xi,t)|ds \\
&\le C + |\xi|\int_0^t \|u(s)\|^2 ds + C|\xi|\int_0^t \|\nabla u(s)\|_{p-1}^{p-1} ds.
\end{aligned} \tag{2.14}$$

Now we estimate $\int_0^t \|\nabla u(s)\|_{p-1}^{p-1}ds$ in three cases (note that the case $p=2, n=2$ is the Navier-Stokes equations [10, 16]):

(i) $2 < p < 3$, $n = 2$;
(ii) $1 + \frac{2n}{n+2} \le p < 3$, $n \ge 3$;
(iii) $p \ge 3$, $n \ge 2$.

Case (i): $2 < p < 3, n = 2$. By Gagliardo-Nirenberg inequality (refer to [9]),

$$\begin{aligned}
\int_0^t \|\nabla u(s)\|_{p-1}^{p-1}ds &\le \int_0^t |\xi|\|u(s)\|\|D^2u(s)\|^{p-2}ds \\
&\le \Big(\int_0^t \|u(s)\|^{\frac{2}{4-p}}ds\Big)^{\frac{4-p}{2}}\Big(\int_0^\infty \|D^2u(t)\|^2dt\Big)^{\frac{p-2}{2}} \\
&\le \Big(\int_0^t \|u(s)\|^{\frac{2}{4-p}}ds\Big)^{\frac{4-p}{2}},
\end{aligned}$$

noting that $\int_0^\infty \|D^2u(t)\|^2dt \le C$.

Case (ii): $1 + \frac{2n}{n+2} \le p < 3, n \ge 3$.

$$\begin{aligned}
\int_0^t \|\nabla u(s)\|_{p-1}^{p-1}ds &\le \int_0^t \|u(s)\|^\alpha\|D^2u(s)\|^\beta ds \\
&\le \Big(\int_0^t \|u(s)\|^{\frac{2\alpha}{2-\beta}}ds\Big)^{\frac{2-\beta}{2}}\Big(\int_0^\infty \|D^2u(t)\|^2dt\Big)^{\frac{\beta}{2}} \\
&\le \Big(\int_0^t \|u(s)\|^{\frac{2\alpha}{2-\beta}}ds\Big)^{\frac{2-\beta}{2}},
\end{aligned} \tag{2.15}$$

where $\alpha = \frac{2n-(n-2)(p-1)}{4}$, $\beta = \frac{(n+2)(p-1)-2n}{4}$, and $0 < \beta < 1$.

Case (iii): $p \ge 3$, $n \ge 2$. With the above definition of the weak solution, we know also that $\nabla u \in L^2((0,\infty)\times\mathbb{R}^n) \cap L^p((0,\infty)\times\mathbb{R}^n)$, so by using the interpolation technology, $\nabla u \in L^{p-1}((0,\infty)\times\mathbb{R}^n)$, i.e.

$$\int_0^\infty \|\nabla u(s)\|_{p-1}^{p-1}ds \le C. \tag{2.16}$$

Hence (2.14)-(2.16) imply the assertions of lemma 2.2 and the proof is complete. □

3. Decay estimates of the non-Newtonian flows

As is well known that the weak solutions of Navier-Stokes equations have the optimal decay estimates in the whole space [8, 15, 17]. In this section, we show that the non-Newtonian flow has also the same optimal L^2 time decay estimates. The results read as follows.

Theorem 3.1. *Assume that $u_0 \in \mathbf{H} \cap L^1(\mathbb{R}^n)$. Let $u(t)$ be a weak solution of (1.2)-(1.4) and (2.1). Then, for $n = 2$, $p > 2$ and for $n \geq 3$, $p \geq 1 + \frac{2n}{n+2}$, we have*

$$\|u(t)\| \leq C(1+t)^{-n/4}, \quad \forall\, t > 0. \tag{3.1}$$

Proof. From the energy inequality (2.3), it follows that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} |u|^2 dx + \int_{\mathbb{R}^n} |\nabla u|^2 \leq 0. \tag{3.2}$$

Applying Plancherel's theorem to (3.2) yields

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} |\hat{u}(\xi,t)|^2 d\xi + \int_{\mathbb{R}^n} |\xi|^2|\hat{u}(\xi,t)|^2 d\xi \leq 0. \tag{3.3}$$

Let $f(t)$ be a smooth function of t with $f(0) = 1, f(t) > 0$ and $f'(t) > 0$, then

$$\frac{d}{dt}\Big(f(t)\int_{\mathbb{R}^n} |\hat{u}(\xi,t)|^2 d\xi\Big) + 2f(t)\int_{\mathbb{R}^n} |\xi|^2|\hat{u}(\xi,t)|^2 d\xi \leq f'(t)\int_{\mathbb{R}^n} |\hat{u}(\xi,t)|^2 d\xi.$$

Set $B(t) = \{\xi \in \mathbb{R}^n : 2f(t)|\xi|^2 \leq f'(t)\}$. Then

$$\begin{aligned}
&2f(t)\int_{\mathbb{R}^n} |\xi|^2|\hat{u}(\xi,t)|^2 d\xi \\
&= 2f(t)\int_{B(t)} |\xi|^2|\hat{u}(\xi,t)|^2 d\xi + 2f(t)\int_{B(t)^c} |\xi|^2|\hat{u}(\xi,t)|^2 d\xi \\
&\geq 2f(t)\int_{B(t)^c} |\xi|^2|\hat{u}(\xi,t)|^2 d\xi \\
&\geq f'(t)\int_{\mathbb{R}^n} |\hat{u}(\xi,t)|^2 d\xi - f'(t)\int_{B(t)} |\hat{u}(\xi,t)|^2 d\xi.
\end{aligned}$$

Therefore,

$$\frac{d}{dt}\Big(f(t)\int_{\mathbb{R}^n} |\hat{u}(\xi,t)|^2 d\xi\Big) \leq f'(t)\int_{B(t)} |\hat{u}(\xi,t)|^2 d\xi.$$

Integrating in time yields

$$f(t)\int_{\mathbb{R}^n} |\hat{u}(\xi,t)|^2 d\xi \leq \int_{\mathbb{R}^n} |\hat{u}_0|^2 d\xi + C\int_0^t f'(s)\int_{B(s)} |\hat{u}(\xi,s)|^2 d\xi ds. \tag{3.4}$$

Now we study three cases: (i) $2 < p < 3$, $n = 2$; (ii) $1 + \frac{2n}{n+2} \leq p < 3$, $n \geq 3$; (iii) $p \geq 3$, $n \geq 2$.

(i) Case $2 < p < 3$**,** $n = 2$**.** Let $A^2 = \frac{f'(t)}{2f(t)}$, and ω_n be volume of unit ball in $\mathbb{R}^n$. According to (2.6),

$$
\begin{aligned}
& f(t)\int_{\mathbb{R}^2}|\hat{u}(\xi,t)|^2 d\xi \\
& \le \int_{\mathbb{R}^2}|\hat{u}_0|^2 d\xi + C\omega_n \int_0^t f'(s)\int_0^A \Big\{1 \\
& \quad + \rho\int_0^s \|u(\tau)\|^2 d\tau + \rho\Big(\int_0^s \|u(\tau)\|^{\frac{2}{4-p}} d\tau\Big)^{\frac{4-p}{2}}\Big\}^2 \rho\, d\rho\, ds \\
& \le C + C\int_0^t f'(s)\Big\{\frac{f'(s)}{2f(s)} + \Big(\frac{f'(s)}{2f(s)}\Big)^2\Big(\int_0^s \|u(\tau)\|^2 d\tau\Big)^2\Big\} ds \\
& \quad + C\int_0^t f'(s)\Big\{\Big(\frac{f'(s)}{2f(s)}\Big)^2\Big(\int_0^s \|u(\tau)\|^{\frac{2}{4-p}} d\tau\Big)^{4-p}\Big\} ds.
\end{aligned}
\tag{3.5}
$$

On the one hand, using (2.5) on the right hand side of (3.5), we obtain

$$
f(t)\int_{\mathbb{R}^2}|\hat{u}(\xi,t)|^2 d\xi \le C + C\int_0^t f'(s)\Big\{\frac{f'(s)}{2f(s)} + \big(\frac{f'(s)}{2f(s)}\big)^2 (s^2 + s^{4-p})\Big\} ds. \tag{3.6}
$$

Let $f(t) = (\ln(e+t))^5$. Then $f'(t) = \frac{5(\ln(e+t))^4}{e+t}$, and $\frac{f'(t)}{f(t)} = \frac{5}{(e+t)\ln(e+t)}$. By (3.6) and an elementary calculation based on Plancherel's theorem, we have

$$
\begin{aligned}
& (\ln(e+t))^5\int_{\mathbb{R}^2}|u(x,t)|^2 dx \\
& = (\ln(e+t))^5\int_{\mathbb{R}^2}|\hat{u}(\xi,t)|^2 d\xi \\
& \le C + C\int_0^t \Big\{\frac{(\ln(e+s))^3}{(e+s)^2} + \frac{s^2(\ln(e+s))^2}{(e+s)^3} + \frac{s^{4-p}(\ln(e+s))^2}{(e+s)^3}\Big\} ds \\
& \le C + C\int_0^t \frac{(\ln(e+s))^2}{e+s} ds \quad (\text{because } 1 < 4-p < 2) \\
& \le C(\ln(e+t))^3,
\end{aligned}
$$

and so

$$
\|u(t)\| \le C(\ln(e+t))^{-1}. \tag{3.7}
$$

By the inductive argument, we suppose that

$$
\|u(t)\| \le C(\ln(e+t))^{-m} \quad \forall\, m \in \mathbf{N}. \tag{3.8}
$$

Inserting (3.8) into the right hand side of (3.5), letting $f(t) = (\ln(e+t))^{2m+3}$, and using (3.5) and the inequality $\int_0^t (\ln(e+s))^{-m} ds \le C(e+t)\ln(e+t))^{-m}$, thus from

(3.5)

$$
\begin{aligned}
&(\ln(e+t))^{2m+3}\int_{\mathbb{R}^2}|u(x,t)|^2dx\\
&=(\ln(e+t))^{2m+3}\int_{\mathbb{R}^2}|\hat{u}(\xi,t)|^2d\xi\\
&\le C+C\int_0^t\Big\{\frac{(\ln(e+s))^{2m+1}}{(e+s)^2}+\frac{1}{e+s}+\frac{(\ln(e+s))^{2m-(4-p)m}}{(e+s)(e+s)^{p-2}}\Big\}ds\\
&\le C+C\int_0^t\frac{1}{e+s}ds\\
&\le C\ln(e+t).
\end{aligned}
$$

Here we used that $(\ln(e+s))^k\le c(k)(e+s)$, for all $k>0$. The above inequality implies

$$\|u(t)\|\le C(\ln(e+t))^{-m-1}\quad\forall\, m\in\mathbf{N}. \tag{3.9}$$

On the other hand, from (3.5) and the Hölder inequality,

$$
\begin{aligned}
&f(t)\int_{\mathbb{R}^2}|\hat{u}(\xi,t)|^2d\xi\\
&\le C+C\int_0^t f'(s)\frac{f'(s)}{2f(s)}ds+\int_0^t sf'(s)\Big(\frac{f'(s)}{2f(s)}\Big)^2ds\int_0^t\|u(s)\|^4ds\\
&\quad+C\int_0^t s^{\frac{4-p}{2}}f'(s)\Big(\frac{f'(s)}{2f(s)}\Big)^2ds\Big(\int_0^t\|u(s)\|^{\frac{4}{4-p}}ds\Big)^{\frac{4-p}{2}}.
\end{aligned}
$$

Let $f(t)=(1+t)^2$. Then

$$
\begin{aligned}
&(1+t)^2\int_{\mathbb{R}^2}|\hat{u}(\xi,t)|^2d\xi\\
&\le C(1+t)+C(1+t)\int_0^t\|u(s)\|^4ds+C(1+t)^{\frac{4-p}{2}}\Big(\int_0^t\|u(s)\|^{\frac{4}{4-p}}ds\Big)^{\frac{4-p}{2}}.
\end{aligned}\tag{3.10}
$$

Noting that $\frac{1}{2}<\frac{4-p}{2}<1$ and applying the Young inequality to the last term of (3.10), we have

$$\Big(\int_0^t\|u(s)\|^{\frac{4}{4-p}}ds\Big)^{\frac{4-p}{2}}\le C\int_0^t\|u(s)\|^{\frac{4}{4-p}}ds+C. \tag{3.11}$$

Inserting (3.8) and (3.11) into (3.10), we get the following estimate

$$
\begin{aligned}
(1+t)\int_{\mathbb{R}^2}|\hat{u}(\xi,t)|^2d\xi\le C+C\int_0^t\|u(s)\|^2(1+s)\Big\{&(1+s)^{-1}(\ln(e+s))^{-m}\\
&+(1+s)^{-1}(\ln(e+s))^{-\frac{m(2p-4)}{4-p}}\Big\}ds.
\end{aligned}
$$

Let

$$
\begin{gathered}
g(t)=(1+t)\int_{\mathbb{R}^2}|\hat{u}(\xi,t)|^2d\xi=(1+t)\int_{\mathbb{R}^2}|u(x,t)|^2dx,\, f(t)=C,\\
h(t)=(1+t)^{-1}(\ln(e+t))^{-m}+(1+t)^{-1}(\ln(e+t))^{-\frac{m(2p-4)}{4-p}}.
\end{gathered}
$$

When the integer m is suitable large, it is simple to deduce that $\int_0^\infty h(t)dt < \infty$. Applying Lemma 2.1, we have

$$g(t) \le C\exp\Big(\int_0^\infty h(t)dt\Big) \le C,$$

and thus

$$\|u(t)\| \le C(1+t)^{-1/2}.$$

(ii) Case $1+\frac{2n}{n+2} \le p < 3$, $n \ge 3$. Inserting (2.7) into the right hand of (3.4) and using Hölder inequality and Young inequality, we have

$$\begin{aligned}
&f(t)\int_{\mathbb{R}^n} |\hat{u}(\xi,t)|^2 d\xi\\
&\le \int_{\mathbb{R}^n} |\hat{u}_0| d\xi + C\omega_n \int_0^t f'(s)\int_0^A \Big\{1\\
&\quad + \rho\int_0^s \|u(\tau)\|^2 d\tau + \rho\Big(\int_0^s \|u(\tau)\|^{\frac{2\alpha}{2-\beta}} d\tau\Big)^{\frac{2-\beta}{2}}\Big\}^2 \rho^{n-1} d\rho ds\\
&\le C + C\int_0^t f'(s)\Big\{\Big(\frac{f'(s)}{2f(s)}\Big)^{n/2} + \Big(\frac{f'(s)}{2f(s)}\Big)^{\frac{n+2}{2}} s\int_0^s \|u(\tau)\|^4 d\tau\Big\} ds\\
&\quad + C\int_0^t f'(s)\Big(\frac{f'(s)}{2f(s)}\Big)^{\frac{n+2}{2}} s^{\frac{2-\beta}{2}} ds\Big(\int_0^t \|u(s)\|^{\frac{4\alpha}{2-\beta}} ds\Big)^{\frac{2-\beta}{2}}\\
&\le C + C\int_0^t f'(s)\Big\{\Big(\frac{f'(s)}{2f(s)}\Big)^{n/2} + \Big(\frac{f'(s)}{2f(s)}\Big)^{\frac{n+2}{2}} s\int_0^t \|u(s)\|^4 ds\Big\} ds\\
&\quad + C\int_0^t f'(s)\Big(\frac{f'(s)}{2f(s)}\Big)^{\frac{n+2}{2}} s^{\frac{2-\beta}{2}} ds\Big(\int_0^t \|u(s)\|^{\frac{4\alpha}{2-\beta}} ds + C\Big).
\end{aligned}$$

Let $f(t) = (1+t)^n$. Noting that $\frac{1}{2} < \frac{2-\beta}{2} < 1$, $\frac{4\alpha}{2-\beta} > 2$ and $\|u(t)\| \le C$, we have

$$\begin{aligned}
&(1+t)^n \int_{\mathbb{R}^n} |\hat{u}(\xi,t)|^2 d\xi\\
&\le C(1+t)^{n/2} + C(1+t)^{n/2}\int_0^t \|u(s)\|^4 ds + C(1+t)^{n/2}\int_0^t \|u(s)\|^{\frac{4\alpha}{2-\beta}} ds\\
&\le C(1+t)^{n/2} + C(1+t)^{n/2}\int_0^t \|u(s)\|^2 ds.
\end{aligned}$$

This yields

$$\begin{aligned}
(1+t)^{n/2}\|u(t)\|^2 &= (1+t)^{n/2}\int_{\mathbb{R}^n} |\hat{u}(\xi,t)|^2 d\xi\\
&\le C + \int_0^t (1+s)^{n/2}\|u(s)\|^2 (1+s)^{-\frac{n}{2}} ds.
\end{aligned}$$

Letting

$$f(t) = C, \quad g(t) = (1+t)^{n/2}\|u(t)\|^2, \quad h(t) = (1+t)^{-n/2},$$

applying Lemma 2.1 and the bound $\int_0^\infty h(s)ds \le C$, we deduce readily that

$$\|u(t)\| \le C(1+t)^{-n/4}.$$

(iii) Case $p \geq 3$, $n \geq 2$. Inserting (2.8) into the right hand of (3.4),

$$\begin{aligned} &f(t)\int_{\mathbb{R}^n}|\hat{u}(\xi,t)|^2 d\xi \\ &\leq C + C\omega_n \int_0^t f'(s)\int_0^A \Big(1+\rho\int_0^s \|u(\tau)\|^2 d\tau + \rho\Big)^2 \rho^{n-1} d\rho\, ds \\ &\leq C + C\int_0^t f'(s)\Big\{\Big(\frac{f'(s)}{2f(s)}\Big)^{n/2} + \Big(\frac{f'(s)}{2f(s)}\Big)^{\frac{n+2}{2}}\Big(\Big(\int_0^s \|u(\tau)\|^2 d\tau\Big)^2 + 1\Big)\Big\}\, ds. \end{aligned} \tag{3.12}$$

First, we discuss the case $n = 2$. It follows from (3.12) and the bound $\|u(t)\| \leq C$ that

$$f(t)\int_{\mathbb{R}^2}|\hat{u}(\xi,t)|^2 d\xi \leq C + C\int_0^t f'(s)\Big(\frac{f'(s)}{2f(s)} + \Big(\frac{f'(s)}{2f(s)}\Big)^2(s^2+1)\Big)ds.$$

Let $f(t) = (\ln(e+t))^5$. By the same calculation as that of (3.7), we have

$$\|u(t)\| \leq C(\ln(e+t))^{-1}. \tag{3.13}$$

Hence, letting $f(t) = (1+t)^2$ in (3.10) and using the Hölder inequality,

$$\begin{aligned} (1+t)^2\int_{\mathbb{R}^2}|\hat{u}(\xi,t)|^2 d\xi &\leq C(1+t) + C\int_0^t (1+s)^{-1}\Big(\int_0^s \|u(\tau)\|^2 d\tau\Big)^2 ds \\ &\leq C(1+t) + C\int_0^t\int_0^s \|u(\tau)\|^4 d\tau ds \\ &\leq C(1+t) + C(1+t)\int_0^t \|u(s)\|^4 ds. \end{aligned}$$

By (3.11), we obtain the inequality

$$(1+t)\int_{\mathbb{R}^2}|\hat{u}(\xi,t)|^2 d\xi \leq C + C\int_0^t \|u(s)\|^2(1+s)\left((1+s)^{-1}(\ln(e+s))^{-2}\right) ds.$$

Let

$$\begin{gathered} g(t) = (1+t)\int_{\mathbb{R}^2}|\hat{u}(\xi,t)|^2 d\xi = (1+t)\int_{\mathbb{R}^2}|u(x,t)|^2 dx, \\ h(t) = (1+t)^{-1}(\ln(e+t))^{-2}, \quad f(t) = C\,. \end{gathered}$$

Applying Lemma 2.1, we have

$$g(t) \leq C\exp\Big(\int_0^\infty h(t)dt\Big) \leq C,$$

and so

$$\|u(t)\| \leq C(1+t)^{-1/2}.$$

Next, we carry out the proof in the case $n \geq 3$. Letting $f(t) = (1+t)^n$ in (3.10) and using the Hölder inequality, we have, similar to the argument in the case of $n = 2$,

$$\begin{aligned} &(1+t)^n\int_{\mathbb{R}^n}|\hat{u}(\xi,t)|^2 d\xi \\ &\leq C + C(1+t)^{-\frac{n}{2}} + C(1+t)^{-\frac{n+2}{2}} + C(1+t)^{-\frac{n}{2}}\int_0^t \|u(s)\|^4\, ds. \end{aligned}$$

Thus

$$(1+t)^{n/2}\int_{\mathbb{R}^n}|\hat{u}(\xi,t)|^2d\xi \le C + C\int_0^t \|u(s)\|^2(1+s)^{n/2}(1+s)^{-\frac{n}{2}}ds.$$

Since $\int_0^\infty (1+s)^{-\frac{n}{2}}ds \le C$ when $n\ge 3$. Thus applying Lemma 2.1, we have

$$(1+t)^{n/2}\int_{\mathbb{R}^n}|\hat{u}(\xi,t)|^2d\xi \le C\exp\Big(\int_0^\infty (1+t)^{-\frac{n}{2}}dt\Big) \le C.$$

Hence $\|u(t)\|\le C(1+t)^{-n/4}$. The proof of Theorem 3.1 is complete. □

4. Error estimates for Newtonian and non-Newtonian flows

Theorem 4.1. *In addition to the assumption of Theorem 3.1, suppose that $\tilde{u}$ denotes the weak solution of the Newtonian system* (1.1)-(1.4) *with $\mu_1=\mu_2=0$. Then*

$$\|u(t)-\tilde{u}(t)\| = o\Big((1+t)^{-n/4}\Big), \quad \text{as } t\to\infty.$$

Note that the estimates of Theorem 4.1 with $u(t)-\tilde{u}(t)$ replaced by $e^{t\Delta}u_0-\tilde{u}(t)$ also hold (see Kajikiya and Miyakawa [8]). Thus from the inequality

$$\|u(t)-\tilde{u}(t)\| \le \|e^{t\Delta}u_0-\tilde{u}(t)\| + \|e^{t\Delta}u_0-u(t)\|,$$

we need to prove the validity of the estimates of Theorem 4.1 with $u(t)-\tilde{u}(t)$ replaced by $e^{t\Delta}u_0-u(t)$. So we only need the following lemma.

lemma 4.2. *In addition to the assumption of Theorem 3.1, let $v(t)=e^{t\Delta}u_0$ be the solution of the linear heat equation with the same initial data u_0, then for $t\ge 1$,*

$$\|u(t)-v(t)\|^2 \le C\begin{cases}(1+t)^{-p/2}, & 2<p<3,\ n=2\\ (1+t)^{-\frac{n}{2}-\frac{1}{2}}, & 1+\frac{2n}{n+2}\le p<3,\ n\ge 3\\ (1+t)^{-\frac{n}{2}-\frac{1}{2}}, & p\ge 3,\ n\ge 2.\end{cases}$$

We remark that from Lemma 4.2, it is readily seen that when $u_0\in\mathbf{H}\cap L^1$,

$$\|u(t)-v(t)\| = o\Big((1+t)^{-n/4}\Big), \quad t\to\infty.$$

Proof of Lemma 4.2. Denote the difference $w(t)=u(t)-v(t)$. Thus $w(t)$ satisfies

$$w_t - \triangle w + \Delta^2 w = B(u,v), \quad w(x,0)=0, \tag{4.1}$$

where $B(u,v) = -(u\cdot\nabla)u + \nabla\cdot(|e(u)|^{p-2}e(u)) - \Delta^2 v - \nabla\pi$. Since $u_0\in\mathbf{H}$, v is divergence free, and so is w.

Multiplying by w and integrating with respect to $\mathbb{R}^n$, it follows that

$$\frac{d}{dt}\|w\|^2 + 2\|\nabla w\|^2 + 2\|\Delta w\|^2 =: 2B(u,v,w), \tag{4.2}$$

where

$$\begin{aligned}B(u,v,w) &= -((u\cdot\nabla)u,w) - ((|e(u)|^{p-2}e(u)),\nabla w) - (\Delta^2 v,w)\\ &= ((u\cdot\nabla)w,w+v) + (|e(u)|^{p-2}e(u),\nabla v) - \|\nabla u\|_p^p + (\Delta v,\Delta w) \quad (4.3)\\ &= ((u\cdot\nabla)w,v) + (|e(u)|^{p-2}e(u),\nabla v) + (\Delta v,\Delta w) - \|\nabla u\|_p^p.\end{aligned}$$

Since, for $1\le q\le\infty$ and $k\in N$,

$$\|D^k v(t)\|_q \le (1+t)^{-\frac{n}{2}(1-\frac{1}{q})-\frac{k}{2}}\|u_0\|_1 \quad \forall\, t\ge 1, \tag{4.4}$$

which follows from the properties of the heat kernel (see Kajikiya and Miyakawa [8]). Thus we estimate the first three terms. Noting that $\|u(t)\| \le C(1+t)^{-n/4}$, we have

$$\begin{aligned}
&2|((u\cdot\nabla)w,v)+(|e(u)|^{p-2}e(u),\nabla v)+(\Delta v,\Delta w)|\\
&\le 2\|u\|\|\nabla w\|\|v\|_\infty+2\|\nabla u\|_{p-1}^{p-1}\|\nabla v\|_\infty+2\|\Delta v\|\|\Delta w\|\\
&\le \|\nabla w\|^2+\|u\|^2\|v\|_\infty^2+2\|\nabla u\|_{p-1}^{p-1}\|\nabla v\|_\infty+2\|\Delta w\|^2+\frac{1}{2}\|\Delta v\|^2\\
&\le \|\nabla w\|^2+2\|\Delta w\|^2+(1+t)^{-\frac{3n}{2}}+2(1+t)^{-\frac{n}{2}-\frac{1}{2}}\|\nabla u\|_{p-1}^{p-1}+\frac{1}{2}(1+t)^{-\frac{n}{2}-2}.
\end{aligned}\tag{4.5}$$

Hence (4.2)-(4.5) yield

$$\frac{d}{dt}\|w\|^2+\|\nabla w\|^2\le 2(1+t)^{-\frac{n}{2}-\frac{1}{2}}\|\nabla u\|_{p-1}^{p-1}+\frac{1}{2}(1+t)^{-\frac{n}{2}-2}.\tag{4.6}$$

Similar to (3.4) with $f(t)=(1+t)^{2n}$, the derivation of (4.6) implies

$$\begin{aligned}
&(1+t)^{2n}\int_{\mathbb{R}^n}|\hat{w}(\xi,t)|^2d\xi\\
&\le C(1+t)^{2n}\int_{B(t)}|\hat{w}(\xi,s)|^2d\xi+C(1+t)^{\frac{3n}{2}-1}+C(1+t)^{\frac{3n}{2}-\frac{1}{2}}\int_0^t\|\nabla u\|_{p-1}^{p-1}ds.
\end{aligned}$$

Similar to the proof of the Lemma 2.2, we use (4.1) and $\|v(t)\|_1\le\|u_0\|_1\le C$ to obtain

$$\begin{aligned}
|\hat{w}(\xi,t)|&\le C|\xi|\int_0^t\|u(s)\|^2ds+C|\xi|\int_0^t\|\nabla u\|_{p-1}^{p-1}ds+C|\xi|^4\int_0^t\|v(s)\|_1ds\\
&\le C|\xi|\int_0^t\|u(s)\|^2ds+C|\xi|\int_0^t\|\nabla u\|_{p-1}^{p-1}ds+C|\xi|^4t.
\end{aligned}\tag{4.7}$$

Therefore, (4.6) and (4.7) yield

$$\begin{aligned}
\|w(t)\|^2=\int_{\mathbb{R}^n}|\hat{w}(\xi,t)|^2d\xi&\le C(1+t)^{-\frac{n}{2}-1}\Big(\int_0^t\|u(s)\|^2ds\Big)^2+C(1+t)^{-\frac{n}{2}-1}\\
&\quad+C(1+t)^{-\frac{n}{2}-\frac{1}{2}}\int_0^t\|\nabla u\|_{p-1}^{p-1}ds+C(1+t)^{-\frac{n}{2}-1}\Big(\int_0^t\|\nabla u\|_{p-1}^{p-1}ds\Big)^2.
\end{aligned}$$

When $2<p<3$, $n=2$, it follows from (2.15) and $\|u(t)\|\le C(1+t)^{1/2}$ that

$$\begin{aligned}
\|w(t)\|^2&\le C(1+t)^{-2}(\ln(1+t))^2+C(1+t)^{-(p-1)}+C(1+t)^{-\frac{p}{2}}\\
&\le C(1+t)^{-\frac{p}{2}}.
\end{aligned}$$

When $1+\frac{n+2}{2n}\le p<3$ and $n\ge 3$, equation (2.15) and the inequalities $\|u(t)\|\le C(1+t)^{-n/4}$ and $\frac{n\alpha}{4-2\beta}>1$ imply

$$\int_0^t\|\nabla u\|_{p-1}^{p-1}ds=\int_0^t(1+s)^{-\frac{n\alpha}{4-2\beta}}ds\le C,$$

which yields

$$\|w(t)\|^2\le C(1+t)^{-\frac{n}{2}-1}+C(1+t)^{-\frac{n}{2}-\frac{1}{2}}\le C(1+t)^{-\frac{n}{2}-\frac{1}{2}}.$$

Similarly, for the case of $p\ge 3$ and $n\ge 2$, we derive from (2.16) that

$$\|w(t)\|^2\le C(1+t)^{-\frac{n}{2}-\frac{1}{2}}.$$

Hence the proof of Lemma 4.2 is complete. □

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