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EXAMPLE OF AN ∞-HARMONIC FUNCTION WHICH IS NOT C^2 ON A DENSE SUBSET

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ABSTRACT. We show that for certain boundary values, McShane-Whitney's minimal-extension-like function is ∞-harmonic near the boundary and is not C^2 on a dense subset.

1. RESULTS

Let us consider the strip $\{(u, v) \in \mathbb{R}^2 : 0 < v < \delta\}$, which is going to be the domain for a function constructed in this article. Take a function $f \in C^{1,1}(\mathbb{R})$ and let $L_f := \|f'\|_\infty$ and $L'_f := \mathrm{Lip}(f')$. Let us consider an analogue of the minimal extension of McShane and Whitney,

$$u(x, d) := \sup_{y\in\mathbb{R}}[f(y) - L|(x, d) - (y, 0)|], \tag{1.1}$$

where $0 < d < \delta$ and $L > L_f$. Note that to obtain the classical minimal extension of McShane and Whitney we have to take $L = L_f$.

For the rest of this article we fix the function f and the constants $L > L_f$, $\delta > 0$. We will find conditions on $\delta > 0$, which make our statements true. The real number x will be associated with the point $(x, \delta) \in \Gamma_\delta := \{(u, v) \in \mathbb{R}^2 : v = \delta\}$, and the real number y with the point $(y, 0) \in \Gamma_0$. In the sequel the values of u on the line Γ_δ will be of our interest and we write $u(x)$ for $u(x, \delta)$ (see Figure 1).

Proposition 1.1. *The function u defined above satisfies*

$$u(x) = \sup_{y\in\mathbb{R}}[f(y) - L\sqrt{\delta^2 + (x - y)^2}] = \max_{|y-x|\le D\delta}[f(y) - L\sqrt{\delta^2 + (x - y)^2}], \tag{1.2}$$

where $D := \frac{2LL_f}{L^2 - L_f^2}$.

Proof. From the definition of u we have $f(x) - L\delta \le u(x)$ so it is sufficient to show that if $|x - y| > D\delta$ then

$$f(y) - L\sqrt{\delta^2 + (x - y)^2} < f(x) - L\delta.$$

On the other hand, from the bound of f' we have

$$f(y) - L\sqrt{\delta^2 + (x - y)^2} \le f(x) + L_f|x - y| - L\sqrt{\delta^2 + (x - y)^2}.$$

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Thus we note that all values of y for which

$$f(x) + L_f|x-y| - L\sqrt{\delta^2 + (x-y)^2} < f(x) - L\delta$$

can be ignored in taking supremum in the definition of u. We write

$$L_f|x-y| + L\delta < L\sqrt{\delta^2 + (x-y)^2}$$

and arrive at

$$L_f^2|x-y|^2 + 2LL_f\delta|x-y| + L^2\delta^2 < L^2\delta^2 + L^2|x-y|^2\,.$$

Therefore,

$$2LL_f\delta < (L^2 - L_f^2)|x-y| \quad\Longleftrightarrow\quad |x-y| > D\delta.$$

□

Let $y(x)$ be one of the points in $\{|y-x| \le D\delta\}$, where the maximum in (1.2) is achieved,

$$u(x) = f(y(x)) - L\sqrt{\delta^2 + (x-y(x))^2}. \tag{1.3}$$

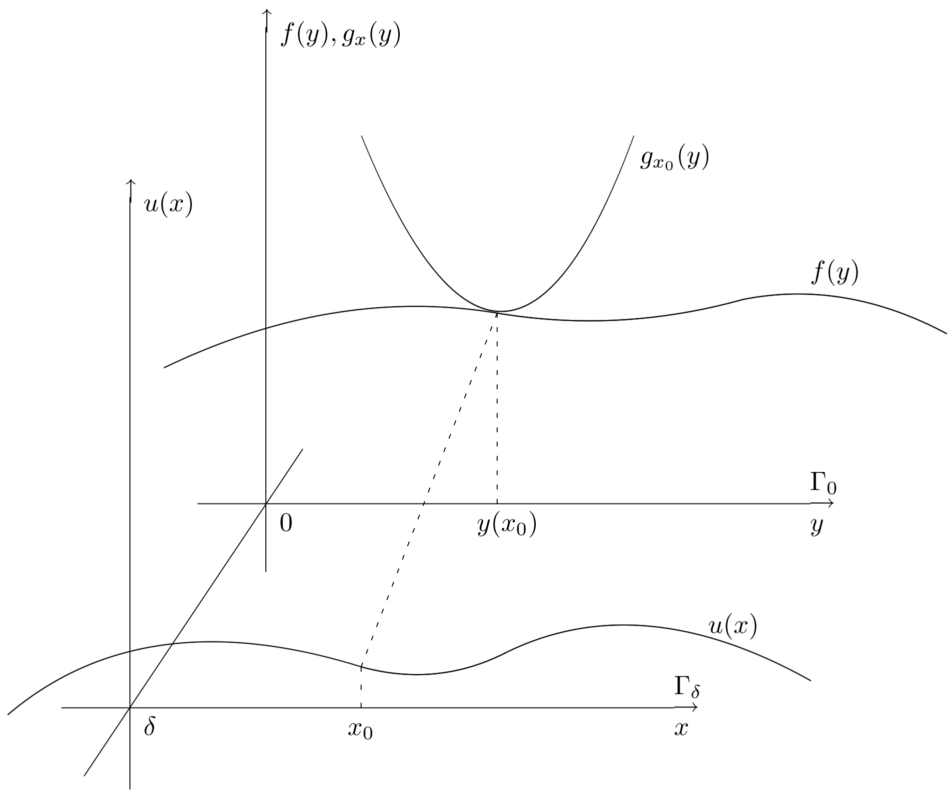

FIGURE 1. Touched by hyperbola

Lemma 1.2. *If $\delta > 0$ is small enough then for every $x \in \Gamma_\delta$ the point $y(x)$ is unique and $y(x) : \mathbb{R} \to \mathbb{R}$ is a bijective Lipschitz map.*

Proof. For each $x \in \Gamma_\delta$ consider the function $g_x(y) := u(x) + L\sqrt{\delta^2 + (x-y)^2}$ defined on Γ_0 (see Figure 1). The graph of g_x is a hyperbola and the graph of any other function $g_{x'}$ can be obtained by a translation. Obviously $f(y) \le g_x(y)$ on Γ_0 and $g_x(y(x)) = f(y(x))$. If at every point $y \in \Gamma_0$ the graph of f can be touched from above by some hyperbola $g_x(y)$ then we will get the surjectivity of $y(x)$. To obtain this result, the following will be sufficient

$$g_x''(y) > L_f', \quad \text{for all } |y-x| \le D\delta. \tag{1.4}$$

For a fixed $y_0 \in \Gamma_0$, we can find a hyperbola $h_{x_0}(y) = C + L\sqrt{\delta^2 + (x_0-y)^2}$ such that $h_{x_0}(y_0) = f(y_0)$ and $h_{x_0}'(y_0) = f'(y_0)$; then obviously $f(y) \le h_{x_0}(y)$ for $|y-x_0| \le D\delta$ (see (1.4)) and for $|y-x_0| > D\delta$ (see Proposition 1.1). In other words, $h_{x_0}(y) = g_{x_0}(y)$. So (1.4) gives us

$$\delta < \frac{L}{L_f'(1+D^2)^{3/2}}, \tag{1.5}$$

where D is defined in Proposition 1.1.

Note that also uniqueness of $y(x)$ follows from (1.4); assume we have $y(x)$ and $\tilde{y}(x)$, then

$$L_f'|y(x) - \tilde{y}(x)| < \Big|\int_{y(x)}^{\tilde{y}(x)} g_x''(t)dt\Big| = |f'(y(x)) - f'(\tilde{y}(x))| \le L_f'|y(x) - \tilde{y}(x)|.$$

We have used here that

$$f'(y(x)) = g_x'(y(x)) = \frac{L(y(x)-x)}{\sqrt{\delta^2 + (y(x)-x)^2}} \tag{1.6}$$

(derivatives in y at the point $y(x)$).

The injectivity of the map $y(x)$ follows from differentiability of f. Assume $y_0 = y(x) = y(\tilde{x})$, so we have $f(y_0) = g_x(y_0) = g_{\tilde{x}}(y_0)$. On the other hand, $f(y) \le \min(g_x(y), g_{\tilde{x}}(y))$; this contradicts differentiability of f at y_0.

The monotonicity of $y(x)$ can be obtained using the same arguments; if $x < \tilde{x}$ then the 'left' hyperbola $g_x(y)$ touches the graph of f 'lefter' than the 'right' hyperbola $g_{\tilde{x}}(y)$, since both hyperbolas are above the graph of f.

Now we will prove that $y(x)$ is Lipschitz. From (1.6) it follows that

$$y(x) - x = \frac{\delta f'(y(x))}{\sqrt{L^2 - (f'(y(x)))^2}}. \tag{1.7}$$

Taking $Y(x) := y(x) - x$ we can rewrite this as

$$Y(x) = \frac{\delta f'(Y(x)+x)}{\sqrt{L^2 - (f'(Y(x)+x))^2}} = \delta\Phi(f'(Y(x)+x)), \tag{1.8}$$

where $\Phi(t) = t/\sqrt{L^2 - t^2}$. For $\delta < \frac{(L^2-L_f^2)^{\frac{3}{2}}}{L^2 L_f'}$, we can use Banach's fix point theorem and get that this functional equation has unique continuous solution. On the other hand, it is not difficult to check that

$$\Big|\frac{Y(x_2) - Y(x_1)}{x_2 - x_1}\Big| \le \frac{\delta C}{1 - \delta C},$$

where $C = \frac{L^2 L_f'}{(L^2 - L_f^2)^{3/2}}$. □

Corollary 1.3. *If δ is as small as in the previous Lemma, then the function u is ∞-harmonic in the strip between Γ_0 and Γ_δ.*

Proof. This follows from the fact that if we take the strip with boundary values f on Γ_0 and u on Γ_δ then McShane-Whitney's minimal and maximal solutions will coincide, obviously with u. $\square$

Remark 1.4. We can rewrite (1.7) in the form

$$x(y) = y - \frac{\delta f'(y)}{\sqrt{L^2 - (f'(y))^2}}, \tag{1.9}$$

where $x(y)$ is the inverse of $y(x)$. This together with (1.3) gives us

$$u(x(y)) = f(y) - \frac{\delta L^2}{\sqrt{L^2 - (f'(y))^2}}.$$

Using the recent result of O.Savin that u is C^1, we conclude that function $x(y)$ is as regular as f', so we cannot expect to have better regularity than Lipschitz.

Lemma 1.5. *If $\delta > 0$ is as small as above and function f is not twice differentiable at y_0, then the function u is not twice differentiable at $x_0 := x(y_0)$.*

Proof. First note that for all x and y, such that $x = x(y)$ we have

$$u'(x) = f'(y).$$

This can be checked analytically but actually is a trivial geometrical fact; the hyperbola 'slides' in the direction of the growth of f at point y, thus the cone which generates this hyperbola and 'draws' with its peak the graph of u moves in same direction which is the direction of the growth of u at point $x = x(y)$.

Now assume we have two sequences $y_k \to y_0$ and $\tilde{y}_k \to y_0$ such that

$$\frac{f'(y_k) - f'(y_0)}{y_k - y_0} \to \underline{f''}(y_0) \quad \text{and} \quad \frac{f'(\tilde{y}_k) - f'(y_0)}{\tilde{y}_k - y_0} \to \overline{f''}(y_0)$$

and $\underline{f''}(y_0) < \overline{f''}(y_0)$. Let us define appropriate sequences on Γ_δ denoting by $x_k := x(y_k)$ and by $\tilde{x}_k := x(\tilde{y}_k)$ and compute the limits of

$$\frac{u'(x_k) - u'(x_0)}{x_k - x_0} \quad \text{and} \quad \frac{u'(\tilde{x}_k) - u'(x_0)}{\tilde{x}_k - x_0}.$$

We have

$$\frac{u'(x_k) - u'(x_0)}{x_k - x_0} = \frac{f'(y_k) - f'(y_0)}{y_k - y_0} \frac{y_k - y_0}{x_k - x_0}$$

the first multiplier converges to $\underline{f''}(y_0)$, let us compute the limit of the second one. From (1.9) we get that

$$\frac{x_k - x_0}{y_k - y_0} \to 1 - \delta \Phi'(f'(y_0)) \underline{f''}(y_0),$$

where $\Phi(t) = t/\sqrt{L^2 - t^2}$. Thus

$$\frac{u'(x_k) - u'(x_0)}{x_k - x_0} \to \frac{\underline{f''}(y_0)}{1 - \delta \Phi'(f'(y_0)) \underline{f''}(y_0)},$$

and analogously

$$\frac{u'(\tilde{x}_k) - u'(x_0)}{\tilde{x}_k - x_0} \to \frac{\overline{f''}(y_0)}{1 - \delta \Phi'(f'(y_0)) \overline{f''}(y_0)}.$$

To complete the proof we need to use the monotonicity of the function

$$\frac{t}{1-\delta C t}, \quad -L'_f < t < L'_f,$$

where $\frac{1}{L} < C < L^2/(L^2 - L_f^2)^{3/2}$. □

Note that if the function f is not C^2 at a point y then u constructed here is not C^2 on the whole line connecting y and $x(y)$. So choosing f to be not twice differentiable on a dense set we can get a function u which is not C^2 on the collection of corresponding line-segments. A similar example is the distance function from a convex set, whose boundary is C^1 and not C^2 on a dense subset. Then the distance function is ∞-harmonic and is not C^2 on appropriate lines.

2. Motivation

Our example u has the property of having constant $|\nabla u|$ on gradient flow curves (lines in our case). It would be interesting to find a general answer to the question:

> What geometry do the gradient flow curves of an ∞-harmonic function u have, on which $|\nabla u|$ is not constant?

From Aronsson's results we know that u is not C^2 on such a curve. This is our motivation for the investigation of C^2-differentiability of ∞-harmonic functions.

The author has only one item in the list of references. The history and the recent developments of the theory of ∞-harmonic functions, as well as a complete reference list could be found in that paper.

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