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POSITIVE SOLUTIONS TO QUASILINEAR EQUATIONS INVOLVING CRITICAL EXPONENT ON PERTURBED ANNULAR DOMAINS

CLAUDIANOR O. ALVES

Abstract. In this paper we study the existence of positive solutions for the problem

$$-\Delta_p u = u^{p^*-1} \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega$$

where Ω is a perturbed annular domain (see definition in the introduction) and $N > p \geq 2$. To prove our main results, we use the Concentration-Compactness Principle and variational techniques.

1. Introduction

Consider the problem

$$\begin{gathered}
-\Delta_p u = \lambda |u|^{p-2}u + |u|^{p^*-2}u, \quad \text{in } \Omega \\
u > 0, \quad \text{in } \Omega \\
u = 0, \quad \text{on } \partial\Omega
\end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $\lambda \geq 0$, $p^* = \frac{Np}{N-p}$, $N > p \geq 2$ and

$$\Delta_p u = \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\Big(|\nabla u|^{p-2}\frac{\partial u}{\partial x_j}\Big).$$

We recall that the weak solutions of (1.1) are critical points, on $W_0^{1,p}(\Omega)$, of the energy functional

$$I_\lambda(u) = \frac{1}{p}\int_\Omega (|\nabla u|^p - \lambda(u_+)^p)dx - \frac{1}{p^*}\int_\Omega (u_+)^{p^*}dx\,,$$

where $u_+(x) = \max\{u(x), 0\}$. Using Sobolev embedding it follows that $I_\lambda \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$.

An important point related with problem (1.1) it is Pohozaev's identity (see [12] and [17]), which implies that (1.1) does not have a solution if Ω is strictly star-shaped with respect to the origin in $\mathbb{R}^N$ and $\lambda \leq 0$.

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Since the embedding $W_0^1(\Omega) \hookrightarrow L^{p^*}(\Omega)$ is not compact, we encounter serious difficulties in applying standard variational techniques to problem (1.1). The lack of compactness can be understood by the fact that I_λ does not satisfy the so-called Palais-Smale (PS) condition in the whole $\mathbb{R}$.

Brezis and Nirenberg in [6] studied (1.1) for the case $p = 2$ and $\lambda > 0$,they used the fact that the (PS) condition holds in some energy range, for example in the interval $(-\infty, \frac{1}{N} S^{N/2})$, where S is the best constant of the embedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ given by

$$S = \min_{u \in D^{1,p}(\mathbb{R}^N),\ u \neq 0} \frac{\int_{\mathbb{R}^N} |\nabla u|^p dx}{\left(\int_{\mathbb{R}^N} |u|^{p^*} dx\right)^{p/p^*}}.$$

Using the family of functions

$$\Phi_{\delta,y}(x) = \frac{\left[N\left(\frac{N-p}{p-1}\right)\delta\right]^{\frac{N-p}{p^2}}}{\left[\delta + |x-y|^{\frac{p}{p-1}}\right]^{\frac{N-p}{p}}}, \quad x, y \in \mathbb{R}^N,\ \delta > 0$$

which satisfies

$$\|\Phi_{\delta,y}(x)\|_{1,p}^p = |\Phi_{\delta,y}(x)|_{p^*}^{p^*} = S^{N/p}$$

(see Talenti [20]), where

$$\|u\|_{1,p} = \Big(\int_{\mathbb{R}^N} |\nabla u|^p dx \Big)^{1/p} \quad \text{and} \quad |u|_{p^*} = \Big(\int_{\mathbb{R}^N} |u|^{p^*} dx \Big)^{1/p^*},$$

the authors in [6] showed that the minimization problem

$$S_\lambda = \min_{u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega (|\nabla u|^p dx - \lambda |u|^p) dx}{(\int_\Omega |u|^{p^*} dx)^{\frac{p}{p^*}}}$$

has a solution, hence (1.1) has a solution.

After the results obtained in [6], several authors have considered (1.1), for instance, Struwe in [18] (see also [19]) studied the behaviour of the Palais-Smale sequence of I_λ for the case $p = 2$ showing a result of Global Compactness. In his arguments, he used strongly some estimates for the Laplacian operator proved by Lions and Magenes in [14]. In [8], Coron used the study made in [18] and proved that $(P)_0$ has a solution for a class of annular-shaped domains. In the papers of Bahri and Coron [4], Benci and Cerami [5] and Willem [24] some results of existence of solution depending of the topology of Ω were proved.For the case $p \geq 2$, Gueda and Veron [12] and Garcia Azorero and Peral Alonso [11] showed that the results obtained in [6] are true for p-Laplacian operator. There exists a rich literature involving the problem (1.1) with $p \geq 2$,we refer the reader to Peral Alonso [16] and references therein.

The main purpose of the present paper is to show that the result proved by Struwe in [18] holds for the p-Laplacian operator and as a consequence the result obtained by Coron in [8] is also true for the p-Laplacian operator with $p \geq 2$.

To state our main result we need some definitions and notation.

An important problem in this paper is the limit problem in $\mathbb{R}^N$ given by

$$\begin{gathered} -\Delta_p w = w^{p^*-1} \quad \text{in } \mathbb{R}^N \\ w > 0 \quad \text{in } \mathbb{R}^N \\ w \in D^{1,p}(\mathbb{R}^N). \end{gathered} \tag{1.2}$$

Hereafter, let us denote by $I_\infty : D^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ the energy functional related to limit problem, that is

$$I_\infty(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{1}{p^*}\int_{\mathbb{R}^N} (u_+)^{p^*} dx.$$

We say that a domain Ω is a *Perturbed Annular Domain* (PAD) if there exist $R_1, R_2 > 0$ such that

$$\Omega \supset \{x \in \mathbb{R}^N : R_1 < |x| < R_2\} \quad \text{and} \quad \overline{\Omega} \not\supset \{x \in \mathbb{R}^N : |x| < R_1\}.$$

Our main results are stated in the following two theorems.

Theorem 1.1. *Let $\{u_n\}$ be a $(PS)_c$ sequence to I_λ with $u_n \rightharpoonup u_0$ in $W_0^{1,p}(\Omega)$. Then, the sequence $\{u_n\}$ satisfies either*

(a) *$u_n \to u_0$ in $W_0^{1,p}(\Omega)$, or*
(b) *There exist $k \in \mathbb{N}$ and non-trivial solutions $z_1, \dots, z_k$ for the problem* (1.2) *such that*

$$\|u_n\|^p \to \|u_0\|^p + \sum_{j=1}^{k} \|z_j\|_{1,p}^p\,,$$

$$I_\lambda(u_n) \to I_\lambda(u_0) + \sum_{j=1}^{k} I_\infty(z_j)\,.$$

Theorem 1.2. *Let Ω be a (PAD) in $\mathbb{R}^N$. Then, if $\frac{R_2}{R_1}$ is sufficiently large, problem* (1.1) *with $\lambda = 0$ has a positive solution $u \in W_0^{1,p}(\Omega)$.*

Theorem 1.1 was proved by Struwe in [18] in the particular case $p = 2$. To prove Theorem 1.1 in the general case $p \geq 2$, we make a similar study to the one found in [18], because here we also need to understand the behaviour of Palais-Smale sequence of I_λ. However, we will use different arguments because some estimates explored in [18] are not clear to hold for p-Laplacian operator. Here, the main tool employed is the Concentration-Compactness Principle by Lions [13], which overcome in some sense the lack of estimates of the type Lions and Magenes [14] to p-Laplacian operator and we also use some arguments explored by the author in [2, 3].

Theorem 1.2 was studied by Coron in [8] in the case $p = 2$. In this paper, we prove Theorem 1.2 as a good application of Theorem 1.1 and in its proof we use some ideas found in [8] (see also [18]), that is, we use the information obtained about the behavior of the $(PS)_c$ sequences, the deformation lemma on manifolds and some estimates involving the family of functions related with the best constant S.

In the what follows, we denote by $\|w\|$ the usual norm of a function $w \in W_0^{1,p}(\Omega)$ and by $I'_\lambda(w, \Theta)$ where Θ is a bounded domain the Frechet Derivative of I_λ on $W_0^{1,p}(\Theta)$ at w. Now, if $\Theta = \Omega$ we will use only $I'_\lambda(w)$. Moreover, $u_-(x) = \max\{-u(x), 0\}$, $B_s(y)$ with $s > 0$ is a ball with center at $y \in \mathbb{R}^N$ and radius s and $B_s = B_s(0)$. For each $a \in \mathbb{R}^N$ we define the following sets: $\{x^N > a\} = \{x = (x^1, \dots, x^N) \in \mathbb{R}^N; x^N > a\}$, $\{x^N = a\} = \{x = (x^1, \dots, x^N) \in \mathbb{R}^N; x^N = a\}$ and $\{x^N < a\} = \{x = (x^1, \dots, x^N) \in \mathbb{R}^N; x^N < a\}$.

2. Preliminary Results

In this section we will recall and show some lemmas that are crucial in the proofs of Theorems 1.1 and 1.2. We begin by recalling the following Lemma by Lions [13].

Lemma 2.1. *Let $(u_n) \subset D^{1,p}(\mathbb{R}^N)$ with $u_n \rightharpoonup u$ in $D^{1,p}(\mathbb{R}^N)$. Then, there exist $\{y_i\}_{i\in\Lambda} \subset \mathbb{R}^N$ and $\{\nu_i\}_{i\in\Lambda} \subset \mathbb{R}$, where Λ is at most a countable set such that*

$$\int_{\mathbb{R}^N} |u_n|^{p^*}\phi dx \to \int_{\mathbb{R}^N} |u|^{p^*}\phi dx + \sum_{i\in\Lambda} \phi(y_i)\nu_i \quad \forall \phi \in C_0^\infty(\mathbb{R}^N).$$

The next lemma was proved by Alves [3], using arguments found in Brezis and Lieb [7].

Lemma 2.2. *Let $\eta_n : \mathbb{R}^N \to \mathbb{R}^K$ $(K \geq 1)$ with $\eta_n \subset L^p(\mathbb{R}^N) \times \cdots \times L^p(\mathbb{R}^N)$ $(p \geq 2)$, $\eta_n(x) \to 0$ a.e. in $\mathbb{R}^K$ and $A(y) = |y|^{p-2}y$ for all $y \in \mathbb{R}^K$. Then, if $|\eta_n|_{L^p(\mathbb{R}^N)} \leq C$ $\forall n \in \mathbb{N}$, we have*

$$\int_{\mathbb{R}^N} |A(\eta_n + w) - A(\eta_n) - A(w)|^{\frac{p}{p-1}} dx = o_n(1)$$

for each $w \in L^p(\mathbb{R}^N) \times \cdots \times L^p(\mathbb{R}^N)$ fixed.

Proposition 2.3. *Let $v \in D_0^{1,p}(\{x^N > a\})$ be a nonnegative solution of the problem*

$$\begin{gathered} -\Delta_p v = v^{p^*-1} \quad \text{in } \{x^N > a\} \\ v \geq 0 \quad \text{in } \{x^N > a\} \\ v = 0 \quad \text{on } \{x^N = a\}. \end{gathered}$$

Then $v = 0$.

Proof. By results showed by Trudinger [22], Guedda and Veron [12], DiBenedetto [9], and Tolksdorf [23], we have

$$v \in D_0^{1,2}(\{x^N > 0\}) \cap C^1(\{x^N \geq 0\})$$

and adapting the ideas explored by Li and Shusen [15]

$$v(x) \to 0 \text{ as } |x| \to \infty.$$

Moreover, with suitable modifications, the arguments used by Esteban and Lions [10] and Gueda and Veron [12, Theorem 1.1] show that

$$\int_{x^N=0} \langle x - x_0, \eta\rangle |v_\eta|^p d\sigma = 0$$

where x_0 is a point fixed in $\{x^N > 0\}$ and η is the forward normal to $\{x^N = 0\}$. Hence $v_\eta = 0$ on $\{x^N = 0\}$ and by a result showed in Vàsquez [21] we have $v \equiv 0$. $\square$

Remark 2.4. Proposition 2.3 holds for sets of the form $\{x^N < a\}$ and for more general half-planes.

Throughout this paper, we assume that all $(PS)_c$ sequences of I_λ are nonnegative functions, since by using the definition of I_λ it follows that $I'_\lambda(u_n)(u_{n-}) \to 0$, thus $\|u_{n-}\| \to 0$. Consequently the sequence $\{u_{n+}\}$ is also a $(PS)_c$ sequence for I_λ.

Lemma 2.5. *Suppose $\{u_n\}$ is a $(PS)_c$ sequence for I_0 in $W_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup 0$ weakly. Then there exist a sequence (x_n) of points in $\mathbb{R}^N$ with $x_n \to x_0 \in \overline{\Omega}$, a real sequence (λ_n) with $\lambda_n \to 0$, a non-trivial solution v_0 of (1.2) and a $(PS)_c$ sequence $\{w_n\}$ for I_0 in $W_0^{1,p}(\Omega)$ such that for a subsequence $\{u_n\}$ there holds*

$$w_n = u_n - \lambda_n^{\frac{p-N}{p}} v_0(\frac{1}{\lambda_n}(. - x_n)) + o_n(1),$$

where $o_n(1) \to 0$ in $D^{1,p}(\mathbb{R}^N)$ as $m \to \infty$. In particular, $w_n \rightharpoonup 0$ weakly. Furthermore,

$$I_0(w_n) = I_0(u_n) - I_\infty(v_0) + o_n(1).$$

Moreover, $\frac{1}{\lambda_n}\operatorname{dist}(x_n, \partial\Omega) \to \infty$.

Proof. Without loss of generality we will suppose that $c \geq \frac{1}{N}S^{N/p}$, because if $c \in (0, \frac{1}{N}S^{N/p})$, u_n is strongly convergent (see [11]). Let the Lévy concentration function be

$$Q_n(\lambda) = \sup_{y\in\mathbb{R}^N} \int_{B_\lambda(y)} (u_n)^{p^*} dx.$$

Note that there exists $(x_n, \lambda_n) \in \mathbb{R}^N \times (0, \infty)$ such that

$$Q_n(\lambda_n) = \int_{B_{\lambda_n}(x_n)} (u_n)^{p^*} dx = \frac{1}{2}S^{N/p}.$$

Setting

$$v_n(x) = \lambda_n^{\frac{N-p}{p}} u_n(\lambda_n x + x_n),$$

we have

$$\sup_{y\in\mathbb{R}^N} \int_{B_1(y)} (v_n)^{p^*} dx = \int_{B_1} (v_n)^{p^*} dx = \frac{1}{2}S^{N/p}.$$

Moreover,

$$\int_{\Omega_n} (v_n)^{p^*} dx = \int_{\Omega} (u_n)^{p^*} dx \quad \text{and} \quad \int_{\Omega_n} |\nabla v_n|^p dx = \int_{\Omega} |\nabla u_n|^p dx$$

where $\Omega_n = \frac{1}{\lambda_n}(\Omega - x_n)$. Here and in what follows, Ω_∞ is the limit set of Ω_n when n goes to infinity. For each $\{\Phi_n\} \subset W_0^{1,p}(\Omega_n)$ with bounded norm in $D^{1,p}(\mathbb{R}^N)$, we get

$$\int_{\mathbb{R}^N} |\nabla v_n|^{p-2} \nabla v_n \nabla \Phi_n dx - \int_{\mathbb{R}^N} (v_n)^{p^*-1} \Phi_n dx = o_n(1), \tag{2.1}$$

since by considering the sequence $\overline{\Phi_n}(x) = \lambda_n^{\frac{p-N}{p}} \Phi_n(\frac{1}{\lambda_n}(x - x_n))$, we have that (2.1) is equivalent to

$$I_0'(u_n)(\overline{\Phi_n}) = o_n(1).$$

Let v_0 be the weak limit of $\{v_n\} \in D^{1,p}(\mathbb{R}^N)$. Now, we will show that $v_0 \neq 0$. Applying Lemma 2.1 for the sequence $\{v_n\}$, we conclude by arguments explored in [11],[16],[12] and [2] that there is not $y_i \in \overline{\Omega_\infty}^c$ and Λ is finite or empty. Here, we have that Λ is empty, because if $\nu_i > 0$ by well known arguments we get that $\nu_i \geq S^{N/p}$. From the definition of the function v_n

$$\frac{1}{2}S^{N/p} = \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} (v_n)^{p^*} dx \geq \int_{B_1(y_i)} (v_n)^{p^*} dx$$

then passing to the limit in the above inequality and using again Lemma 2.1, we obtain a contradiction. Thus, Λ is empty and

$$\int_{\mathbb{R}^N}(v_n)^{p^*}\Phi dx \to \int_{\mathbb{R}^N}(v_0)^{p^*}\Phi dx \quad \forall \Phi\in C_0^\infty(\mathbb{R}^N) \text{ as } n\to\infty$$

which implies $v_n \to v_0$ in $L^{p^*}_{\rm loc}(\mathbb{R}^N)$, consequently

$$\int_{B_1}(v_0)^{p^*}dx = \frac{1}{2}S^{N/p}$$

and $v_0 \neq 0$. Using the fact the v_0 is not zero we have that $\lambda_n \to 0$, because if there exists $\delta > 0$ such that $\lambda_n \geq \delta$, we have the following inequality

$$\int_{\mathbb{R}^N}(v_n)^p dx = \frac{1}{\lambda_n^p}\int_{\mathbb{R}^N}(u_n)^p dx \leq C_1\int_\Omega (u_n)^p dx$$

and by the fact that $u_n \to 0$ in $L^p(\Omega)$ it follows that

$$\int_{\mathbb{R}^N}(v_0)^p dx = 0,$$

which is a contradiction. Now, using the fact that $\lambda_n \to 0$ we may assume that there exists $x_0 \in \overline{\Omega}$ such that $x_n \to x_0 \in \overline{\Omega}$. By weak continuity of v_n and (2.1), the function v_0 is a solution of the problem

$$\begin{gathered} -\Delta_p v = v^{p^*-1}, \quad \text{in } \Omega_\infty \\ v \geq 0, v \not\equiv 0 \quad \text{in } \Omega_\infty \\ v = 0, \quad \text{on } \partial\Omega_\infty. \end{gathered}$$

To determine Ω_∞, we have to consider two cases:

(A) $\frac{1}{\lambda_n}\operatorname{dist}(x_n,\partial\Omega)\to\infty$ as $n\to\infty$
(B) $\frac{1}{\lambda_n}dist(x_n,\partial\Omega)\leq\alpha$ for all $n\in\mathbb{N}$ and some $\alpha>0$.

Claim: Case (B) above does not hold. In fact, assume by contradiction that (B) holds and that without loss of generality $x_n \to 0 \in \partial\Omega$. Moreover, we will suppose also that $0, \Omega$ and $\partial\Omega$ are described in the following form (see more details in Adimurthi, Pacella and Yadava [1]):

> There exist $\delta > 0$, an open neighborhood $\mathcal{N}$ of 0 and a diffeomorphism $\Psi : B_\delta(0) \to \mathcal{N}$ which has a jacobian determinant at 0 equal to one, with $\Psi(B_\delta^+) = \mathcal{N}\cap\Omega$ where $B_\delta^+ = B_\delta(0)\cap\{x^N > 0\}$.

Now, let us define the function $\xi_n \in D^{1,p}(\mathbb{R}^N)$ given by

$$\xi_n(x) = \begin{cases} \lambda_n^{\frac{N-p}{p}} u_n(\Psi(\lambda_n x + P_n))\chi(\Psi(\lambda_n x+P_n)), & x\in B_{\frac{\delta}{\lambda_n}}(-\frac{P_n}{\lambda_n}) \\ 0, & x\in\mathbb{R}^N\setminus B_{\frac{\delta}{\lambda_n}}(-\frac{P_n}{\lambda_n}) \end{cases}$$

where $\Psi(P_n) = x_n$, $\chi\in C_0^\infty(\mathbb{R}^N)$, $0\leq\chi(x)\leq 1$ for all $x\in\mathbb{R}^N$, $\chi(x)=1$ for all $x\in\mathcal{O}_{\frac{\delta}{2}}$, $\chi(x)=0$ for all $x\in\mathcal{O}_{\frac{3\delta}{4}}$, $\mathcal{O}_{\frac{\delta}{2}}=\Psi(B_{\frac{\delta}{2}})$, and $\mathcal{O}_{\frac{3\delta}{4}}=\Psi(B_{\frac{3\delta}{4}})$. By a simple computation, it is possible to show that for some subsequence

$$\frac{P_n^N}{\lambda_n}\to\alpha_0 \quad \text{for some } \alpha_0\geq 0 \text{ as } n\to\infty$$

and that there exists a nonnegative function $\xi \in D_0^{1,p}(\{x^N > -\alpha_0\})$ such that $\xi_n(x) \rightharpoonup \xi$ in $D^{1,p}(\mathbb{R}^N)$ which satisfies

$$\begin{gathered} -\Delta_p \xi = \xi^{p^*-1} \quad \text{in } \{x^N > -\alpha_0\} \\ \xi = 0 \quad \text{on } \{x^N = -\alpha_0\}. \end{gathered} \tag{2.2}$$

From Proposition 2.3, we have that $\xi \equiv 0$. On the other hand,

$$\int_{B_1} v_n^p dx \le C \int_{\mathcal{A}} \xi_n^p dx$$

for all large n where $\mathcal{A} \subset \{x^N > -\alpha_0\}$ is a bounded domain. Since $\{\xi_n\}$ is a bounded sequence in $W^{1,p}(\mathcal{A})$, we obtain by Sobolev embedding

$$\int_{\mathcal{A}} \xi_n^p dx \to 0$$

thus

$$\int_{B_1} v_n^p dx \to 0,$$

and so $v_0 \equiv 0$ in B_1, which is a contradiction. Thus Case (A) holds, so $\Omega_\infty = \mathbb{R}^N$ and v_0 is a solution of (1.2).

To conclude, we consider $\Phi \in C_0^\infty(\mathbb{R}^N)$ verifying $0 \le \Phi(x) \le 1$, $\Phi \equiv 1$ in B_1 and $\Phi = 0$ in B_2^c. Let

$$w_n = u_n(x) - \lambda_n^{\frac{p-N}{p}} v_0(\frac{1}{\lambda_n}(x - x_n))\Phi(\frac{1}{\overline{\lambda_n}}(x - x_n))$$

where we choose $\overline{\lambda_n}$ verifying $\widetilde{\lambda}_n = \frac{\lambda_n}{\overline{\lambda_n}} \to 0$. Considering

$$\widetilde{w}_n(x) = \lambda_n^{\frac{N-p}{p}} w_n(\lambda_n x + x_n) = v_n(x) - v_0(x)\Phi(\widetilde{\lambda}_n x)$$

and by repeating of the same arguments explored by Struwe in [18], we complete the proof of Lemma 2.5. $\square$

3. Proof of Theorem 1.1

By hypothesis we have $u_n(x) \to u_0(x)$ a.e in Ω. Thus using standard arguments found in [11, 12, 16, 2], we have $I'_\lambda(u_0) = 0$. Suppose that u_n does not converge to u_0 in $W_0^{1,p}(\Omega)$ and let $\{z_{n,1}\} \subset W_0^{1,p}(\Omega)$ be given by $z_{n,1} = u_n - u_0$. Then

$$z_{n,1} \rightharpoonup 0 \quad \text{but} \quad z_{n,1} \not\to 0 \text{ in } W_0^{1,p}(\Omega)\,.$$

By Brezis and Lieb [7] and by Lemma 2.2 it follows that

$$I_0(z_{n,1}) = I_\lambda(u_n) - I_\lambda(u_0) + o_n(1)\,, \tag{3.1}$$

$$I'_0(z_{n,1}) = I'_\lambda(u_n) - I'_\lambda(u_0) + o_n(1)\,. \tag{3.2}$$

From these two equations, we conclude that $\{z_{n,1}\}$ is a $(PS)_c$ sequence for I_0. By Lemma 2.5, there exist $(\lambda_{n,1}) \subset \mathbb{R}$, $(x_{n,1}) \subset \mathbb{R}^N$, $z_1 \in D^{1,p}(\mathbb{R}^N)$ a non-trivial solution of (1.2) and a $(PS)_c$ sequence $\{z_{n,2}\}$ in $W_0^{1,p}(\Omega)$ for I_0 given by

$$z_{n,2}(x) = z_{n,1}(x) - \lambda_{n,1}^{\frac{p-N}{p}} z_1(\frac{1}{\lambda_{n,1}}(x - x_{n,1})) + o_n(1).$$

If we define

$$v_{n,1}(x) = \lambda_{n,1}^{\frac{p-N}{p}} z_{n,1}(\lambda_{n,1} x + x_{n,1})$$

and $\{\widetilde{z}_{n,2}\}$ by
$$\widetilde{z}_{n,2}(x) = v_{n,1}(x) - z_1(x) + o_n(1),$$
we conclude by arguments explored in the proof of Lemma 2.5 that $v_{n,1} \rightharpoonup z_1$ in $D^{1,p}(\mathbb{R}^N)$,
$$I_\infty(v_{n,1}) = I_0(z_{n,1})$$
and
$$\|I_0'(v_{n,1}, \Omega_{n,1})\| = o_n(1),$$
where $\Omega_{n,1} = \frac{1}{\lambda_{n,1}}(\Omega - x_{n,1})$. Using again [7] and Lemma 2.2, we conclude that
$$I_\infty(\widetilde{z}_{n,2}) = I_\infty(v_{n,1}) - I_\infty(z_1) + o_n(1) = I_\lambda(u_n) - I_\lambda(u_0) - I_\infty(z_1) + o_n(1)$$
and
$$\|I_0'(\widetilde{z}_{n,2}, \Omega_{n,1})\| \le \|I_0'(v_{n,1}, \Omega_{n,1})\| + \|I_\infty'(z_1)\| + o_n(1),$$
consequently $\|I_0'(\widetilde{z}_{n,2}, \Omega_{n,1})\| = o_n(1)$ and $\|I_0'(z_{n,2})\| = o_n(1)$. If $z_{n,2} \to 0$ in $W_0^{1,p}(\Omega)$ the theorem finishes. Now, if $\{z_{n,2}\}$ does not converge to 0 in $W_0^{1,p}(\Omega)$, we apply again Lemma 2.5 and find $(\lambda_{n,2}) \subset \mathbb{R}$, $(x_{n,2}) \subset \mathbb{R}^N, z_2 \in D^{1,p}(\mathbb{R}^N)$ a non-trivial solution of (1.2) and a $(PS)_c$ sequence $\{z_{n,3}\}$ in $W_0^{1,p}(\Omega)$ for I_0 given by
$$z_{n,3}(x) = \widetilde{z}_{n,2}(x) - \lambda_{n,2}^{\frac{p-N}{p}} z_2(\frac{1}{\lambda_{n,2}}(x - x_{n,2})) + o_n(1).$$
Considering the sequences $\{v_{n,2}\}$ and $\{\widetilde{z}_{n,3}\}$ given by
$$v_{n,2}(x) = \lambda_{n,2}^{\frac{p-N}{p}} \widetilde{z}_{n,2}(\lambda_{n,2}x + x_{n,2}) \quad \text{and} \quad \widetilde{z}_{n,3}(x) = v_{n,2}(x) - z_2(x) + o_n(1)$$
we have $v_{n,2} \rightharpoonup z_2$ in $D^{1,p}(\mathbb{R}^N)$ and
$$I_\infty(\widetilde{z}_{n,3}) = I_\infty(\widetilde{z}_{n,2}) - I_\infty(z_2) + o_n(1) = I_\lambda(u_n) - I_\lambda(u_0) - I_\infty(z_1) - I_\infty(z_2) + o_n(1)$$
and
$$\|I_0'(\widetilde{z}_{n,3}, \Omega_{n,2})\| \le \|I_0'(v_{n,2}, \Omega_{n,2})\| + \|I_\infty'(z_2)\| + o_n(1),$$
whence $\|I_0'(\widetilde{z}_{n,3}, \Omega_{n,2})\| = o_n(1)$ and $\|I_0'(z_{n,3})\| = o_n(1)$. If $z_{n,3} \to 0$ the proof is done, if not, we repeat the arguments used, and then we will find $z_1, \dots, z_k$ non-trivial solutions to (1.2) satisfying
$$\|\widetilde{z}_{n,k}\|^p = \|u_n\|^p - \|u_0\|^p - \sum_{j=1}^{k-1} \|z_j\|_{1,p}^p + o_n(1)\,, \tag{3.3}$$
$$I_\infty(\widetilde{z}_{n,k}) = I_\lambda(u_n) - I_\lambda(u_0) - \sum_{j=1}^{k-1} I_\infty(z_j) + o_n(1)\,. \tag{3.4}$$
Now, we recall that
$$\|z_j\|_{1,p}^p \ge S^{\frac{N}{p}} \quad j = 1, \dots, k\,. \tag{3.5}$$
Combining (3.3) and (3.5),
$$0 \le \|\widetilde{z}_{n,k}\|^p \le \|u_n\|^p - \|u_0\|^p - \sum_{j=1}^{k-1} S^{\frac{N}{p}} = \|u_n\|^p - \|u_0\|^p - (k-1)S^{\frac{N}{p}} + o_n(1). \tag{3.6}$$
Since $\{u_n\}$ is bounded, from (3.6) there exists $k \in \mathbb{N}$ such that $\limsup_{n\to\infty} \|\widetilde{z}_{n,k}\|^p \le 0$. Consequently, $\widetilde{z}_{n,k} \to 0$ in $W_0^{1,p}(\Omega)$ and this concludes the proof.

Corollary 3.1. *Let $\{u_n\}$ be a $(PS)_c$ sequence for I_λ with $c \in (0, \frac{1}{N}S^{\frac{N}{p}})$. Then $\{u_n\}$ contains a subsequence strongly convergent in $W_0^{1,p}(\Omega)$.*

Corollary 3.2. *The functional $I_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ satisfies the $(PS)_c$ condition in the interval $(\frac{1}{N}S^{N/p}, \frac{2}{N}S^{N/p})$.*

Corollary 3.3. *Let $\{u_n\}$ be a $(PS)_c$ sequence for I_λ with $c \in (\frac{k}{N}S^{\frac{N}{p}}, \frac{(k+1)}{N}S^{N/p})$ and $k \in \mathbb{N}$. Then the weak limit u_0 of $\{u_n\}$ is not zero.*

Hereafter we denote by $f_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ the functional

$$f_\lambda(u) = \int_\Omega (|\nabla u|^p - \lambda(u_+)^p)dx$$

and by $\mathcal{M} \subset W_0^{1,p}(\Omega)$ the manifold

$$\mathcal{M} = \{u \in W_0^{1,p}(\Omega); \int_\Omega (u_+)^{p^*}dx = 1\}.$$

We remark that if $\{u_n\} \subset \mathcal{M}$ satisfies

$$f_\lambda(u_n) \to c \quad \text{and} \quad f'_\lambda|_{\mathcal{M}}(u_n) \to 0$$

it follows that $\{v_n\} = \{c^{\frac{N-p}{p^2}}u_n\} \subset W_0^{1,p}(\Omega)$ satisfies the limits

$$I_\lambda(v_n) \to \frac{1}{N}c^{\frac{N}{p}} \quad \text{and} \quad I'_\lambda(v_n) \to 0.$$

Corollary 3.4. *If there exist $\{u_n\} \subset \mathcal{M}$ and $c \in (S, 2^{\frac{p}{N}}S)$ such that $f_\lambda(u_n) \to c$ and $f'_\lambda|_{\mathcal{M}}(u_n) \to 0$, then f_λ has a critical point $u \in \mathcal{M}$ with $f_\lambda(u) = c$.*

Remark 3.5. Corollary 3.4 implies that (1.1) has at least a positive solution.

4. Proof of Theorem 1.2

Postponing the proof of Theorem 1.2 for a moment, we first fix some notations and show some technical lemmas. In this section, we assume that $R_1 = (4R)^{-1} < 1 < 4R = R_2$ and denote by Σ the unit sphere on $\mathbb{R}^N$,

$$\Sigma = \{x \in \mathbb{R}^N : |x| = 1\}.$$

For each $\sigma \in \Sigma$ and $t \in [0,1)$, we define the function $u_t^\sigma \in D^{1,p}(\mathbb{R}^N)$ by

$$u_t^\sigma(x) = \Big[\frac{1-t}{(1-t)^{\frac{p}{p-1}} + |x - t\sigma|^{\frac{p}{p-1}}}\Big]^{\frac{N-p}{p}}.$$

Using the well known result obtained in [20], it follows that S is attained on any such function u_t^σ.Moreover, letting $t \to 0$ we have

$$u_t^\sigma \to u_0 = \Big[\frac{1}{1 + |x|^{\frac{p}{p-1}}}\Big]^{\frac{N-p}{p}} \quad \text{in } D^{1,p}(\mathbb{R}^N)$$

for any $\sigma \in \Sigma$. In the sequel $\phi \in C_0^\infty(\Omega)$ is a radially symmetric function such that $0 \le \phi \le 1$ on $\Omega, \phi \equiv 1$ on the annulus $\{x \in \mathbb{R}^N : \frac{1}{2} < |x| < 2\}$ and $\phi \equiv 0$ outside the annulus $\{x \in \mathbb{R}^N : \frac{1}{4} < |x| < 4\}$. Let us consider for $R \ge 1$ the functions

$$\phi_R(x) = \begin{cases} \phi(Rx), & 0 \le |x| < R^{-1} \\ 1, & R^{-1} \le |x| < R \\ \phi(\frac{x}{R}), & |x| \ge R. \end{cases}$$

and $w_t^\sigma = u_t^\sigma \phi_R$, $w_0 = u_0\phi_R \in W_0^{1,p}(\Omega)$.

Lemma 4.1. *For each $\epsilon > 0$ there exists $R > 0$ such that*

$$\int_{B_{R^{-1}}} (u_t^\sigma)^{p^*} dx, \int_{B_{R^{-1}}} |\nabla u_t^\sigma|^p dx, \int_{B_R^c} (u_t^\sigma)^{p^*} dx, \int_{B_R^c} |\nabla u_t^\sigma|^p dx < \epsilon$$

uniformly in $\sigma \in \Sigma$ and $t \in [0,1)$.

Proof. Using the definition of u_t^σ, we obtain

$$\int_{B_{R^{-1}}} (u_t^\sigma)^{p^*} dx = (1-t)^N \int_{B_{R^{-1}}} \frac{dx}{\big[(1-t)^{\frac{p}{p-1}} + |x - t\sigma|^{\frac{p}{p-1}}\big]^N}$$

or equivalently

$$\int_{B_{R^{-1}}} (u_t^\sigma)^{p^*} dx = (1-t)^N \int_{B_{R^{-1}}(-t\sigma)} \frac{dy}{\big[(1-t)^{\frac{p}{p-1}} + |y|^{\frac{p}{p-1}}\big]^N}.$$

Thus given $\epsilon > 0$, there exists $\delta > 0$ such that for all $t \in [1-\delta, 1]$ and for all $R \ge R_0$, we have

$$\int_{B_{R^{-1}}} (u_t^\sigma)^{p^*} dx \le (1-t)^N \int_{B_{R^{-1}}(-t\sigma)} \frac{dy}{|y|^{\frac{Np}{p-1}}} < \frac{\epsilon}{2} \ \forall \sigma \in \Sigma. \tag{4.1}$$

On the other hand, there exists $R_0 > 0$ such that for all $R \ge R_0$,

$$\int_{B_{1/(1-t)R}(\frac{-t\sigma}{1-t})} \frac{dw}{\big[1 + |w|^{\frac{p}{p-1}}\big]^N} < \frac{\epsilon}{2} \quad \forall \sigma \in \Sigma \quad \text{and} \quad \forall t \in [0, 1-\delta] \tag{4.2}$$

Hence, if R_0 is sufficiently large, from (4.1) and (4.2)

$$\int_{B_{R^{-1}}} (u_t^\sigma)^{p^*} dx < \epsilon \ \forall t \in [0,1) \quad \text{and} \quad \forall \sigma \in \Sigma \text{ if } R \ge R_0\,.$$

Now, we estimate the integral $\int_{B_{R^c}} (u_t^\sigma)^{p^*} dx$: Note that

$$\int_{B_R^c} (u_t^\sigma)^{p^*} dx = (1-t)^{\frac{N(p-2)}{p-1}} \int_{\Theta_t^c} \frac{dy}{\big[1 + |y|^{\frac{p}{p-1}}\big]^N}\,,$$

where $\Theta_t = B_{\frac{R}{(1-t)}}(\frac{-t\sigma}{1-t})$; thus

$$\int_{B_R^c} (u_t^\sigma)^{p^*} dx \le C \int_{B_{R-1}^c} \frac{dy}{\big[1 + |y|^{\frac{p}{p-1}}\big]^N}$$

then for R large,

$$\int_{B_R^c} (u_t^\sigma)^{p^*} dx \le \epsilon \quad \forall \sigma \in \Sigma,\ \forall t \in [0,1).$$

The estimates for the two integrals involving gradient of u_t^σ follow with the same type of argument. $\square$

As a consequence of the above lemma, we get the following result

Lemma 4.2. *The functions $\{w_t^\sigma\}$ are strongly convergent in $D^{1,p}(\mathbb{R}^N)$ to $\{u_t^\sigma\}$ as $R \to \infty$ uniformly in $\sigma \in \Sigma$ and $t \in [0,1]$. Moreover, for each $R > 0$ fixed, we have that $\{w_t^\sigma\}$ is strongly convergent in $D^{1,p}(\mathbb{R}^N)$ to $\{u_t^\sigma\}$ as $t \to 1$, uniformly in $\sigma \in \Sigma$.*

Remark 4.3. Lemma 4.2 also holds for the normalized functions $v_t^\sigma = w_t^\sigma/|w_t^\sigma|_{p^*}$; that is, $\|v_t^\sigma - \frac{u_t^\sigma}{|u_t^\sigma|_{p^*}}\|_{1,p} \to 0$ as $R \to \infty$ uniformly in $\sigma \in \Sigma$ and $t \in [0,1)$.

Hereafter we define the function $\beta : \mathcal{M} \to \mathbb{R}^N$ namely "Barycenter", by setting

$$\beta(u) = \int_\Omega x(u_+)^{p^*}dx.$$

Proposition 4.4. *If $(u_n) \subset \mathcal{M}$ is such that $\|u_n\|^p \to S$, then* $\operatorname{dist}(\beta(u_n), \overline{\Omega}) \to 0$.

Proof. Note that the sequence $w_n = S^{\frac{N-p}{p^2}} u_n$ satisfies

$$I_0(w_n) \to \frac{1}{N} S^{N/p} \quad \text{and} \quad I_0'(w_n) \to 0\,.$$

Using the fact that S is never attained in a bounded domain, we get by Theorem 1.1 that $w_n \rightharpoonup 0$ in $W_0^{1,p}(\Omega)$ and that there exists $(\lambda_n) \subset \mathbb{R}$, $(x_n) \subset \mathbb{R}^N$ with $x_n \to x_0 \in \overline{\Omega}$ and $v_0 \in \mathcal{M}$ such that

$$u_n(x) = \lambda_n^{\frac{p-N}{p}} v_0(\frac{1}{\lambda_n}(x - x_n)) + o_n(1)\,.$$

Then

$$\beta(u_n) = \int_\Omega \frac{x}{\lambda_n^N} v_0(\frac{1}{\lambda_n}(x - x_n))^{p^*}dx + o_n(1).$$

If $\phi \in C_0^\infty(\mathbb{R}^N, \mathbb{R}^N)$ is a function with $\phi(x) = x$ for $x \in \overline{\Omega}$, we get

$$\beta(u_n) = \int_{\mathbb{R}^N} \phi(\lambda_n x + x_n) v_0^{p^*}dx + o_n(1)\,.$$

Then by Lebesgue's Theorem,

$$\int_{\mathbb{R}^N} \phi(\lambda_n x + x_n) v_0^{p^*}dx \to \int_{\mathbb{R}^N} \phi(x_0) v_0^{p^*}dx = x_0 \in \overline{\Omega}$$

whence $\operatorname{dist}(\beta(u_n), \overline{\Omega}) \to 0$. □

Proof of Theorem 1.2. Observe that by Lemma 4.2 $f_0(v_t^\sigma) \to S$ as $R \to \infty$ uniformly in $\sigma \in \Sigma$ and $t \in [0,1)$. In particular, if $R \geq 1$ is sufficiently large, we have

$$\sup_{\sigma\in\Sigma,\ t\in[0,1)} f_0(v_t^\sigma) < S_1 < 2^{\frac{p}{N}} S$$

for some constant $S_1 \in (0, \infty)$. Suppose by contradiction that $(P)_0$ does not admit a positive solution, this is equivalent to the fact that

$$I_0(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{p^*}\int_\Omega (u_+)^{p^*}dx$$

does not admit a critical point $u > 0$. Thus, f_0 does not have a critical value in the interval $(S, 2^{\frac{p}{N}} S)$. Moreover, by Theorem 1.1, f_0 verifies on $\mathcal{M}$ the $(PS)_c$ condition in $(S, 2^{\frac{p}{N}} S)$. Using the same arguments explored in [19], there exist $\delta > 0$ and a flow $\Phi : \mathcal{M} \times [0,1] \to \mathcal{M}$ such that

$$\Phi(\mathcal{M}_{S_1}, 1) \subset \mathcal{M}_{S+\delta}$$

where

$$\mathcal{M}_c = \{u \in \mathcal{M} : f_0(u) \leq c\}, \quad \Phi(u,t) = u \quad \forall u \in \mathcal{M}_{S+\frac{\delta}{2}}\,.$$

Using Proposition 4.4, we can assume that $\beta(\mathcal{M}_{S+\delta}) \subset U$, where U is a neighborhood of $\overline{\Omega}$ such that any point $p \in U$ has a unique nearest neighbor $q = \pi(p) \in \Omega$ and such that the projection π is continuous .

The map $h : \Sigma \times [0,1] \to \Omega$ given by

$$h(\sigma, t) = \begin{cases} \pi(\beta(\Phi(v_t^\sigma, 1))), & t \in [0,1) \\ \sigma, & t = 1 \end{cases}$$

is well-defined. Furthermore, h is a continuous function in $\Sigma \times [0,1]$, which is obvious for $t \in [0,1)$, now for the case $t = 1$ we use the following argument: Note that for each $(\sigma_n, t_n) \in \Sigma \times [0,1]$, we compute

$$\begin{aligned} \int_\Omega x(u_{t_n}^{\sigma_n})^{p^*} dx = (1-t_n)^{\frac{N(p-2)}{p-1}} t_n \sigma_n \int_{\Omega_{t_n}} \frac{dx}{\left[1 + |w|^{\frac{p}{p-1}}\right]^N} \\ + (1-t_n)^{\frac{N(p-2)}{p-1}} (1-t_n) \int_{\Omega_{t_n}} \frac{w dx}{\left[1 + |w|^{\frac{p}{p-1}}\right]^N} \,, \end{aligned}$$

where $\Omega_{t_n} = \frac{(\Omega - t_n\sigma_n)}{1-t_n}$. Since

$$|u_{t_n}^{\sigma_n}|_{p^*}^{p^*} = (1-t_n)^{\frac{N(p-2)}{p-1}} \int_{\mathbb{R}^N} \frac{dx}{\left[1 + |w|^{\frac{p}{p-1}}\right]^N},$$

if $(\sigma_n, t_n) \to (\sigma, 1)$ as $n \to \infty$ we get

$$\beta\Big(\frac{u_{t_n}^{\sigma_n}}{|u_{t_n}^{\sigma_n}|_{p^*}}\Big) \to \sigma.$$

Using the limit above together with Lemma 4.2, we conclude that

$$\lim_{n\to\infty} h(\sigma_n, t_n) = \sigma = h(\sigma, 1) \,.$$

Therefore, h is a continuous functions in $\Sigma \times [0,1]$. Also observe that

$$h(\sigma, 0) = \pi(\beta(\Phi(v_0, 1))) = x_0 \in \Omega, \quad \forall \sigma \in \Sigma$$
$$h(\sigma, 1) = \sigma, \quad \forall \sigma \in \Sigma$$

hence h is a contraction of Σ in Ω, contradicting the hypotheses on Ω. $\square$

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Claudianor O. Alves

Universidade Federal de Campina Grande, Departamento de Matemática e Estatística, CEP: 58109-970 Campina Grande - PB, Brazil

E-mail address: `coalves@dme.ufcg.edu.br`