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EXISTENCE OF POSITIVE SOLUTIONS FOR NONLINEAR BOUNDARY-VALUE PROBLEMS IN UNBOUNDED DOMAINS OF $\mathbb{R}^n$

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Abstract. Let D be an unbounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with a nonempty compact boundary ∂D. We consider the following nonlinear elliptic problem, in the sense of distributions,

$$\begin{gathered}\Delta u = f(.,u), \quad u > 0 \quad \text{in } D,\\ u\big|_{\partial D} = \alpha\varphi,\\ \lim_{|x|\to+\infty} \frac{u(x)}{h(x)} = \beta\lambda,\end{gathered}$$

where α, β, λ are nonnegative constants with $\alpha + \beta > 0$ and φ is a nontrivial nonnegative continuous function on ∂D. The function f is nonnegative and satisfies some appropriate conditions related to a Kato class of functions, and h is a fixed harmonic function in D, continuous on $\overline{D}$. Our aim is to prove the existence of positive continuous solutions bounded below by a harmonic function. For this aim we use the Schauder fixed-point argument and a potential theory approach.

1. Introduction

In this paper, we are concerned with the existence and asymptotic behavior of positive solutions for the following nonlinear elliptic equation, in the sense of distributions,

$$\Delta u = f(.,u) \quad \text{in } D, \tag{1.1}$$

where D is an unbounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with a nonempty compact boundary ∂D and f is a nonnegative measurable function on D that may be singular or sublinear with respect to the second variable. More precisely we will study the problem

$$\begin{gathered}\Delta u = f(.,u), \quad u > 0 \quad \text{in } D,\\ u\big|_{\partial D} = \alpha\varphi,\\ \lim_{|x|\to+\infty} \frac{u(x)}{h(x)} = \beta\lambda \geq 0,\end{gathered} \tag{1.2}$$

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where $\alpha \geq 0$, $\beta \geq 0$, φ is a nontrivial nonnegative continuous function on ∂D, h is the harmonic function in D given by (2.2) below and f satisfies some appropriate conditions related to a Kato class (see Definition 2.3) introduced by Bachar et al in [3] for $n \geq 3$ and Mâagli and Maâtoug in [11] for $n = 2$.

In [2], Athreya considered (1.1) with a special case of nonlinearity $f(x,u) = g(u) \leq \max(1, u^{-\alpha})$ for $0 < \alpha < 1$, on a simply connected bounded C^2-domain Ω. He showed that if h_0 is a fixed positive harmonic function in Ω and φ is a nontrivial nonnegative continuous function on $\partial\Omega$, there exists a constant $c > 1$ such that if $\varphi \geq c\, h_0$ on $\partial\Omega$, then (1.1) has a positive continuous solution u satisfying $u = \varphi$ on $\partial\Omega$ and $u \geq h_0$ in Ω.

This result was extended by Bachar et al [5] on the half space $\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$ $(n \geq 2)$. More precisely, they proved that the problem

$$
\begin{gathered}
\Delta u = f(.,u) \quad \text{in } \mathbb{R}^n_+, \\
u = \varphi \quad \text{in } \partial\mathbb{R}^n_+, \\
\lim_{x_n \to +\infty} \frac{u(x)}{x_n} = c \geq 0,
\end{gathered}
$$

has a positive solution u satisfying $u(x) \geq c x_n + \rho_0(x)$ in $\mathbb{R}^n_+$, where ρ_0 is a fixed positive continuous bounded harmonic function in $\overline{\mathbb{R}^n_+}$.

In the sublinear case where $f(x,u) = p(x)u^\alpha$, $0 < \alpha \leq 1$, Lair and Wood [8] studied the existence of positive large solutions and bounded ones for the equation (1.1). In particular they proved the existence of entire bounded nonnegative solutions in $\mathbb{R}^n$ provided that p is locally hölder continuous and satisfies $\int_0^\infty t \max_{|x|=t}(p(x))dt < \infty$.

This result was extended by Bachar and Zeddini in [4] to more general function $f(x,u) = q(x)g(u)$. More precisely it is shown in [4] that the equation (1.1) has at least one positive continuous bounded solution in $\mathbb{R}^n$, provided that the Green potential of q is continuous bounded in $\mathbb{R}^n$ and for all $\alpha > 0$, there exists a constant $k > 0$ such that the function $x \to kx - g(x)$ is nondecreasing on $[\alpha, \infty)$.

In this work, we will give two existence results for the problem (1.2). For this aim, we fix a positive harmonic function h_0 in D, which is continuous and bounded in $\overline{D}$ such that $\lim_{|x|\to+\infty} h_0(x) = 0$, whenever $n \geq 3$. We suppose that the function f satisfies combinations of the following hypotheses:

- (H1) $f : D \times (0, +\infty) \to [0, +\infty)$ is measurable, continuous with respect to the second variable.
- (H2) There exists a nonnegative measurable function θ on $D \times (0, +\infty)$ such that the function $t \mapsto \theta(x,t)$ is nonincreasing on $(0, +\infty)$, and satisfies
$$f(x,t) \leq \theta(x,t), \text{ for } \ (x,t) \in D \times (0, +\infty).$$
- (H3) The function ψ defined on D by $\psi(x) = \frac{\theta(x, h_0(x))}{h_0(x)}$ belongs to the class $K^\infty(D)$.
- (H4) For each $\alpha \geq 0$ and $\beta \geq 0$ with $\alpha + \beta > 0$, there exists a nonnegative function $q_{\alpha,\beta} = q \in K^\infty(D)$ such that for each $x \in D$ and $t \geq s \geq \alpha h_0(x) + \beta h(x)$, we have
$$f(x,t) - f(x,s) \leq (t-s)q(x), \tag{1.3}$$
$$f(x,t) \leq t q(x). \tag{1.4}$$

For the rest of this paper, we denote by $H_D\varphi$ the bounded continuous solution of the Dirichlet problem

$$\begin{gathered}\Delta w = 0 \quad \text{in } D,\\ w\big|_{\partial D} = \varphi,\\ \lim_{|x|\to+\infty}\frac{w(x)}{h(x)} = 0,\end{gathered}\tag{1.5}$$

where φ is a nonnegative continuous function on ∂D and h is the harmonic function given by (2.2).

The outline of this paper is as follows. In the second section we recall and improve some useful results concerning estimates on the Green function G_D of the Laplace operator Δ in D and some properties of functions belonging to the Kato class $K^\infty(D)$. In section 3, we will prove a first existence result for the problem (1.2), by using the Schauder fixed-point theorem. More precisely, we prove the following

Theorem 1.1. *Under the assumptions (H1)–(H3), there exists a constant $c > 1$ such that if $\varphi \ge ch_0$ on ∂D, then for each $\lambda \ge 0$, the problem (1.2) with $\alpha = \beta = 1$ has a positive continuous solution u satisfying for each $x \in D$,*

$$\lambda h(x) + h_0(x) \le u(x) \le \lambda h(x) + H_D\varphi(x).$$

In the last section, we use a potential theory approach to prove a second existence result for the problem (1.2). More precisely, we will prove the following result.

Theorem 1.2. *Under the assumptions (H1) and (H4), if $\alpha \ge 0$ and $\beta \ge 0$ with $\alpha + \beta > 0$, then there exists a constant $c_1 > 1$ such that if $\varphi \ge c_1 h_0$ on ∂D and $\lambda \ge c_1$, the problem (1.2) has a positive continuous solution u satisfying: For each $x \in D$,*

$$\alpha h_0(x) + \beta h(x) \le u(x) \le \alpha H_D\varphi(x) + \beta\lambda h(x).$$

Notation and preliminaries. Throughout this paper, we will adopt the following notation.

i. D is an unbounded domain in $\mathbb{R}^n$ $(n \ge 2)$ such that the complement of $\overline{D}$ in $\mathbb{R}^n$, $\overline{D}^c = \bigcup_{j=1}^d D_j$ where D_j is a bounded $C^{1,1}$-domain and $\overline{D}_i \bigcap \overline{D}_j = \emptyset$, for $i \ne j$.

ii. For a metric space S, we denote by $\mathcal{B}(S)$ the set of Borel measurable functions and $\mathcal{B}_b(S)$ the set of bounded ones. $\mathcal{C}(S)$ will denote the set of continuous functions on S. The exponent $+$ means that only the nonnegative functions are considered.

iii. $\mathcal{C}_0(\overline{D}) = \{f \in \mathcal{C}(\overline{D}) : \lim_{|x|\to+\infty} f(x) = 0\}$.

iv. $\mathcal{C}_b(D) = \{f \in \mathcal{C}(D) : f \text{ is bounded in } D\}$. We note that $\mathcal{C}_0(\overline{D})$ and $\mathcal{C}_b(D)$ are two Banach spaces endowed with the uniform norm

$$\|f\|_\infty = \sup_{x\in D}|f(x)|.$$

v. For $x \in D$, we denote by $\delta_D(x)$ the distance from x to ∂D,

$$\rho_D(x) = \frac{\delta_D(x)}{\delta_D(x)+1}, \quad \lambda_D(x) = \delta_D(x)(\delta_D(x)+1).$$

vi. Let f and g be two positive functions on a set S. We denote $f \sim g$, if there exists a constant $c > 0$ such that
$$\frac{1}{c} g(x) \leq f(x) \leq c g(x) \quad \text{for all } x \in S.$$

vii. For $f \in \mathcal{B}^+(D)$, we denote by Vf the Green potential of f defined on D by
$$Vf(x) = \int_D G_D(x,y) f(y) dy.$$
Recall that if $f \in L^1_{\rm loc}(D)$ and $Vf \in L^1_{\rm loc}(D)$, then we have in the distributional sense (see [6, p. 52])
$$\Delta(Vf) = -f \quad \text{in } D. \tag{1.6}$$
Furthermore, we recall that for $f \in \mathcal{B}^+(D)$, the potential Vf is lower semi-continuous in D and if $f = f_1 + f_2$ with $f_1, f_2 \in \mathcal{B}^+(D)$ and $Vf \in \mathcal{C}^+(D)$, then $Vf_i \in \mathcal{C}^+(D)$ for $i \in \{1,2\}$.

viii. Let $(X_t, t > 0)$ be the Brownian motion in $\mathbb{R}^n$ and P^x be the probability measure on the Brownian continuous paths starting at x. For $q \in \mathcal{B}^+(D)$, we define the kernel V_q by
$$V_q f(x) = E^x\Big(\int_0^{\tau_D} e^{-\int_0^t q(X_s)ds} f(X_t) dt\Big), \tag{1.7}$$
where E^x is the expectation on P^x and $\tau_D = \inf\{t > 0 : X_t \notin D\}$.

If $q \in \mathcal{B}^+(D)$ such that $Vq < \infty$, the kernel V_q satisfies the resolvent equation (see [6, 9])
$$V = V_q + V_q(qV) = V_q + V(qV_q). \tag{1.8}$$
So for each $u \in \mathcal{B}(D)$ such that $V(q|u|) < \infty$, we have
$$(I - V_q(q.))(I + V(q.))u = (I + V(q.))(I - V_q(q.))u = u. \tag{1.9}$$

ix. We recall that a function $f : [0,\infty) \to \mathbb{R}$ is called completely monotone if $(-1)^n f^{(n)} \geq 0$, for each $n \in \mathbb{N}$. Moreover, if f is completely monotone on $[0,\infty)$ then by [15, Theorem 12a] there exists a nonnegative measure μ on $[0,\infty)$ such that
$$f(t) = \int_0^\infty \exp(-tx) d\mu(x).$$
So, using this fact and the Holder inequality we deduce that if f is completely monotone from $[0,\infty)$ to $(0,\infty)$, then $\log f$ is a convex function.

x. Let $f \in \mathcal{B}^+(D)$ be such that $Vf < \infty$. From (1.7), it is easy to see that for each $x \in D$, the function $F : \lambda \to V_{\lambda q} f(x)$ is completely monotone on $[0,\infty)$.

xi. Let $a \in \mathbb{R}^n \backslash \overline{D}$ and $r > 0$ such that $\overline{B(a,r)} \subset \mathbb{R}^n \backslash \overline{D}$. Then we have
$$G_D(x,y) = r^{2-n} G_{\frac{D-a}{r}}\big(\frac{x-a}{r}, \frac{y-a}{r}\big), \quad \text{for } x, y \in D,$$
$$\delta_D(x) = r \delta_{\frac{D-a}{r}}\big(\frac{x-a}{r}\big), \quad \text{for } x \in D,$$
So without loss of generality, we may suppose that $\overline{B(0,1)} \subset \mathbb{R}^n \diagdown \overline{D}$. Moreover, we denote by D^* the open set
$$D^* = \{x^* \in B(0,1) : x \in D \cup \{\infty\}\},$$

where $x^* = x/|x|^2$ is the Kelvin inversion from $D \cup \{\infty\}$ onto D^* (see [3, 11]). Then for $x, y \in D$,

$$G_D(x,y) = |x|^{2-n}|y|^{2-n} G_{D^*}(x^*, y^*).$$

Also we mention that the letter C will denote a generic positive constant which may vary from line to line.

2. Properties of the Green function and the Kato class

In this section, we recall and improve some results concerning the Green function $G_D(x,y)$ and the Kato class $K^\infty(D)$, which are stated in [3] for $n \geq 3$ and in [11] for $n = 2$.

Theorem 2.1 (3G-Theorem). *There exists a constant $C_0 > 0$ depending only on D such that for all x, y and z in D*

$$\frac{G_D(x,z)G_D(z,y)}{G_D(x,y)} \leq C_0 \Big(\frac{\rho_D(z)}{\rho_D(x)} G_D(x,z) + \frac{\rho_D(z)}{\rho_D(y)} G_D(y,z) \Big).$$

Proposition 2.2. *On D^2 (that is $x, y \in D$), we have*

$$G_D(x,y) \sim \begin{cases} \frac{1}{|x-y|^{n-2}} \min\Big(1, \frac{\lambda_D(x)\lambda_D(y)}{|x-y|^2}\Big), & n \geq 3, \\ \log(1 + \frac{\lambda_D(x)\lambda_D(y)}{|x-y|^2}), & n = 2. \end{cases}$$

Moreover, for $M > 1$ and $r > 0$ there exists a constant $C > 0$ such that for each $x \in D$ and $y \in D$ satisfying $|x - y| \geq r$ and $|y| \leq M$, we have

$$G_D(x,y) \leq C \frac{\rho_D(x)\rho_D(y)}{|x-y|^{n-2}}. \tag{2.1}$$

Definition 2.3. A Borel measurable function q in D belongs to the Kato class $K^\infty(D)$ if q satisfies

$$\lim_{\alpha \to 0} (\sup_{x \in D} \int_{D \cap B(x,\alpha)} \frac{\rho_D(y)}{\rho_D(x)} G_D(x,y)|q(y)|dy) = 0,$$

$$\lim_{M \to \infty} (\sup_{x \in D} \int_{D \cap (|y| \geq M)} \frac{\rho_D(y)}{\rho_D(x)} G_D(x,y)|q(y)|dy) = 0.$$

In this paper, h denotes the function defined, on D, by

$$h(x) = c_n |x|^{2-n} G_{D^*}(x^*, 0) = c_n \lim_{|y| \to +\infty} |y|^{n-2} G_D(x,y), \tag{2.2}$$

where $c_n = \begin{cases} 2\pi & \text{for } n = 2, \\ \frac{4\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}-1)} & \text{for } n \geq 3. \end{cases}$. Then we have the following statement.

Proposition 2.4. *The function h defined by (2.2) is harmonic in D and satisfies $\lim_{x \to z \in \partial D} h(x) = 0$,*

$$\lim_{|x| \to \infty} \frac{h(x)}{\log |x|} = 1 \quad \text{for } n = 2,$$

$$\lim_{|x| \to \infty} h(x) = 1 \quad \text{for } n \geq 3.$$

Moreover,

$$h(x) \sim \begin{cases} \rho_D(x) & \text{for } n \geq 3, \\ \log(1 + \rho_D(x)) & \text{for } n = 2. \end{cases} \tag{2.3}$$

The proof of the above proposition can be found in [11, Lemma 4.1] and in [13].

Remark 2.5 ([7, p.427])**.** The function $H_D\varphi$ defined in (1.5) belongs to $\mathcal{C}(\overline{D}\cup\{\infty\})$ and satisfies

$$\lim_{|x|\to+\infty}|x|^{n-2}H_D\varphi(x)=C>0.$$

In the sequel, we use the notation

$$\|q\|_D=\sup_{x\in D}\int_D\frac{\rho_D(y)}{\rho_D(x)}G_D(x,y)|q(y)|dy \tag{2.4}$$

$$\alpha_q=\sup_{x,y\in D}\int_D\frac{G_D(x,z)G_D(z,y)}{G_D(x,y)}|q(z)|dz. \tag{2.5}$$

It is shown in [3, 11] that if $q\in K^\infty(D)$, then

$$\|q\|_D<\infty. \tag{2.6}$$

Now, we remark that from the 3G-Theorem,

$$\alpha_q\le 2C_0\|q\|_D,$$

where C_0 is the constant. Next, we prove that $\alpha_q\sim\|q\|_D$.

Proposition 2.6. *The following assertions hold*

(i) *For any nonnegative superharmonic function v in D and any q in $K^\infty(D)$,*

$$\int_D G_D(x,y)v(y)|q(y)|dy\le\alpha_q v(x),\quad\forall x\in D. \tag{2.7}$$

(ii) *There exists a constant $C>0$ such that for each $q\in K^\infty(D)$,*

$$C\|q\|_D\le\alpha_q.$$

Proof. (i) Let v be a nonnegative superharmonic function in D. Then by [14, Theorem 2.1] there exists a sequence $(f_k)_k$ of nonnegative measurable functions in D such that the sequence $(v_k)_k$ defined on D by

$$v_k(y):=\int_D G_D(y,z)f_k(z)dz$$

increases to v. Since for each $x\in D$, we have

$$\int_D G_D(x,y)v_k(y)|q(y)|dy\le\alpha_q v_k(x),$$

the result follows from the monotone convergence theorem.

(*ii*) We will discuss two cases: Case 1 ($n\ge3$). Using Fatou's Lemma and (2.2) we obtain

$$\int_D\frac{h(z)}{h(x)}G_D(x,z)|q(z)|dz\le\liminf_{|y|\to+\infty}\int_D\frac{G_D(x,z)G_D(z,y)}{G_D(x,y)}|q(z)|dz\le\alpha_q.$$

Hence, the result follows from (2.3).

Case 2 ($n=2$). Let φ_1 be a positive eigenfunction associated to the first eigenvalue of the Laplacian in D^*. From [10, Proposition 2.6], we have

$$\varphi_1(\xi)\sim\delta_{D^*}(\xi),\quad\forall\xi\in D^*.$$

Let $v(x)=\varphi_1(x^*)$ for $x\in D$. Then v is superharmonic in D and

$$v(x)\sim\delta_{D^*}(x^*)\sim\rho_D(x).$$

Applying the assertion (i) to this function v we deduce the result. $\square$

Proposition 2.7 ([3, 11])**.** *Let q be a function in $K^\infty(D)$ and v be a positive superharmonic function in D.*

(a) *Let $x_0 \in \overline{D}$. Then*

$$\lim_{r\to 0}\big(\sup_{x\in D}\int_{B(x_0,r)\cap D}\frac{v(y)}{v(x)}G_D(x,y)|q(y)|dy\big)=0, \tag{2.8}$$

$$\lim_{M\to+\infty}\big(\sup_{x\in D}\int_{(|y|\geq M)\cap D}\frac{v(y)}{v(x)}G_D(x,y)|q(y)|dy\big)=0. \tag{2.9}$$

(b) *The potential Vq is in $\mathcal{C}_b(D)$, $\lim_{x\to z\in\partial D}Vq(x)=0$, and for $n\geq 3$, $\lim_{|x|\to+\infty}Vq(x)=0$.*

(c) *The function $x\to\frac{\delta_D(x)}{|x|^{n-1}}q(x)$ is in $L^1(D)$.*

Example 2.8. Let $p>n/2$ and $\gamma,\mu\in\mathbb{R}$ such that $\gamma<2-\frac{n}{p}<\mu$. Then using the Hölder inequality and the same arguments as in [3, Proposition 3.4] and [11, Proposition 3.6], it follows that for each $f\in L^p(D)$, the function defined in D by $\frac{f(x)}{|x|^{\mu-\gamma}(\delta_D(x))^\gamma}$ belongs to $K^\infty(D)$. Moreover, by taking $p=+\infty$, we find again the results of [3, 11].

Proposition 2.9. *Let v be a nonnegative superharmonic function in D and $q\in K^\infty_+(D)$. Then for each $x\in D$ such that $0<v(x)<\infty$, we have*

$$\exp(-\alpha_q)v(x)\leq v(x)-V_q(qv)(x)\leq v(x).$$

Proof. Let v be a nonnegative superharmonic function in D. Then by [14, Theorem 2.1] there exists a sequence $(f_k)_k$ of nonnegative measurable functions in D such that the sequence $(v_k)_k$ given in D by

$$v_k(x):=\int_D G_D(x,y)f_k(y)dy$$

increases to v. Let $x\in D$ such that $0<v(x)<\infty$. Then there exists $k_0\in\mathbb{N}$ such that $0<Vf_k(x)<\infty$, for $k\geq k_0$.

Now, for a fixed $k\geq k_0$, we consider the function $\chi(t)=V_{tq}f_k(x)$. Since the function χ is completely monotone on $[0,\infty)$, then $\log\chi$ is convex on $[0,\infty)$. Therefore,

$$\chi(0)\leq\chi(1)\exp\big(-\frac{\chi'(0)}{\chi(0)}\big),$$

which implies

$$Vf_k(x)\leq V_qf_k(x)\exp\big(\frac{V(qVf_k)(x)}{Vf_k(x)}\big).$$

Hence, it follows from Proposition 2.6(i) that

$$\exp(-\alpha_q)Vf_k(x)\leq V_qf_k(x).$$

Consequently, from (1.8) we obtain

$$\exp(-\alpha_q)Vf_k(x)\leq Vf_k(x)-V_q(qVf_k(x))(x)\leq Vf_k(x).$$

By letting $k\to\infty$, we deduce the result. □

3. Proof of Theorem 1.1

Recall that h_0 is a fixed positive harmonic function in D, which is continuous and bounded in $\overline{D}$ and h is the function defined by (2.2). For a fixed nonnegative function $q \in K^\infty(D)$, we define

$$\Gamma_q = \{p \in K^\infty(D) : |p| \leq q\}.$$

To prove Theorem 1.1 we need the following result.

Lemma 3.1. *Let q be a nonnegative function belonging to $K^\infty(D)$. Then the family of functions*

$$F_q = \Big\{ \int_D G_D(.,y)h_0(y)p(y)dy : p \in \Gamma_q \Big\}$$

is uniformly bounded and equicontinuous in $\overline{D} \cup \{\infty\}$. Consequently, it is relatively compact in $\mathcal{C}(\overline{D} \cup \{\infty\})$.

Proof. Let $q \in K^\infty_+(D)$ and L the operator defined on Γ_q by

$$Lp(x) = \int_D G_D(x,y)h_0(y)p(y)\,dy.$$

Then by (2.7), we have for each $p \in \Gamma_q$ and $x \in D$,

$$|Lp(x)| \leq \int_D G_D(x,y)h_0(y)q(y)dy \leq \alpha_q h_0(x) \leq \alpha_q \|h_0\|_\infty.$$

Hence the family $F_q := L(\Gamma_q)$ is uniformly bounded.

Now, let us prove that $L(\Gamma_q)$ is equicontinuous on $\overline{D} \cup \{\infty\}$. Let $x_0 \in D$ and $r > 0$. Let $x \in B(x_0,r) \cap D$ and $p \in \Gamma_q$. Since h_0 is bounded, for $M > 0$ we have

$$\begin{aligned}
\frac{1}{\|h_0\|_\infty}|Lp(x) - Lp(x_0)| &\leq \int_D |G_D(x,y) - G_D(x_0,y)|q(y)dy \\
&\leq 2\sup_{z\in D}\int_{B(x_0,2r)\cap D} G_D(z,y)q(y)dy \\
&\quad + 2\sup_{z\in D}\int_{(|y|\geq M)\cap D} G_D(z,y)q(y)dy \\
&\quad + \int_\Omega |G_D(x,y) - G_D(x_0,y)|q(y)dy,
\end{aligned}$$

where $\Omega = B^c(x_0,2r) \cap B(0,M) \cap D$. On the other hand, for every $y \in \Omega$ and $x \in B(x_0,r) \cap D$, using (2.1), we obtain

$$\begin{aligned}
|G_D(x,y) - G_D(x_0,y)| &\leq C\Big[\frac{\rho_D(x)}{|x-y|^{n-2}} + \frac{\rho_D(x_0)}{|x_0-y|^{n-2}}\Big]\rho_D(y) \\
&\leq C\delta_D(y) \leq C\frac{\delta_D(y)}{|y|^{n-1}}.
\end{aligned}$$

Now, since G_D is continuous outside the diagonal, we deduce by the dominated convergence theorem and Proposition 2.7 (c) that

$$\int_\Omega |G_D(x,y) - G_D(x_0,y)|q(y)dy \to 0 \quad \text{as } |x-x_0| \to 0.$$

So, using Proposition 2.7(a) for $v \equiv 1$, we deduce that $|Lp(x) - Lp(x_0)| \to 0$ as $|x - x_0| \to 0$, uniformly for all $p \in \Gamma_q$. On the other hand, on D, we have

$$|Lp(x)| \le \|h_0\|_\infty Vq(x), \tag{3.1}$$

which tends to zero as $x \to \partial D$. Hence, $L(\Gamma_q)$ is equicontinuous on $\overline{D}$.

Next, we shall prove that $L(\Gamma_q)$ is equicontinuous at ∞. First, we claim that

$$\lim_{|x|\to\infty} Lp(x) = \begin{cases} 0 & \text{for } n \ge 3, \\ \int_D h_0(y)p(y)h(y)dy & \text{for } n = 2. \end{cases}$$

Using (3.1) and Proposition 2.7(b), we obtain $Lp(x) \to 0$ as $|x| \to \infty$, for $n \ge 3$, uniformly in $p \in \Gamma_q$.

Finally, we assume that $n = 2$ and we put $l = \int_D h_0(y)p(y)h(y)dy$. Since $\lim_{|x|\to+\infty} G_D(x,y) = h(y)$, then using Fatou's lemma and Proposition 2.7(b), we obtain

$$\begin{aligned} |l| &\le \int_D h_0(y)q(y)h(y)dy \\ &\le \liminf_{|x|\to+\infty} \int_D G(x,y)h_0(y)q(y)dy \\ &\le \|h_0\|_\infty \|Vq\|_\infty < +\infty. \end{aligned}$$

Now, we shall prove that $\lim_{|x|\to+\infty} Lp(x) = l$. Let $\varepsilon > 0$, then by (2.9), there exists $M > 1$ such that for each $x \in D$ with $|x| \ge 1 + M$ we have

$$\begin{aligned} |Lp(x) - l| &\le \int_D |G_D(x,y) - h(y)| h_0(y)q(y)dy \\ &\le \varepsilon + \int_{B(0,M)\cap D} |G_D(x,y) - h(y)| h_0(y)q(y)dy. \end{aligned}$$

On the other hand, using (2.1), for $y \in B(0,M) \cap D$ and $|x| \ge 1 + M$, we have

$$|G_D(x,y) - h(y)| h_0(y) \le C\Big(\frac{\delta_D(y)}{|y|} + h(y)\Big).$$

We deduce from Proposition 2.7(c) and Lebesgue's theorem that $\lim_{|x|\to+\infty} Lp(x) = l$, uniformly in $p \in \Gamma_q$. Thus by Ascoli's theorem F_q is relatively compact in $\mathcal{C}(\overline{D} \cup \{\infty\})$. This completes the proof. $\square$

Proof of Theorem 1.1. We shall use a fixed-point argument. Let $c = 1 + \alpha_\psi$, where α_ψ is the constant defined by (2.5) associated to the function ψ given in (H3) and suppose that

$$\varphi(x) \ge c h_0(x), \quad \forall x \in \partial D.$$

Since h_0 is a harmonic function in D, continuous and bounded in $\overline{D}$, then the function $w := H_D\varphi - ch_0$ is a solution to the problem

$$\begin{gathered} \Delta w = 0 \quad \text{in } D, \\ w\big|_{\partial D} = \varphi - ch_0 \ge 0, \\ \lim_{|x|\to+\infty} \frac{w(x)}{h(x)} = 0, \end{gathered}$$

and by the maximum principle it follows that

$$H_D\varphi(x) \ge ch_0(x), \quad \forall x \in \overline{D}. \tag{3.2}$$

Let $\lambda \geq 0$ and let Λ be the non-empty closed bounded convex set

$$\Lambda = \{v \in C(\overline{D} \cup \{\infty\}) : h_0 \leq v \leq H_D\varphi\}.$$

Let S be the operator defined on Λ by

$$Sv(x) = H_D\varphi(x) - \int_D G_D(x,y) f(y, v(y) + \lambda h(y)) dy.$$

We shall prove that the family $S\Lambda$ is relatively compact in $C(\overline{D} \cup \{\infty\})$. Let $v \in \Lambda$, then by (H2) and (H3) and the fact that h_0 is positive in D, we have for each $y \in D$,

$$\frac{1}{h_0(y)} f(y, v(y) + \lambda h(y)) \leq \frac{\theta(y, h_0(y))}{h_0(y)} = \psi(y).$$

Hence, we deduce that the function

$$y \mapsto \frac{1}{h_0(y)} f(y, v(y) + \lambda h(y)) \in \Gamma_\psi.$$

It follows that the family

$$\{\int_D G_D(.,y) f(y, v(y) + \lambda h(y)) dy : v \in \Lambda\} \subseteq F_\psi.$$

Thus, from Lemma 3.1, the family $\{\int_D G_D(.,y) f(y, v(y) + \lambda h(y)) dy : v \in \Lambda\}$ is relatively compact in $C(\overline{D} \cup \{\infty\})$. Since $H_D\varphi \in C(\overline{D} \cup \{\infty\})$, we deduce that the family $S(\Lambda)$ is relatively compact in $C(\overline{D} \cup \{\infty\})$.

Next, we shall prove that S maps Λ to itself. It's clear that for all $v \in \Lambda$ we have $Sv(x) \leq H_D\varphi(x), \forall x \in D$. Moreover, from hypothesis (H2) and (2.7), it follows that

$$\begin{aligned}
\int_D G_D(x,y) f(y, v(y) + \lambda h(y)) dy &\leq \int_D G_D(x,y) \theta(y, h_0(y)) dy \\
&= \int_D G_D(x,y) \psi(y) h_0(y) dy \\
&\leq \alpha_\psi h_0(x).
\end{aligned}$$

Hence, using (3.2) we obtain $Sv(x) \geq H_D\varphi(x) - \alpha_\psi h_0(x) \geq h_0(x)$, which proves that $S(\Lambda) \subset \Lambda$.

Now, we prove the continuity of the operator S in Λ in the supremum norm. Let $(v_k)_k$ be a sequence in Λ which converges uniformly to a function v in Λ. Then, for each $x \in D$, we have

$$|Sv_k(x) - Sv(x)| \leq \int_D G_D(x,y) |f(y, v_k(y) + \lambda h(y)) - f(y, v(y) + \lambda h(y))| dy.$$

On the other hand, by hypothesis (H2), we have

$$|f(y, v_k(y) + \lambda h(y)) - f(y, v(y) + \lambda h(y))| \leq 2h_0(y)\psi(y) \leq 2\|h_0\|_\infty \psi(y).$$

Since by Proposition 2.7(b), $V\psi$ is bounded, we conclude by the continuity of f with respect to the second variable and by the dominated convergence theorem that for all $x \in D$,

$$Sv_k(x) \to Sv(x) \quad \text{as } k \to +\infty.$$

Consequently, as $S(\Lambda)$ is relatively compact in $C(\overline{D} \cup \{\infty\})$, we deduce that the pointwise convergence implies the uniform convergence, namely,

$$\|Sv_k - Sv\|_\infty \to 0 \quad \text{as } k \to +\infty.$$

Therefore, S is a continuous mapping of Λ to itself. So since $S\Lambda$ is relatively compact in $C(\overline{D}\cup\{\infty\})$ it follows that S is compact mapping on Λ.

Finally, the Schauder fixed-point theorem implies the existence of $v\in\Lambda$ such that

$$v(x)=H_D\varphi(x)-\int_D G_D(x,y)f(y,v(y)+\lambda h(y))dy.$$

Put $u(x)=v(x)+\lambda h(x)$, for $x\in D$. Then $u\in C(\overline{D})$ and u satisfies

$$u=H_D\varphi+\lambda h-\int_D G_D(.,y)f(y,u(y))dy. \tag{3.3}$$

Now, we verify that u is a solution of (1.2) with $\alpha=\beta=1$. Since $\psi\in K^\infty(D)$, it follows from Proposition 2.7(c), that $\psi\in L^1_{\rm loc}(D)$. Furthermore, by hypotheses (H2) and (H3) we have $f(.,u)\le h_0\psi$. This shows that $f(.,u)\in L^1_{\rm loc}(D)$ and $V(f(.,u))\in F_\psi$. Then, from Lemma 3.1, we have $V(f(.,u))\in C(\overline{D}\cup\{\infty\})\subset L^1_{\rm loc}(D)$. Thus, by applying Δ on both sides of (3.3) and using (1.6), we obtain that u satisfies the elliptic equation (in the sense of distributions)

$$\Delta u=f(.,u)\quad\text{in } D\,.$$

Since $H_D\varphi=\varphi$ on $\partial D, \lim_{x\to z\in\partial D}h(x)=0$, and $\lim_{x\to z\in\partial D}V(f(.,u))(x)=0$, we conclude that $\lim_{x\to z\in\partial D}u(x)=\varphi(z)$. On the other hand, since

$$\lambda h(x)+h_0(x)\le u(x)\le\lambda h(x)+H_D\varphi(x)$$

and $\lim_{|x|\to+\infty}\frac{H_D\varphi(x)}{h(x)}=\lim_{|x|\to+\infty}\frac{h_0(x)}{h(x)}=0$, we deduce $\lim_{|x|\to+\infty}\frac{u(x)}{h(x)}=\lambda$. This completes the proof. □

Example 3.2. Let $D=B^c(0,1)$, $p>\frac{n}{2}$, $\sigma>0$ and $\nu>0$. Let φ and g in $\mathcal{C}^+(\partial D)$ and put $h_0=H_Dg$. Then from [1, p. 258], there exists a constant $c_0>0$ such that for each $x\in D$,

$$\frac{c_0(|x|-1)}{|x|^{n-1}}\le h_0(x)$$

Moreover, suppose that the function f satisfies (H1) and

$$f(x,t)\le t^{-\sigma}\frac{v(x)}{|x|^{\nu-1+n(\sigma+1)}(|x|-1)^{1-2\sigma-\frac{n}{p}}},$$

where $v\in L^p_+(D)$. Then, there exists a constant $c>1$ such that if $\varphi\ge cg$ on ∂D, the problem (1.2) with $\alpha=\beta=1$ has a positive solution u in $\mathcal{C}(\overline{D})$ satisfying that for each $x\in D$,

$$\lambda h(x)+h_0(x)\le u(x)\le\lambda h(x)+H_D\varphi(x),$$

where h is the function given by (2.2).

Indeed, (H1) and (H2) are satisfied and by taking $\gamma=2-\sigma-\frac{n}{p}$ and $\mu=2-\frac{n}{p}+\nu$ in Example 2.8, we deduce that the function

$$x\mapsto(h_0(x))^{-1-\sigma}\frac{v(x)}{|x|^{\nu-1+n(\sigma+1)}(|x|-1)^{1-2\sigma-\frac{n}{p}}}\in K^\infty(D),$$

which implies that hypothesis (H3) is satisfied.

4. Proof of Theorem 1.2

Recall that for a fixed nonnegative function $q \in K^\infty(D)$, we have defined the set $\Gamma_q = \{p \in K^\infty(D) : |p| \le q\}$. Using Propositions 2.6 and 2.7, with similar arguments as in [11, Lemma 4.3], we establish the following lemma.

Lemma 4.1. *Let q be a nonnegative function in $K^\infty(D)$ and let h be the function given by* (2.2)*. Then the family of functions*

$$\mathfrak{F}_q(h) = \Big\{ \frac{1}{h} \int_D G(.,y)h(y)p(y)dy : p \in \Gamma_q \Big\}$$

is uniformly bounded and equicontinuous in $\overline{D} \cup \{\infty\}$. Consequently, it is relatively compact in $\mathcal{C}_0(\overline{D})$.

Proof of Theorem 1.2. Let $\alpha \ge 0$, $\beta \ge 0$ with $\alpha + \beta > 0$ and let $q := q_{\alpha,\beta}$ be the function in $K^\infty(D)$ given by (H4). Let $c_1 := e^{\alpha_q} > 1$, where α_q is the constant given by (2.5). Suppose that

$$\varphi(x) \ge c_1 h_0(x), \quad \forall x \in \partial D.$$

Then by the maximum principle it follows that

$$H_D\varphi(x) \ge c_1 h_0(x), \quad \forall x \in \overline{D}. \tag{4.1}$$

Now, let $\lambda \ge c_1$ and put

$$\begin{gathered} w(x) := \beta\lambda h(x) + \alpha H_D\varphi(x), \text{ for } x \in D, \\ v(x) := \alpha h_0 + \beta h(x), \text{ for } x \in D. \end{gathered} \tag{4.2}$$

Consider the nonempty convex set

$$\Omega := \{u \in \mathcal{B}(D) : v \le u \le w\}.$$

Let T be the operator defined on Ω by

$$Tu(x) := w(x) - V_q(qw)(x) + V_q(qu - f(.,u))(x).$$

From hypothesis (H4) we have for each $u \in \Omega$

$$0 \le f(.,u) \le uq. \tag{4.3}$$

Let us prove that the operator T maps Ω to itself. By (2.7), it follows that

$$\int_D G_D(x,y)w(y)q(y)dy \le \alpha_q w(x). \tag{4.4}$$

Since w is a harmonic function in D and $Vq < \infty$, by (4.3) and Proposition 2.9, we have for each $x \in D$,

$$Tu(x) \ge w(x) - V_q(qw)(x) \ge e^{-\alpha_q} w(x) = e^{-\alpha_q}(\beta\lambda h(x) + \alpha H_D\varphi(x)).$$

Therefore, as $\lambda \ge c_1$ and by (4.1) we obtain

$$Tu(x) \ge \beta h(x) + \alpha h_0(x) = v(x).$$

On the other hand, we have for each $x \in D$,

$$Tu(x) \le w(x) - V_q(qw)(x) + V_q(qu)(x) \le w(x).$$

So $T(\Omega) \subset \Omega$. Now, let $u_1, u_2 \in \Omega$ such that $u_1 \ge u_2$, then by (H4) we have

$$Tu_1 - Tu_2 = V_q(q[u_1 - u_2] - [f(.,u_1) - f(.,u_2)]) \ge 0.$$

Hence, T is a nondecreasing operator on Ω.

Next, we consider the sequence $(u_m)_{m\in\mathbb{N}}$ defined by

$$u_0 = \beta h + \alpha h_0 \quad \text{and} \quad u_{m+1} = Tu_m \quad \text{for } m \in \mathbb{N}.$$

Since Ω is invariant under T, we obtain $v = u_0 \le u_1 \le w$. Therefore, from the monotonicity of T on Ω, we have

$$v = u_0 \le u_1 \le \dots \le u_m \le u_{m+1} \le w.$$

Thus, from the monotone convergence theorem and the fact that f is continuous with respect to the second variable, the sequence $(u_m)_{m\in\mathbb{N}}$ converges to a function u satisfying

$$u = (I - V_q(q.))w + V_q(qu - f(.,u)). \tag{4.5}$$

By (2.6) and (2.7), we obtain for each $x \in D$,

$$0 \le V(qu)(x) \le V(qw)(x) \le \alpha_q w(x) < \infty.$$

Applying $(I + V(q.))$ on both sides of (4.5), it follows from (1.8) and (1.9) that

$$u = \beta\lambda h + \alpha H_D\varphi - V(f(.,u)). \tag{4.6}$$

Now, let us verify that u is a solution of the problem (1.2). Since $q \in K^\infty(D)$ then by Proposition 2.7, we obtain $q \in L^1_{\rm loc}(D)$. By (4.3) we have

$$f(.,u) \le qu \le qw. \tag{4.7}$$

Therefore, since w is continuous in D, we obtain that $f(.,u) \in L^1_{\rm loc}(D)$. Using Proposition 2.6 and (4.7), for each $x \in D$, we have

$$V(f(.,u))(x) \le \int_D G_D(x,y)w(y)q(y)dy \le \alpha_q w(x).$$

Then $V(f(.,u)) \in L^1_{\rm loc}(D)$. Thus, by applying Δ on both sides of (4.6), we deduce that u is a solution of

$$\Delta u = f(.,u) \quad \text{in } D$$

(in the sense of distributions). Using (4.7) we obtain that

$$f(.,u) \le \beta\lambda hq + \alpha q H_D\varphi \le \beta\lambda hq + \alpha\|\varphi\|_\infty q\,.$$

Let $g := \beta\lambda hq + \alpha\|\varphi\|_\infty q$. Since $f(.,u)$ and $(g - f(.,u))$ are in $\mathcal{B}^+(D)$ then $V(f(.,u))$ and $V(g - f(.,u))$ are two lower semi-continuous functions.

On the other hand, by Proposition 2.7(b) we have $V(q) \in \mathcal{C}(D)$ and by Lemma 4.1 the function $\frac{1}{h}V(hq) \in \mathcal{C}_0(\overline{D})$. So Vg is a continuous function. This implies that $V(g - f(.,u)) = Vg - V(f(.,u))$ is also an upper semi-continuous function. Consequently $V(g - f(.,u))$ is in $\mathcal{C}(D)$. Thus $V(f(.,u)) = Vg - V(g - f(.,u)) \in \mathcal{C}(D)$. Therefore u is in $\mathcal{C}(D)$.

Now using Proposition 2.6(i) and the fact that $\lim_{x\to z\in\partial D} h(x) = 0$ we deduce that $\lim_{x\to\partial D} V(hq)(x) = 0$. In addition from Proposition 2.7(b) we have $\lim_{x\to\partial D} V(q)(x) = 0$. So that $\lim_{x\to\partial D} V(g)(x) = 0$. This in turn implies that $\lim_{x\to\partial D} V(f(.,u)) = 0$. Then by (4.6), we obtain that $u\big|_{\partial D} = \alpha\varphi$. On the other hand, we have

$$\frac{1}{h}V(f(.,u)) \le \beta\lambda\frac{1}{h}V(hq) + \alpha\|\varphi\|_\infty\frac{1}{h}Vq.$$

Using Propositions 2.4 and 2.7(b), we obtain that $\frac{1}{h(x)}V(f(.,u))(x)$ tends to 0 as $|x| \to +\infty$ and consequently $\lim_{|x|\to+\infty}\frac{u(x)}{h(x)} = \beta\lambda$. Hence, u is a positive continuous solution in D of the problem (1.2). This completes the proof. $\square$

Example 4.2. Let $D = B^c(0,1)$ and $0 < \gamma \leq 1$. Let p be a nonnegative function such that the function $q(x) = (\frac{|x|^{n-1}}{|x|-1})^{1-\gamma}p(x)$ is in $K^\infty(D)$. Let $\varphi \in \mathcal{C}^+(\partial D)$ and h_0 be a positive harmonic function in D, which belongs to $\mathcal{C}_b(\overline{D})$. Then, for each $\alpha \geq 0$ and $\beta \geq 0$ with $\alpha+\beta > 0$, there exists a constant $c_1 > 1$ such that if $\varphi \geq c_1 h_0$ on ∂D and $\lambda \geq c_1$, the problem

$$\begin{gathered}\Delta u = p(x)u^\gamma \quad \text{in } D,\\ u\big|_{\partial D} = \alpha\varphi,\\ \lim_{|x|\to+\infty} \frac{u(x)}{h(x)} = \beta\lambda \geq 0,\end{gathered}$$

has a positive continuous solution on D satisfying that for each $x \in D$,

$$\beta h(x) + \alpha h_0(x) \leq u(x) \leq \beta\lambda h(x) + \alpha H_D\varphi(x).$$

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