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LOCAL STABILITY OF SPIKE STEADY STATES IN A SIMPLIFIED GIERER-MEINHARDT SYSTEM

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Abstract. In this paper we study the stability of the single internal spike solution of a simplified Gierer-Meinhardt' system of equations in one space dimension. The linearization around this spike consists of a selfadjoint differential operator plus a non-local term, which is a non-selfadjoint compact integral operator. We find the asymptotic behaviour of the small eigenvalues and we prove stability of the steady state for the parameter (p, q, r, μ) in a four-dimensional region (the same as for the shadow equation, [8]) and for any finite D if ε is sufficiently small. Moreover, there exists an exponentially large $D(\varepsilon)$ such that the stability is still valid for $D < D(\varepsilon)$. Thus we extend the previous results known only for the case $r = p + 1$ or $r = 2, 1 < p < 5$.

1. Introduction

Based on pioneering ideas of Turing [15] about pattern formation by interaction of diffusing chemical substances, Gierer & Meinhardt proposed and studied in [4] the following system of reaction diffusion equations on a spatial domain Ω

$$
\begin{gathered}
U_t = \varepsilon^2 \Delta U - U + U^p H^{-q} \quad x \in \Omega,\ t > 0, \\
\tau H_t = D\Delta H - \mu H + U^r H^{-s} \quad x \in \Omega,\ t > 0\,, \\
\partial_n U = 0 = \partial_n H \quad x \in \partial\Omega,\ t \geq 0,
\end{gathered}
\tag{1.1}
$$

where U and H represent activator and inhibitor concentrations, ε and D their diffusivities, and where τ and μ are the reaction time rate and the decay rate of the inhibitor; D is assumed to be positive and ε and τ small (positive). Ω is a bounded domain; we shall restrict our analysis to one space dimension and choose $\Omega := [-1, 1]$. The exponents $\{p > 1\,,\ q > 0\,,\ r > 1\}$ satisfy the inequality

$$
\gamma_r := \frac{qr}{p-1} > 1\,. \tag{1.2}
$$

Iron, Ward & Wei [17] analyze by formal asymptotic expansions the stability of approximate $N-$spike solutions for the simplified system, namely, that obtained by taking $\tau = 0$. Rigorous results are obtained in [8] for the case of the so-called

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shadow equation

$$U_t = \varepsilon^2 U_{xx} - U + 2^q U^p \Big(\int_{-1}^{1} U^r dx \Big)^{-q}, \quad U_x(-1,t) = U_x(1,t),$$

derived from the system in the limit $D \to \infty$ and $\tau \to 0$.

In this paper we propose to study rigourously the simplified system when $s = 0$. After rescaling $U \to \varepsilon^{-\nu_1} U$, $H \to \varepsilon^{-\nu_2} H$, $\nu_1 := \frac{q}{1-p+qr}$, $\nu_2 := \frac{p-1}{1-p+qr}$, we get the system

$$\begin{gathered} U_t = \varepsilon^2 \Delta U - U + U^p H^{-q} \quad x \in \Omega,\ t > 0, \\ 0 = D\Delta H - \mu H + \varepsilon^{-1} U^r \quad x \in \Omega,\ t > 0 \\ \partial_n U = 0 = \partial_n H \quad x \in \partial\Omega,\ t \geq 0 \end{gathered} \tag{1.3}$$

Our goal is a rigorous study of stability of the single internal spike solution via the spectrum of the linearized operator. In [14] it is shown by the implicit function theorem, that such a spike solution exists for

$p > 1$, $r > 1$, $q > 0$, $qr \neq (p-1)(s+1)$
and D exponentially large w.r.t. $\varepsilon > 0$,

which is close to the shadow spike corresponding to $D = \infty$. A different approach, based on geometric singular perturbation theory, is applied in [3] for the same problem on the whole line. In [21] a rigorous treatment of the stability of multiple-peaked spike solutions is given, based on the Liapunov-Schmidt reduction method. See also [18], [19], where stability and Hopf bifurcation of the one-spike solution is studied. In this paper we construct a single spike solution (on a bounded interval) by fix-point iteration and we establish stability by a rigorous analysis of the spectrum of the first variation around this spike.

In section 2 we construct a positive (stationary) solution with a single internal spike for $p > 1$, $r > 1$, $q > 0$, $qr \neq (p-1)(s+1)$ and for any fixed D, using contraction around another shadow spike that exists for all $D > 0$. The existence of such a solution is proved in [20] in a larger domain $\sqrt{D} \gg \varepsilon$. In section 3 we study the spectrum of the differential operator L_ε. The eigenvalues are estimated using Rayleigh's quotient. In section 4 we make a detailed study of the influence of the nonlocal term on the eigenvalues as a function of the parameters p, q, r and D using perturbational methods. We construct an asymptotic approximation of the small eigenvalue λ_ε of the perturbed non-selfadjoint operator and show that $Re\lambda_\varepsilon > 0$ for any finite D and for sufficiently small ε. We cover not only the usual known cases $r = p+1$, or $r = 2$, $1 < p < 5$ (see [17]), but also $r = (p+3)/2$, and using perturbational techniques, some wide areas around all these cases (see details in [8]). Moreover, we show that there is a critical value $D(\varepsilon)$, which is exponentially large w.r.t. $\varepsilon \in (0, \varepsilon_0)$, such that $Re\lambda_\varepsilon > 0$ for $D < D(\varepsilon)$ an may be negative above $D(\varepsilon)$, implying that stability may be lost for too large D. The same value for $D(\varepsilon)$ was obtained in [17] by formal asymptotic methods. Finally, we study in section 5 the stability of the spike solution along the lines of [8].

2. A spike solution and linearization around it

2.1. **Existence of a stationary one-spike solution and its asymptotics.** Let $(S(x), H(X))$ be a steady state of equations 1.3, i.e.

$$\begin{gathered}\varepsilon^2 S'' - S + S^p H^{-q} = 0\,, \quad S'(\pm 1) = 0\,,\\ H'' - \delta^2 H + \delta^2\mu^{-1}\varepsilon^{-1} S^r = 0\,, \quad H'(\pm 1) = 0\,,\end{gathered} \tag{2.1}$$

where $\delta^2 := \mu/D$. We shall prove the existence of such a spike solution for any fixed $\delta > 0$ and all ε, $0 < \varepsilon < \varepsilon_0(\delta)$ if $p > 1, r > 1, q > 0, qr \neq p-1$. By definition, the (single) spike solution (or spike) is such a steady state for which $S(x) = O(1)$ as $\varepsilon \to 0$ in a neighbourhood of the origin and $S(x)$ is exponentially small outside.

Let h be the solution of the linear equation

$$h'' - \delta^2 h = -f,\ h'(\pm 1) = 0\,,$$

then

$$h(x) = \int_{-1}^{1} \widetilde{G}_\delta(x,y) f(y) dy,$$

where Green's function $\widetilde{G}_\delta$ is given by

$$\widetilde{G}_\delta(x,y) = \frac{1}{\delta \sinh 2\delta} \cosh \delta(1+x) \cosh \delta(1-y) \quad \text{if } x < y. \tag{2.2}$$

This function is even: $\widetilde{G}_\delta(-x,-y) = \widetilde{G}_\delta(x,y)$. We can solve the second equation of the system (2.1) using Green's function and eliminate H from the first equation by

$$H(x) = \frac{1}{\varepsilon} \int_{-1}^{1} G_\delta(x,y) S^r(y) dy,$$

where

$$G_\delta := \frac{\delta^2 \widetilde{G}_\delta}{\mu}, \quad g_\delta := G_\delta(0,0) = \frac{\delta \cosh^2 \delta}{\mu \sinh 2\delta} = \frac{1}{2\mu} + O(\delta^2).$$

Hence the spike solution S satisfies the equation

$$\varepsilon^2 S''(x) - S(x) + S^p(x) \Big(\frac{1}{\varepsilon} \int_{-1}^{1} G_\delta(x,y) S^r(y) dy\Big)^{-q} = 0, \quad S'(\pm 1) = 0. \tag{2.3}$$

It should be positive; however, it is more convenient first to construct a solution of the equation

$$\varepsilon^2 S''(x) - S(x) + |S(x)|^p \Big(\frac{1}{\varepsilon} \int_{-1}^{1} G_\delta(x,y) |S(y)|^r dy\Big)^{-q} = 0, \quad S'(\pm 1) = 0$$

and prove a posteriori that this solutionis positive and hence coincides with the solution of (2.3). In order to find the limit as $\varepsilon \to 0$ we use the stretched variable $\xi = x/\varepsilon$. Setting $\varphi_{\varepsilon,\delta}(\xi) := S(\varepsilon\xi)$ we find

$$\begin{gathered}\varphi''_{\varepsilon,\delta}(\xi) - \varphi_{\varepsilon,\delta}(\xi) + |\varphi_{\varepsilon,\delta}(\xi)|^p \Big(\int_{-1/\varepsilon}^{1/\varepsilon} G_\delta(\varepsilon\xi, \varepsilon\eta) |\varphi_{\varepsilon,\delta}(\eta)|^r d\eta\Big)^{-q} = 0\,,\\ \varphi'_{\varepsilon,\delta}(\pm 1/\varepsilon) = 0\,.\end{gathered} \tag{2.4}$$

Taking the (formal) limit $\varepsilon \to 0$ we get the equation

$$\varphi''_{0,\delta}(\xi) - \varphi_{0,\delta}(\xi) + \varphi^p_{0,\delta}(\xi) \Big(\int_{-\infty}^{\infty} g_\delta \varphi^r_{0,\delta}(\eta) d\eta\Big)^{-q} = 0\,, \quad \varphi'_{0,\delta}(\pm\infty) = 0\,. \tag{2.5}$$

Thus
$$\varphi_{0,\delta} = w_p \Big(\int_{-\infty}^{\infty} g_\delta w_p^r(\eta) d\eta\Big)^{-\alpha_r}, \quad \alpha_r = \frac{q}{1-p+rq},$$
where w_p satisfies
$$w_p'' - w_p + w_p^p = 0, \quad w_p'((\pm\infty) = 0. \tag{2.6}$$
For all $p > 1$ this equation happens to have the closed form solution, cf. [8],
$$w_p(\xi) := \big(\tfrac{p+1}{2}\big)^{\frac{1}{p-1}} \big(\cosh(\tfrac{p-1}{2}\,\xi)\big)^{-\frac{2}{p-1}}, \tag{2.7}$$
which for large $|\xi|$ has the asymptotic behaviour
$$w_p(\xi) = \alpha\, e^{-|\xi|}\,(1 + O(e^{-(p-1)|\xi|})), \quad \alpha := (2p+2)^{\frac{1}{p-1}}. \tag{2.8}$$
Now we want to solve the equation (2.4) for all $\delta > 0$. To this end we introduce an extra parameter $\nu \le \varepsilon$ in the non-linear part of (2.4) defining
$$Q_\nu[\varphi](\xi) := |\varphi(\xi)|^p \Big(\int_{-1/\varepsilon}^{1/\varepsilon} G_\delta(\nu\xi, \nu\eta)|\varphi(\eta)|^r d\eta\Big)^{-q} \tag{2.9}$$
and rewrite (2.4) in the form
$$\varphi''(\xi) - \varphi(\xi) + Q_\varepsilon[\varphi] = 0, \quad \varphi'(\pm 1/\varepsilon) = 0. \tag{2.10}$$
Setting the parameter ν to zero, we get a simplified equation, that we shall call the corresponding *shadow equation* (and which differs from Takagi's [14] by a multiplicative constant):
$$\widetilde{\varphi}''(\xi) - \widetilde{\varphi}(\xi) + Q_0[\widetilde{\varphi}] = 0,\ \widetilde{\varphi}'(\pm 1/\varepsilon) = 0.$$
The solution is given by
$$\widetilde{\varphi} := \Big(\int_{-1/\varepsilon}^{1/\varepsilon} g_\delta \psi_\varepsilon^r(\eta) d\eta\Big)^{-\alpha_r} \psi_\varepsilon, \tag{2.11}$$
where
$$\psi_\varepsilon'' - \psi_\varepsilon + \psi_\varepsilon^p = 0, \quad \psi_\varepsilon'(\pm 1/\varepsilon) = 0.$$
This equation has a unique solution with a single spike located in the interior of the domain; its properties are well known [8] Section 2.1 and [14]. Thus we have constructed a shadow spike solution for any fixed $\delta > 0$, which coincides with the shadow solution from [8], [14] if $\delta = 0$.

The main idea is to find a solution of the problem (2.10) in a small neighbourhood of our shadow spike solution $\widetilde{\varphi}$. If $\varphi = \widetilde{\varphi} + u$ we get an equation for u:
$$\begin{gathered} u'' - u + \{Q_0[\widetilde{\varphi} + u] - Q_0[\widetilde{\varphi}]\} + \{Q_\varepsilon[\widetilde{\varphi} + u] - Q_0[\widetilde{\varphi} + u]\} = 0, \\ u'(\pm 1/\varepsilon) = 0. \end{gathered} \tag{2.12}$$
Using the Taylor formula we can write
$$\begin{gathered} Q_0[\widetilde{\varphi} + u] - Q_0[\widetilde{\varphi}] = Q_0'[\widetilde{\varphi}]u + f(u), \\ f(u) := \int_0^1 \{\partial_\sigma Q_0[\widetilde{\varphi} + \sigma u] - \partial_\sigma Q_0[\widetilde{\varphi} + \sigma u]_{\sigma=0}\} d\sigma, \end{gathered} \tag{2.13}$$
and
$$g_\varepsilon(u) := Q_\varepsilon[\widetilde{\varphi} + u] - Q_0[\widetilde{\varphi} + u] = \int_0^\varepsilon \partial_\nu Q_\nu[\widetilde{\varphi} + u] d\nu, \tag{2.14}$$

where $Q_0'[\widetilde{\varphi}]$ is the (non-local) linear operator

$$\begin{aligned} Q_0'[\widetilde{\varphi}]u &:= p\widetilde{\varphi}^{p-1}u\Big(\int_{-1/\varepsilon}^{1/\varepsilon} g_\delta \widetilde{\varphi}^r(\eta)d\eta\Big)^{-q} \\ &\quad - rq\widetilde{\varphi}^p\Big(\int_{-1/\varepsilon}^{1/\varepsilon} g_\delta \widetilde{\varphi}^r(\eta)d\eta\Big)^{-q-1}\int_{-1/\varepsilon}^{1/\varepsilon} g_\delta \widetilde{\varphi}^{r-1}(\eta)u(\eta)d\eta. \end{aligned}$$

The linear part of the operator in equation (2.12) is given by $\widetilde{A}$,

$$\begin{aligned} \widetilde{A}u &:= -u'' + u - Q_0'[\widetilde{\varphi}]u \\ &= -u'' + u - p\widetilde{\varphi}^{p-1}\Big(\int_{-1/\varepsilon}^{1/\varepsilon} g_\delta \widetilde{\varphi}^r(\eta)d\eta\Big)^{-q}u \\ &\quad + rq\widetilde{\varphi}^p\Big(\int_{-1/\varepsilon}^{1/\varepsilon} g_\delta \widetilde{\varphi}^r(\eta)d\eta\Big)^{-q-1}\int_{-1/\varepsilon}^{1/\varepsilon} g_\delta \widetilde{\varphi}^{r-1}(\eta)u(\eta)d\eta, \end{aligned}$$

or

$$\widetilde{A}u = -u'' + u - p\psi^{p-1}u + \frac{rq\psi_\varepsilon^p\langle u, \psi_\varepsilon^{r-1}\rangle}{\langle 1, \psi_\varepsilon^r\rangle}. \tag{2.15}$$

It is equal to the operator associated with the shadow equation as in [8, eqs. (1.3), (2.1), (2.20), (2.21)]. In [8] it is shown that $\widetilde{L}u := -u'' + u - p\psi^{p-1}u$ restricted to even functions is invertible in L^2. Now we remark that $\widetilde{A}$ is also invertible if $qr \neq p-1$. Indeed, it is sufficient to show that zero is not an eigenvalue of $\widetilde{A}$. Suppose, on the contrary, that $\widetilde{A}u = 0$. Then, since $\widetilde{L}\psi_\varepsilon = (1-p)\psi_\varepsilon^p$, we see that $u = c\psi_\varepsilon$ and $\widetilde{A}\psi_\varepsilon = (1-p+qr)\psi_\varepsilon^p \neq 0$. Thus $\widetilde{A}^{-1}$, restricted to even functions, is a bounded operator in L^2, uniformly w.r.t. ε (cf. [8]). Here L^2 is the space of quadratically integrable functions on the interval $(-1/\varepsilon, 1/\varepsilon)$. Let H^2 be the associated Sobolev space, equipped with the usual norm $\|u\|_2 := \|u''\| + \|u\|$. Since $\|u\|_2 \asymp \|(A+c)u\|$ for some large constant $c > 0$, we conclude that $\widetilde{A}^{-1}$, restricted to even functions, is a bounded operator from L^2 to H^2, uniformly w.r.t. ε. In this way we reduce the problem (2.12) to the integral equation

$$u = Mu, \quad \text{where} \quad Mu := \widetilde{A}^{-1}[f(u) + g_\varepsilon(u)]\,, \tag{2.16}$$

with f and g as defined in (2.13), (2.14).

We are going to apply the contraction method in the ball

$$\begin{aligned} X_\varepsilon := \{u \in H^2(-1/\varepsilon, 1/\varepsilon) : u \text{ is even, } , u'(\pm 1/\varepsilon) = 0\,, \\ \|u\|_\omega := \|u\|_2 + \max|u(\xi)|/\omega(\xi) \le \varrho\}\,, \end{aligned}$$

where $0 < \varepsilon < \varepsilon_0(\delta)$ and where $\omega(\xi)$ is the weight function

$$\omega(\xi) := \begin{cases} e^{-(p-1)|\xi|} & \text{if } 1 < p < 2\,, \\ (1+|\xi|)e^{-|\xi|} & \text{if } p = 2\,, \\ e^{-|\xi|} & \text{if } p > 2\,. \end{cases} \tag{2.17}$$

Since by [8, eq. (2.11)],

$$|\psi_\varepsilon(\xi) - w_p(\xi)| \le c\,e^{-1/\varepsilon}, \quad |\xi| \le 1/\varepsilon, \tag{2.18}$$

we can find a constant $\xi_\varepsilon = \log(C/\varepsilon)$ (where C is a generic positive constant in the sequel) so that $\psi_\varepsilon > \varepsilon^\kappa$ on $[-\xi_\varepsilon, \xi_\varepsilon]$, where κ satisfies $\max(1/2, 1/r) < \kappa < 1$. Then

$$\widetilde{\varphi}(\xi) > C\,g_\delta^{-\alpha_r}\varepsilon^\kappa, \quad \text{if } |\xi| \le \xi_\varepsilon\,.$$

Therefore, choosing

$$\varrho := C\, g_\delta^{-\alpha_r} \varepsilon^\kappa, \tag{2.19}$$

we get

$$\widetilde{\varphi} + \sigma u > 0 \quad \text{on } [-\xi_\varepsilon, \xi_\varepsilon] \text{ for any } u \in X_\varepsilon \text{ and } 0 < \sigma < 1.$$

Hence

$$\begin{aligned} V &:= \int_{-1/\varepsilon}^{1/\varepsilon} G_\delta(\nu\xi, \nu\eta)|\widetilde{\varphi}(\eta) + \sigma u(\eta)|^r d\eta > \\ &> \int_{-\xi_\varepsilon}^{\xi_\varepsilon} G_\delta(\nu\xi, \nu\eta)(\widetilde{\varphi}(\eta) + \sigma u(\eta))^r d\eta - \int_{|\xi|>\xi_\varepsilon} G_\delta(\nu\xi, \nu\eta)|\widetilde{\varphi}(\eta) + \sigma u(\eta)|^r d\eta. \end{aligned}$$

Since $\widetilde{\varphi} + \sigma u > C g_\delta^{-\alpha_r}$ on $[-1, 1]$, since

$$\frac{g_\delta}{\cosh^2 \delta} = \frac{\delta}{\mu \sinh 2\delta} \le G_\delta(x, y) \le 2 g_\delta, \quad \text{if } -1 \le x, y \le 1\,,$$

and since $\xi_\varepsilon > 1$ for $\varepsilon < \varepsilon_0(\delta)$, we get

$$V > C g_\delta^{1-r\alpha_r} [1 - C_\delta \varepsilon^{\kappa r - 1}] \cosh^{-2} \delta > C g_\delta^{1-r\alpha_r} \cosh^{-2} \delta$$

if $\kappa r - 1 > 0$ and if $0 < \varepsilon < \varepsilon_0(\delta)$. Therefore,

$$V > C_\delta \quad \text{if } 0 < \varepsilon < \varepsilon_0(\delta), \tag{2.20}$$

uniformly w.r.t. ε, where the positive quantities C_δ and $\varepsilon_0(\delta)$ are equivalent to 1 w.r.t. δ on any compact interval $[0, \delta_0]$. All other estimates below will be uniform in the same sense. We shall first prove the estimates for $u \in X_\varepsilon$,

$$\|f(u)\| \le C_\delta \varrho^\gamma \|u\|_2^2 \tag{2.21}$$

and

$$\|g_\varepsilon(u)\| \le C_\delta \varepsilon, \tag{2.22}$$

where here and below γ is a generic positive number that depends on p and r.

To prove (2.21) we use the definition of $f(u)$ in (2.13) and write it as a sum of five terms $f(u) = \sum_{j=1}^5 f_j(u)$, where

$$\begin{aligned} f_1(u) &:= pu \int_0^1 \langle |\widetilde{\varphi} + \sigma u|^r, g_\delta\rangle^{-q} \left[|\widetilde{\varphi} + \sigma u|^{p-1} \operatorname{sign}(\widetilde{\varphi} + \sigma u) - \widetilde{\varphi}^{p-1}\right] d\sigma\,, \\ f_2(u) &:= p\widetilde{\varphi}^{p-1} u \int_0^1 \left[\langle |\widetilde{\varphi} + \sigma u|^r, g_\delta\rangle^{-q} - \langle \widetilde{\varphi}^r, g_\delta\rangle^{-q}\right] d\sigma\,, \\ f_3(u) &:= -\, qr \int_0^1 \langle |\widetilde{\varphi} + \sigma u|^r, g_\delta\rangle^{-q-1} |\widetilde{\varphi} + \sigma u|^p \\ &\qquad\qquad \times \langle |\widetilde{\varphi} + \sigma u|^{r-1} \operatorname{sign}(\widetilde{\varphi} + \sigma u) - \widetilde{\varphi}^{r-1}, g_\delta u\rangle\, d\sigma\,, \\ f_4(u) &:= -\, qr \int_0^1 \langle |\widetilde{\varphi} + \sigma u|^r, g_\delta\rangle^{-q-1} [|\widetilde{\varphi} + \sigma u|^p - \widetilde{\varphi}^p] \langle \widetilde{\varphi}^{r-1}, g_\delta u\rangle\, d\sigma\,, \\ f_5(u) &:= -\, qr \int_0^1 \left[\langle |\widetilde{\varphi} + \sigma u|^r, g_\delta\rangle^{-q-1} - \langle \widetilde{\varphi}^r, g_\delta\rangle^{-q-1}\right] \langle \widetilde{\varphi}^{r-1}, g_\delta u\rangle\, \widetilde{\varphi}^p\, d\sigma\,. \end{aligned}$$

Denote the second factor in the integrand of f_1 by

$$f_0(u) := |\widetilde{\varphi} + \sigma u|^{p-1} \operatorname{sign}(\widetilde{\varphi} + \sigma u) - \widetilde{\varphi}^{p-1}.$$

For all $\varepsilon > 0$, $0 \le \sigma \le 1$ and all functions u it satisfies

$$|f_0(u)| \le \begin{cases} 2\min\{\widetilde{\varphi}^{p-2}|u|, |u|^{p-1}\} & \text{if } 1 < p \le 2 \\ 2^{p-1}\max\{\widetilde{\varphi}^{p-2}|u|, |u|^{p-1}\} & \text{if } p > 2 \end{cases} \tag{2.23}$$

This is a consequence of the following inequalities:
If $a > 0$ and $0 < b < 1$ then

(1) $0 \le (a+x)^b - a^b \le bxa^{b-1}$ and $0 \le (a+x)^b - a^b \le x^b$ for all $x \ge 0$
(2) $0 \le a^b - (a-y)^b \le ya^{b-1}$ and $0 \le a^b - (a-y)^b \le y^b$ for all $0 \le y \le a$
(3) $0 \le a^b + t^b \le 2(a+t)^b$ and $0 \le a^b + t^b \le 2a^{b-1}(a+t)$ for all $t \ge 0$.

Note that the inequalities (1) and (2) above are sharp.
If $a > 0$ and $b > 1$ then

(1) $0 \le (a+x)^b - a^b \le \begin{cases} 2^b x a^{b-1} & \text{if } 0 \le x \le a\,, \\ 2^b x^b & \text{if } x \ge a\,, \end{cases}$
(2) $0 \le a^b - (a-y)^b \le bya^{b-1}$ if $0 \le y \le a$
(3) $0 \le a^b + t^b \le (a+t)^b$ if $t \ge 0$

Substituting $y = -x$ and $t = -a-x$, $b = p-1$, $a = \widetilde{\varphi}$ and $x = \sigma u$ this proves (2.23). Restricting this inequality to functions $u \in X_\varepsilon$, which are uniformly bounded by $\varrho\omega$ $\big($with $\varrho < 1$, cf. (2.19)$\big)$, we find the estimate

$$|f_0(u)| \le C|u|^{\sigma_p}, \quad \sigma_p := \min(1, p-1),\ u \in X_\varepsilon\,, \tag{2.24}$$

for some $C > 0$ not depending on δ or ε, cf. (2.20). Essentially, the restriction $|u| \le \varrho\omega$ in this inequality is necessary only if $p > 2$.

Using (2.24), (2.19) and the definitions of $f_j(u)$ we find the following uniform estimates if $|u| \le \varrho\omega$:

$$|f_1(u)| \le C\,|u|^{1+\sigma_p} \le C_\delta \rho^{1+\sigma_p}\omega^{1+\sigma_p} \le C\,\rho^{1+\sigma_p}\widetilde{\omega}, \tag{2.25}$$

where $\widetilde{\omega}(\xi) := \omega(\xi)$ if $1 < p < 2$ and $\widetilde{\omega}(\xi) := e^{-|\xi|}$ if $p \ge 2$,

$$|f_2(u)| \le C\,\widetilde{\phi}^{p-1}|u|\rho \le C\,\rho^2\omega\widetilde{\phi}^{p-1} \le C\,\rho^2\widetilde{\omega}, \tag{2.26}$$

$$|f_3(u)| \le C\,(\widetilde{\phi}^p + |u|^p)\rho^{1+\sigma_r} \le C\,\rho^{1+\sigma_r}\omega(\widetilde{\phi}^p + \omega^p) \le C\,\rho^{1+\sigma_r}\widetilde{\omega}, \tag{2.27}$$

$$|f_4(u)| \le C\,\rho|u|(\widetilde{\phi}^{p-1} + |u|^{p-1}) \le C\,\rho^2\omega(\widetilde{\phi}^{p-1} + \omega^{p-1}) \le C\,\rho^2\widetilde{\omega}, \tag{2.28}$$

$$|f_5(u)| \le C\,\rho^2\widetilde{\phi}^p \le C\,\rho^2\widetilde{\omega}, \tag{2.29}$$

These estimates are uniform w.r.t. $\xi \in [-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}]$, $\delta \in [0, \delta_0]$ and $0 < \varepsilon \le \varepsilon_0(\delta)$. This proves (2.21) for $u \in X_\varepsilon$.

To prove (2.22), we use the estimates

$$|\partial_x G_\delta| \le \delta^2\mu^{-1}\cosh 2\delta, \quad |\partial_y G_\delta| \le \delta^2\mu^{-1}\cosh 2\delta, \tag{2.30}$$

whence we get the uniform estimate

$$\begin{aligned} |g_\varepsilon(u)| &\le C\varepsilon(1+|\xi|)(\widetilde{\varphi}^p + |u|^p)\Big(1 + \int_{-1/\varepsilon}^{1/\varepsilon} |\eta|\,|u|^r d\eta\Big) \\ &\le C\varepsilon(1+|\xi|)(\widetilde{\varphi}^p + \omega^p) \le C\varrho^{1+\gamma}\widetilde{\omega}. \end{aligned} \tag{2.31}$$

Thus (2.22) follows. The estimates (2.21), (2.22) imply the uniform estimate for $u \in X_\varepsilon$

$$\|Mu\|_2 \le C_\delta \varrho^{1+\gamma}, \quad 0 < \varepsilon < \varepsilon_0(\delta). \tag{2.32}$$

Now we prove that for $u \in X_\varepsilon$ the following uniform (pointwise) estimate holds

$$|Mu| \le C_\delta \varrho^{1+\gamma}\omega, \quad 0 < \varepsilon < \varepsilon_0(\delta). \tag{2.33}$$

To this end we write the equation (2.16) in the form

$$(-\partial^2 + 1)Mu = F(u), \quad F(u) := p\psi_\varepsilon^{p-1}Mu + B_\varepsilon(Mu) + f(u) + g_\varepsilon(u). \tag{2.34}$$

We claim that if $u \in X_\varepsilon$ then

$$|F(u)| \le C_\delta \widetilde{\omega}\varrho^{1+\gamma}. \tag{2.35}$$

Indeed, by (2.32) we know that $|Mu| \le C_\delta \varrho^{1+\gamma}$, hence

$$\psi_\varepsilon^{p-1}|Mu| \le C_\delta \varrho^{1+\gamma} e^{-(p-1)|\xi|} \le C_\delta \varrho^{1+\gamma}\widetilde{\omega}.$$

From estimates (2.25)–(2.28), (2.29), (2.31) and $|u| \le \varrho\omega$ we get

$$|B_\varepsilon(Mu)| \le C_\delta \varrho^{1+\gamma}\widetilde{\varphi}^p \le C_\delta \varrho^{1+\gamma}\widetilde{\omega}.$$

Hence (2.35) follows. Further, from (2.34) we have that

$$Mu(\xi) = \varepsilon^{-1}\int_{-1/\varepsilon}^{1/\varepsilon} \widetilde{G}_{1/\varepsilon}(\varepsilon, \xi, \varepsilon\eta)F(u)(\eta)d\eta.$$

Using the explicit formula (2.2) and the estimate (2.35) we can calculate the above integral and derive (2.33). Finally, from (2.32) and (2.33) it follows that M maps X_ε into X_ε uniformly for $0 < \varepsilon < \varepsilon_0(\delta)$.

To prove that M is a contraction we estimate the difference

$$f(u_1) - f(u_2) = \int_0^1 \{Q_0'[\widetilde{\varphi} + u_\sigma] - Q_0'[\widetilde{\varphi}]\}\, d\sigma(u_1 - u_2),$$

where $u_\sigma := u_2 + \sigma(u_1 - u_2)$. As before we write this difference as a sum of five terms, $f(u_1) - f(u_2) = \sum_{j=1}^5 g_j(u_1, u_2)$,

$$g_1(u_1,u_2) := p(u_1-u_2)\int_0^1 \langle\, |\widetilde{\phi}+u_\sigma|^r, g_\delta\,\rangle^{-q}\left\{|\widetilde{\phi}+u_\sigma|^{p-1}sign(\widetilde{\phi}+u_\sigma) - \widetilde{\phi}^{p-1}\right\}d\sigma,$$

$$g_2(u_1,u_2) := p\widetilde{\phi}^{p-1}(u_1-u_2)\int_0^1 \left[\langle\, |\widetilde{\phi}+u_\sigma|^r, g_\delta\,\rangle^{-q} - \langle\, \widetilde{\phi}^r, g_\delta\,\rangle^{-q}\right]d\sigma,$$

$$\begin{aligned} g_3(u_1,u_2) := -\, qr\int_0^1 &\langle\, |\widetilde{\phi}+u_\sigma|^r, g_\delta\,\rangle^{-q-1}|\widetilde{\phi}+u_\sigma|^p \times \\ &\langle\, |\widetilde{\phi}+u_\sigma|^{r-1}sign(\widetilde{\phi}+u_\sigma) - \widetilde{\phi}^{r-1}, g_\delta(u_1-u_2)\,\rangle d\sigma, \end{aligned}$$

$$g_4(u_1,u_2) := -qr\int_0^1 \langle\, |\widetilde{\phi}+\sigma u|^r, g_\delta\,\rangle^{-q-1}\big[|\widetilde{\phi}+u_\sigma|^p - \widetilde{\phi}^p\big]\langle\, \widetilde{\phi}^{r-1}, g_\delta(u_1-u_2)\,\rangle d\sigma,$$

$$g_5(u_1,u_2) := -qr\int_0^1 \big[\langle\, |\widetilde{\phi}+u_\sigma|^r, g_\delta\,\rangle^{-q-1} - \langle\, \widetilde{\phi}^r, g_\delta\,\rangle^{-q-1}\big]\langle\, \widetilde{\phi}^{r-1}, g_\delta(u_1-u_2)\,\rangle\widetilde{\phi}^p d\sigma.$$

Using (2.24), (2.20) we find the following uniform (pointwise) estimates if both u_1 and u_2 are in X_ε and satisfy $|u_j| \le \varrho\omega$ (pointwise):

$$|g_1(u_1,u_2)| \le C_\delta |u|^{\sigma_p}|u_1-u_2| \le C_\delta \rho^{\sigma_p}\omega^{\sigma_p}|u_1-u_2| \le C_\delta \rho^{\sigma_p}|u_1-u_2|\,\widetilde{\omega}/\omega\,, \tag{2.36}$$

$$
\begin{aligned}
|g_2(u_1,u_2)| &\le C_\delta \widetilde{\phi}^{p-1}\big(\rho + \int_{-1/\varepsilon}^{1/\varepsilon} |u|^r d\xi\big)|u_1-u_2| \\
&\le C\rho|u_1-u_2|\widetilde{\phi}^{p-1} \le C_\delta \rho |u_1-u_2|\,\widetilde{\omega}/\omega, \qquad (2.37)\\
|g_3(u_1,u_2)| &\le C_\delta(\widetilde{\phi}^p+|u|^p)\int_{-1/\varepsilon}^{1/\varepsilon}|u|^{\sigma_r}|u_1-u_2|d\xi \le \\
&\le C_\delta \rho^{\sigma_r}\omega(\widetilde{\phi}^p+\omega^p)\|u_1-u_2\| \le C_\delta\rho^{\sigma_r}\|u_1-u_2\|\,\widetilde{\omega}, \qquad (2.38)\\
|g_4(u_1,u_2)| &\le C_\delta |u|(\widetilde{\phi}^{p-1}+|u|^{p-1})|\langle\, \widetilde{\phi}^{r-1}, u_1-u_2\,\rangle| \\
&\le C_\delta\rho\omega(\widetilde{\phi}^{p-1}+\omega^{p-1})\|u_1-u_2\| \le C_\delta\rho\|u_1-u_2\|\,\widetilde{\omega}, \qquad (2.39)\\
|g_5(u_1,u_2)| &\le C_\delta\rho\widetilde{\phi}^p|\langle\, \widetilde{\phi}^{r-1}, u_1-u_2\,\rangle| \le C_\delta\rho\|u_1-u_2\|\,\widetilde{\omega}. \qquad (2.40)
\end{aligned}
$$

Hence if $u \in X_\varepsilon$ we find the uniform estimate

$$
\|f(u_1)-f(u_2)\| \le C_\delta \varrho \|u_1-u_2\|, \quad 0<\varepsilon<\varepsilon_0(\delta). \tag{2.41}
$$

Further we have

$$
g_\varepsilon(u_1)-g_\varepsilon(u_2) = \int_0^\varepsilon \int_0^1 \partial_\sigma \partial_\nu Q_\nu[\widetilde{\varphi}+u_2+\sigma(u_1-u_2)]d\sigma d\nu\,;
$$

defining

$$
Q_\nu[\varphi] := |\varphi|^p q_\nu(\varphi)\,, \quad q_\nu(\varphi) := \Big(\int_{-1/\varepsilon}^{1/\varepsilon} G_\delta(\nu\xi,\nu\eta)|\varphi(\eta)|^r d\eta\Big)^{-q},
$$

we find

$$
\begin{gathered}
g_\varepsilon(u_1)-g_\varepsilon(u_2) = h_1(u_1,u_2)+h_2(u_1,u_2),\\
h_1(u_1,u_2) = |\widetilde{\varphi}+u_1|^p \int_0^\varepsilon [\partial_\nu q_\nu(\widetilde{\varphi}+u_1)-\partial_\nu q_\nu(\widetilde{\varphi}+u_2)]d\nu,\\
h_2(u_1,u_2) = [|\widetilde{\varphi}+u_1|^p - |\widetilde{\varphi}+u_2|^p]\int_0^\varepsilon \partial_\nu q_\nu(\widetilde{\varphi}+u_2)d\nu.
\end{gathered}
$$

We have

$$
\partial_\nu q_\nu(\varphi) = -q\Big(\int_{-1/\varepsilon}^{1/\varepsilon} G_\delta(\nu\xi,\nu\eta)|\varphi(\eta)|^r d\eta\Big)^{-q-1}\int_{-1/\varepsilon}^{1/\varepsilon} G_\delta(\nu\xi,\nu\eta)|\varphi(\eta)|^r d\eta.
$$

Hence,

$$
\begin{aligned}
\partial_\nu q_\nu(\widetilde{\varphi}+u_1)-\partial_\nu q_\nu(\widetilde{\varphi}+u_2) =& -q\Big(\int_{-1/\varepsilon}^{1/\varepsilon} G_\delta(\nu\xi,\nu\eta)|\widetilde{\varphi}(\eta)+u_1(\eta)|^r d\eta\Big)^{-q-1}\\
&\times \int_{-1/\varepsilon}^{1/\varepsilon} \partial_\nu G_\delta(\nu\xi,\nu\eta)\big[|\widetilde{\varphi}(\eta)+u_1(\eta)|^r - |\widetilde{\varphi}(\eta)+u_2(\eta)|^r\big]d\eta\\
-q\int_{-1/\varepsilon}^{1/\varepsilon}\partial_\nu G_\delta(\nu\xi,\nu\eta)|\widetilde{\varphi}(\eta)+u_2(\eta)|^r d\eta&\Big[\Big(\int_{-1/\varepsilon}^{1/\varepsilon} G_\delta(\nu\xi,\nu\eta)|\widetilde{\varphi}(\eta)+u_1(\eta)|^r d\eta\Big)^{-q-1}\\
&\times\Big(\int_{-1/\varepsilon}^{1/\varepsilon} G_\delta(\nu\xi,\nu\eta)|\widetilde{\varphi}(\eta)+u_2(\eta)|^r d\eta\Big)^{-q-1}\Big]
\end{aligned}
$$

If $u_j \in X_\varepsilon$ then using (2.30), (2.24) we get

$$\begin{aligned}
&|\partial_\nu q_\nu(\widetilde{\varphi}+u_1) - \partial_\nu q_\nu(\widetilde{\varphi}+u_2)| \le \\
&\le C_\delta \int_{-1/\varepsilon}^{1/\varepsilon} (|\xi|+|\eta|)|u_1(\eta)-u_2(\eta)\big(\widetilde{\varphi}(\eta)^{r-1}+|u_1(\eta)|^{r-1}+|u_2(\eta)|^{r-1}\big)d\eta \\
&\quad + C_\delta \int_{-1/\varepsilon}^{1/\varepsilon} (|\xi|+|\eta|)|u_1(\eta)-u_2(\eta)\big(\widetilde{\varphi}(\eta)^{r}+|u_1(\eta)|^{r}+|u_2(\eta)|^{r}\big)d\eta\,.
\end{aligned}$$

Hence

$$\begin{aligned}
|h_1(u_1,u_2)| &\le C_\delta \varepsilon |\widetilde{\varphi}+u_1|^p(1+|\xi|)\int_{-1/\varepsilon}^{1/\varepsilon} |u_1(\eta)-u_2(\eta)|(1+|\eta|) \\
&\qquad \times \Big(\widetilde{\varphi}(\eta)^{r-1}+|u_1(\eta)|^{r-1}+|u_2(\eta)|^{r-1}\Big)d\eta \\
&\le C_\delta \varrho^{1+\gamma}(1+|\xi|)(\widetilde{\varphi}^p+\omega^p)\|u_1-u_2\| \le C_\delta \varrho^\gamma \|u_1-u_2\|\widetilde{\omega}\,.
\end{aligned}$$

Analogously,

$$\begin{aligned}
|h_2(u_1,u_2)| &\le C_\delta \varepsilon |u_1-u_2|(1+|\xi|)\big(\widetilde{\varphi}^{p-1}+|u_1|^{p-1}+|u_2|^{p-1}\big) \\
&\qquad \times \int_{-1/\varepsilon}^{1/\varepsilon} (1+|\eta|)(\widetilde{\varphi}(\eta)^r+|u_2(\eta)|^r)d\eta \\
&\le C_\delta \varrho^{1+\gamma}(1+|\xi|)(\widetilde{\varphi}^{p-1}+\omega^{p-1})|u_1-u_2| \le C_\delta \varrho^\gamma |u_1-u_2|\widetilde{\omega}/\omega\,.
\end{aligned}$$

Thus for $u_j \in X_\varepsilon$ we have the uniform (pointwise) estimate

$$|g_\varepsilon(u_1)-g_\varepsilon(u_2)| \le C_\delta \varrho^\gamma \|u_1-u_2\|\widetilde{\omega} + \le C_\delta \varrho^\gamma |u_1-u_2|\widetilde{\omega}/\omega\,. \tag{2.42}$$

In particular,

$$\|g_\varepsilon(u_1)-g_\varepsilon(u_2)\| \le C_\delta \varrho^\gamma \|u_1-u_2\|, \quad \text{if } 0<\varepsilon<\varepsilon_0(\delta). \tag{2.43}$$

Hence (2.41) and (2.43) imply the uniform estimate

$$\|M(u_1)-M(u_2)\|_2 \le C_\delta \varrho^\gamma \|u_1-u_2\|, \quad \text{if } 0<\varepsilon<\varepsilon_0(\delta). \tag{2.44}$$

Next we prove that for $u_j \in X_\varepsilon$ the uniform (pointwise) estimate holds

$$|Mu_1-Mu_2| \le C_\delta \varrho^\gamma \omega \|u_1-u_2\|_\omega, \quad \text{if } 0<\varepsilon<\varepsilon_0(\delta). \tag{2.45}$$

To this end we use the equation (2.34) and write

$$(-\partial^2+1)(Mu_1-Mu_2) = F(u_1,u_2),$$

where

$$F(u_1,u_2) := p\psi_\varepsilon^{p-1}(M_1-Mu_2)+B_\varepsilon(Mu_1-Mu_2)+f(u_1)-f(u_2)+g_\varepsilon(u_1)-g_\varepsilon(u_2).$$

We claim that

$$|F(u_1,u_2)| \le C_\delta \varrho^\gamma \widetilde{\omega}\|u_1-u_2\|_\omega. \tag{2.46}$$

Indeed, if $|u_j| \le \varrho\omega$ then

$$\psi_\varepsilon^{p-1}|Mu_1-Mu_2| \le C_\delta \varrho^\gamma \|u_1-u_2\|\widetilde{\omega}$$

and

$$|f(u_1)-f(u_2)| \le \sum_{j=1}^{5}|g_j(u_1,u_2)|,$$

$$|B_\varepsilon(Mu_1-Mu_2)| \le C_\delta \varrho^\gamma \|u_1-u_2\|\widetilde{\varphi}^p \le C_\delta \varrho^\gamma \|u_1-u_2\|\widetilde{\omega}.$$

Thus estimate (2.46) follows from (2.36)–(2.39), (2.40), (2.42). Then using again the formula

$$Mu_1(\xi) - Mu_2(\xi) = \varepsilon^{-1} \int_{-1/\varepsilon}^{1/\varepsilon} \widetilde{G}_{1/\varepsilon}(\varepsilon, \xi, \varepsilon\eta) F(u_1, u_2)(\eta) d\eta$$

we get (2.45) Finally, from (2.44), (2.45) we find the uniform estimate for $u_j \in X_\varepsilon$:

$$\|Mu_1 - Mu_2\|_\omega \le \frac{1}{2} \|u_1 - u_2\|_\omega, \quad 0 < \varepsilon < \varepsilon_0(\delta).$$

Thus the problem (2.4) has unique solution in X_ε. Moreover, this solution is positive because it also solves the integral equation

$$\varphi = \frac{1}{\varepsilon} \int_{-1/\varepsilon}^{1/\varepsilon} \widetilde{G}_{1/\varepsilon}(\varepsilon\xi, \varepsilon\eta) Q_\varepsilon[\varphi] d\eta. \tag{2.47}$$

Below we need the asymptotic behaviour of the spike solution at the boundary:

$$\varphi(1/\varepsilon) = 2\alpha \Big(\int_{-\infty}^{\infty} g_\delta w_p^r(\eta) d\eta \Big)^{-\alpha_r} e^{-1/\varepsilon} (1 + O(\varepsilon^\gamma)), \quad 0 < \varepsilon < \varepsilon_0(\delta). \tag{2.48}$$

To prove this, we notice first that the shadow spike $\widetilde{\varphi}$ has the same asymptotic behaviour. This follows from (2.11), (2.18) and since (see [8])

$$\psi_\varepsilon(1/\varepsilon) = 2\alpha e^{-1/\varepsilon}(1 + O(e^{-(p-1)/\varepsilon})), \quad p > 1, \ \alpha := (2p+2)^{\frac{1}{p-1}}.$$

Hence it is sufficient to prove the estimate

$$|\varphi(1/\varepsilon) - \widetilde{\varphi}(1/\varepsilon)| \le C_\delta \varepsilon^\gamma e^{-1/\varepsilon}, \quad 0 < \varepsilon < \varepsilon_0(\delta). \tag{2.49}$$

We can estimate this difference using the integral equation (2.47), where $Q_\nu[\varphi]$ is defined by (2.9). We have (using Taylor's formula, (2.20), (2.30)),

$$Q_\nu[\varphi](\eta) = Q_0[\varphi](\eta) + (1 + |\eta|) \varphi^p(\eta) \, O(\nu), \quad |\eta\| \le 1/\varepsilon, \tag{2.50}$$

where

$$Q_0[\varphi](\eta) = \varphi^p(\eta) \Big(\int_{-1/\varepsilon}^{1/\varepsilon} g_\delta \varphi^r(\xi) d\xi \Big)^{-q}.$$

Using also the estimate $|\varphi(\xi) - \widetilde{\varphi}(\xi)| \le \varrho\omega(\xi)$, $0 < \varepsilon < \varepsilon_0(\delta)$, we find

$$Q_0[\varphi](\eta) = Q_0[\widetilde{\varphi}](\eta) + |\varphi^p(\eta) - \widetilde{\varphi}^p(\eta)| \, O(1) + \varphi^p(\eta) \varrho \,.$$

Hence

$$\varphi(\xi) = \widetilde{\varphi}(\xi) + \frac{1}{\varepsilon} \int_{-1/\varepsilon}^{1/\varepsilon} \widetilde{G}_{1/\varepsilon}(\varepsilon\xi, \varepsilon\eta) \Big[|\varphi^p(\eta) - \widetilde{\varphi}^p(\eta)| \, O(1) + (1 + |\eta|) \varphi^p(\eta) \, O(\varrho) \Big] d\eta \,. \tag{2.51}$$

To estimate this integral, we need a better estimate of φ, namely

$$\varphi(\xi) = O_\delta(e^{-|\xi|}), \quad |\xi| \le 1/\varepsilon, \ 0 < \varepsilon < \varepsilon_0(\delta). \tag{2.52}$$

Indeed φ satisfies the equation

$$\varphi'' = q\varphi, \ q = 1 - \varphi^{p-1} \Big(\int_{-1/\varepsilon}^{1/\varepsilon} G_\delta(\varepsilon\xi, \varepsilon\eta) \varphi^r(\eta) d\eta \Big)^{-q}.$$

Since $\varphi \le \widetilde{\varphi} + \varrho\omega$, it follows that

$$q(\xi) = 1 - O(e^{-\gamma|\xi|}),$$

hence applying the classical asymptotic theory we get (2.52).

On the other hand, to estimate the difference $\varphi^p - \widetilde{\varphi}^p$, we use the estimates $|\varphi - \widetilde{\varphi}| \le \varrho\omega$ and $|\varphi|,\ |\widetilde{\varphi}| \le C_\delta e^{-|\eta|}$. Thus

$$|\varphi^p(\eta) - \widetilde{\varphi}^p(\eta)| \le C_\delta |\varphi(\eta) - \widetilde{\varphi}(\eta)| e^{-(p-1)\eta} \le C_\delta |\varphi(\eta) - \widetilde{\varphi}(\eta)|^b e^{-(p-b)\eta}$$

if $0 < b \le 1$. We choose b as follows:

$0 < b < \frac{p-1}{2-p}$ if $1 < p \le 3/2$ and $b = 1$ if $p > 3/2$.

Then

$$|\varphi^p(\eta) - \widetilde{\varphi}^p(\eta)| \le C_\delta \varrho^b \omega^b e^{-(p-b)\eta}. \tag{2.53}$$

Returning now to (2.51) we estimate the integral by (2.53):

$$\begin{aligned} &\frac{1}{\varepsilon}\int_{-1/\varepsilon}^{1/\varepsilon} \widetilde{G}_{1/\varepsilon}(1, \varepsilon\eta) |\varphi^p(\eta) - \widetilde{\varphi}^p(\eta)| d\eta \le \\ &\le \frac{C_\delta \varrho^b}{\sinh 2/\varepsilon} \int_{-1/\varepsilon}^{1/\varepsilon} \cosh(1/\varepsilon + \eta) e^{-|\eta|(p-b+b(p-1))} d\eta \le C_\delta \varrho^b e^{-1/\varepsilon} \end{aligned}$$

if $1 < p < 2$. Analogous estimate is valid for $p \ge 2$. Thus (2.51) becomes

$$\varphi(1/\varepsilon) = \widetilde{\varphi}(1/\varepsilon) + O(\varrho^b e^{-1/\varepsilon}), \quad 0 < \varepsilon < \varepsilon_0(\delta),$$

whence estimate (2.49) follows.

2.2. **Linearization around the one-spike solution.** In order to study stability of the spike solution, we consider the first variation of the system (1.3) around this solution. It is convenient to rewrite this system as one equation, solving first the second equation

$$\begin{gathered} h(x,t) = \frac{1}{\varepsilon}\int_{-1}^{1} G_\delta(x,y) u^r(y,t) dy, \\ u_t = \varepsilon^2 u_{xx} - u + g(u), \\ u_x(\pm 1, t) = 0, \quad u(x,0) = u_0(x), \quad u_0'(\pm 1) = 0\,, \end{gathered} \tag{2.54}$$

where

$$g(u) = u^p \Big(\frac{1}{\varepsilon}\int_{-1}^{1} G_\delta(x,y) u^r(y,t) dy\Big)^{-q}.$$

Let v be the variation around S; set $u(x,t) = S(x,\varepsilon) + v(x,t)$, then v satisfies the equations

$$\begin{gathered} v_t = \varepsilon^2 v_{xx} - v + g(S+v) - g(S), \\ v_x(\pm 1, t) = 0, \quad v(x,0) = v_0(x) := u_0(x) - S(x,\varepsilon), \end{gathered}$$

or written in operator form

$$v_t + Av = f[v]\,, \quad v(x,0) = v_0(x)\,, \tag{2.55}$$

where f is the quadratic term

$$f[v] := \int_0^1 (1-\sigma)\, \partial_\sigma^2\, g(S + \sigma v)\, d\sigma \tag{2.56}$$

and ∂_σ^2 denotes the second derivative of $\sigma \mapsto g(S+\sigma v)$ w.r.t. σ and where $A = L+B$ is the spatial linear operator,

$$\begin{aligned} Lv &:= -\varepsilon^2 v'' + v - pvS^{p-1}\Big(\frac{1}{\varepsilon}\int_{-1}^{1} G_\delta(x,y)S^r(y)dy\Big)^{-q}, \\ Bv &= qr\Big(\frac{1}{\varepsilon}\int_{-1}^{1} G_\delta(x,y)S^r(y)dy\Big)^{-q-1}\Big(\frac{1}{\varepsilon}\int_{-1}^{1} G_\delta(x,y)S^{r-1}(y)v(y)dy\Big)S^p \end{aligned} \tag{2.57}$$

defined on the Sobolev space $H^2(-1,1)$ with boundary conditions $v'(\pm 1) = 0$.

For the study of the spectrum and the study of stability using this spectrum it is convenient to stretch the spatial variable by $x = \varepsilon\,\xi$ and to define the operator $A_\varepsilon = L_\varepsilon + B_\varepsilon$ on the stretched interval $[-1/\varepsilon, 1/\varepsilon]$, where ($\dot{u} := du/d\xi$),

$$\begin{gathered} L_\varepsilon u := -\ddot{u} + u - pu\varphi_\varepsilon^{p-1}\Big(\int_{-1/\varepsilon}^{1/\varepsilon} G_\delta(\varepsilon\xi,\varepsilon\eta)\varphi_\varepsilon^r(\eta)d\eta\Big)^{-q}, \\ \mathcal{D}(L_\varepsilon) := \{u \in H^2([-1/\varepsilon, 1/\varepsilon]) : \dot{u}(\pm 1/\varepsilon) = 0\} \end{gathered}$$

Here and later on we write for simplicity φ_ε instead of $\varphi_{\varepsilon,\delta}$.

The non-local operator B_ε is defined by

$$B_\varepsilon v = qr\varphi_\varepsilon^p\Big(\int_{-1/\varepsilon}^{1/\varepsilon} G_\delta(\varepsilon\xi,\varepsilon\eta)\varphi_\varepsilon^r(\eta)d\eta\Big)^{-q-1}\int_{-1/\varepsilon}^{1/\varepsilon} G_\delta(\varepsilon\xi,\varepsilon\eta)\varphi_\varepsilon^{r-1}(\eta)v(\eta)d\eta.$$

We can calculate the "limiting" operators

$$L_0\,u := -\ddot{u} + u - pw_p^{p-1}u, \quad B_0 u = \frac{qr}{\int_{-\infty}^{\infty} w_p^r(\xi)d\xi}\langle u, w_p^{r-1}\rangle w_p^p.$$

Finally, we can evaluate the integral $\beta_m := \int_{-\infty}^{\infty} w_p^m(\xi)\,d\xi$ in terms of the Gamma function,

$$\beta_m = (\frac{p+1}{2})^{\frac{m}{p-1}}\,\frac{2}{p-1}\,\frac{\sqrt{\pi}\Gamma(\frac{m}{p-1})}{\Gamma(\frac{m}{p-1}+\frac{1}{2})}\,.$$

In section 3 we shall study the spectrum of L_ε. In section 4 we study the way in which the spectrum of L_ε is shifted by adding B_ε.

3. The spectrum of the differential operator

The eigenvalues of a selfadjoint differential operator $L := -d^2/dx^2 + Q(x)$ with domain $\mathcal{D}(L)$ of functions on a bounded or unbounded interval $I \subset \mathbb{R}$ satisfy the minimax property, see [13, theorem XIII.1, p. 76]. If L has isolated eigenvalues $\lambda_0 \le \lambda_1 \le \lambda_2 \le \dots$, ordered in increasing sense and counted according their multiplicity (and below the continuous spectrum if present), these satisfy

$$\lambda_k = \inf_{E\subset\mathcal{C},\ \dim(E)\ge k+1}\ \max_{u\in E,\ \|u\|=1} \langle Lu, u\rangle\,, \tag{3.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product and where $\mathcal{C}$ is the domain of the operator.

The operator L_ε of our study with "potential"

$$Q := 1 - p\,\varphi_\varepsilon^{p-1}\Big(\int_{-1/\varepsilon}^{1/\varepsilon} G_\delta(\varepsilon\xi,\varepsilon\eta)\varphi_\varepsilon^r(\eta)d\eta\Big)^{-q}$$

is a selfadjoint differential operator bounded from below and it has a discrete spectrum consisting of eigenvalues of multiplicity one for each $\varepsilon > 0$: $\lambda_0(\varepsilon) < \lambda_1(\varepsilon) < \lambda_2(\varepsilon) < \dots$ with corresponding eigenfunctions $\psi_0(\cdot,\varepsilon)$, $\psi_1(\cdot,\varepsilon)$, $\psi_2(\cdot,\varepsilon)$, ... Its spectrum converges for $\varepsilon \to 0$ (and for all selfadjoint boundary conditions) to the spectrum of L_0, see e.g. [1, ch. 9]. We shall calculate the rate of convergence.

The "limiting" operator L_0 (on the whole real axis) has the continuous spectrum $[1,\infty)$ and may have discrete eigenvalues below this interval (see [9, p. 140]). Simple calculations show:

$$\begin{aligned}
&\psi_{\mathrm{o}} := w_p^{\frac{p+1}{2}} && L_{\mathrm{o}}\,\psi_{\mathrm{o}} = -\tfrac14(p-1)(p+3)\,\psi_{\mathrm{o}}\,, && p>1\,,\\
&\psi_1 := \dot w_p && L_{\mathrm{o}}\,\psi_1 = 0 && p>1\,,\\
&\psi_2 := w_p^{\frac{3-p}{2}} - \tfrac12\tfrac{p+3}{p+1}\,w_p^{\frac{p+1}{2}} && L_{\mathrm{o}}\,\psi_2 = \tfrac14(p-1)(5-p)\,\psi_2\,, && 1<p<3\,.
\end{aligned} \tag{3.2}$$

Since ψ_0, ψ_1 and ψ_2 have zero, one and two zeros respectively, and since the zeros of the eigenfunctions of second order ordinary differential operators interlace, $\lambda_0 := -\frac14(p-1)(p+3)$, $\lambda_1 := 0$ and (if $p<3$) $\lambda_2 := \frac14(p-1)(5-p)$ are the three smallest eigenvalues of L_0. In order to show that L_0 does not have a second isolated eigenvalue for $p>3$, we substitute $\psi(\xi) = \vartheta(\frac{p-1}{2}\xi)$ in the eigenvalue equation $L_0\psi = \lambda\psi$ using the explicit form of w_p from (2.7). This yields the equation

$$M_p\vartheta := -\ddot\vartheta - 2p(p+1)(p-1)^{-2}\cosh^{-2}(\eta)\vartheta = (\tfrac{2}{p-1})^2\,(\lambda-1)\vartheta = \mu\vartheta\,.$$

Since the "potential" in M_p is an increasing function of p, its eigenvalues are increasing functions of p by the minimax theorem (3.1). Since $\lambda_2 \to 1$ if $p \to 3$ from below, the second eigenvalue of M_p tends to zero for $p \nearrow 3$ and gets absorbed into the continuous spectrum if $p \ge 3$. So L_0 has only two eigenvalues below 1 if $p \ge 3$.

In order to compute the rate of convergence of the smallest eigenvalues $\lambda_0(\varepsilon)$ and $\lambda_1(\varepsilon)$ (and $\lambda_2(\varepsilon)$ if $p<3$) of L_ε, we can use the technique of [6] and [7]. We compute (formally) approximate eigenfunctions and project them onto the true eigenfunctions; the residuals yields estimates for the eigenvalues.

Let $\widetilde L_\varepsilon, \widetilde B_\varepsilon$ be the corresponding operators resulting in linearization around the shadow spike solution $\widetilde\varphi_\varepsilon$. (We use the notation $\widetilde\varphi_\varepsilon$ for $\widetilde\varphi$.) More precisely (see (2.15)) we have,

$$\widetilde L_\varepsilon\, u := -\ddot u + u - p\,\widetilde\varphi_\varepsilon^{p-1}\Big(\int_{-1/\varepsilon}^{1/\varepsilon} G_\delta(0,0)\widetilde\varphi_\varepsilon^r(\eta)d\eta\Big)^{-q} u = -\ddot u + u - \psi_\varepsilon^{p-1}u,$$

$$\begin{aligned}
\widetilde B_\varepsilon v &:= qr\widetilde\varphi_\varepsilon^p\Big(\int_{-1/\varepsilon}^{1/\varepsilon} G_\delta(0,0)\widetilde\varphi_\varepsilon^r(\eta)d\eta\Big)^{-q-1}\Big(\int_{-1/\varepsilon}^{1/\varepsilon} G_\delta(0,0)\widetilde\varphi_\varepsilon^{r-1}(\eta)v(\eta)d\eta\Big)\\
&= qr\psi_\varepsilon^p\frac{\langle u,\psi_\varepsilon^{r-1}\rangle}{\langle\psi_\varepsilon^r,1\rangle}.
\end{aligned}$$

Thus the operator $\widetilde A_\varepsilon = \widetilde L_\varepsilon + \widetilde B_\varepsilon$ does not depend on δ and coincides with the shadow operator from [8]. Since

$$L_\varepsilon = \widetilde L_\varepsilon + O\big(|\varphi_\varepsilon^{p-1} - \widetilde\varphi_\varepsilon^{p-1}| + \big[\varepsilon(1+|\xi|) + |\langle\varphi_\varepsilon^r - \widetilde\varphi_\varepsilon^r, g_\delta\rangle|\big]\widetilde\varphi_\varepsilon^{p-1}\big)$$

it follows the uniform estimate

$$\|L_\varepsilon - \widetilde L_\varepsilon\| = O(\varepsilon^\gamma)\,,\quad 0<\varepsilon<\varepsilon_0(\delta). \tag{3.3}$$

Here and later on the positive quantity $O(1)$ depends on δ and is equivalent to 1 on any compact interval $[0,\delta_0]$. All estimates will be uniform in the same sense. Analogously,

$$\|B_\varepsilon - \widetilde B_\varepsilon\| = O(\varepsilon^\gamma)\,,\quad 0<\varepsilon<\varepsilon_0(\delta). \tag{3.4}$$

In particular, using the asymptotic behaviour of the eigenvalues of $\widetilde{L}_\varepsilon$ [8], we find

$$\lambda_0(\varepsilon) = \lambda_0 + O(\varepsilon^\gamma)\,, \quad \lambda_2(\varepsilon) = \mu_0 + O(\varepsilon^\gamma)\,,$$

where $\mu_0 := \lambda_2$ if $p < 3$ and $\mu_0 := 1$ if $p \geq 3$.

To find the asymptotic behaviour of the small eigenvalue $\lambda_1(\varepsilon)$ we shall use $\dot\varphi_\varepsilon$ as an approximate eigenfunction. Differentiating (2.4), we get

$$L_\varepsilon \dot\varphi_\varepsilon = -q\varepsilon\varphi_\varepsilon^p\Big(\int_{-1/\varepsilon}^{1/\varepsilon} G_\delta(\varepsilon\xi,\varepsilon\eta)\varphi_\varepsilon^r(\eta)d\eta\Big)^{-q-1}\int_{-1/\varepsilon}^{1/\varepsilon}\partial_x G_\delta(\varepsilon\xi,\varepsilon\eta)\varphi_\varepsilon^r(\eta)d\eta.$$

We evaluate this expression as follows. We have

$$\partial_x G_\delta(\varepsilon\xi,\varepsilon\eta) = \pm\delta^2/2\mu + \delta^2\varepsilon(|\xi|+|\eta|)\,O(1), \quad \xi\neq\eta,$$

where "+" corresponds to the region $\xi < \eta$ and "−" corresponds to $\xi > \eta$;

$$\Big(\int_{-1/\varepsilon}^{1/\varepsilon} G_\delta(\varepsilon\xi,\varepsilon\eta)\varphi_\varepsilon^r(\eta)d\eta\Big)^{-q-1} = \Big(\int_{-1/\varepsilon}^{1/\varepsilon} g_\delta\varphi_\varepsilon^r(\eta)d\eta\Big)^{-q-1} + O(\delta^2\varepsilon(1+|\xi|))\,. \quad (3.5)$$

Therefore,

$$L_\varepsilon\dot\varphi_\varepsilon = -\frac{q\,\varepsilon\,\delta^2\varphi_\varepsilon^p}{2\mu}\Big(\int_\xi^{1/\varepsilon}\varphi_\varepsilon^r(\eta)d\eta - \int_{-1/\varepsilon}^{\xi}\varphi_\varepsilon^r(\eta)d\eta\Big) + O(\delta^2\varepsilon^2(1+|\xi|)\varphi_\varepsilon^p)\,.$$

In particular,

$$\langle L_\varepsilon\dot\varphi_\varepsilon,\dot\varphi_\varepsilon\rangle = -\frac{q\,\varepsilon\,\delta^2}{2\mu\,(p+1)}\int_{-1/\varepsilon}^{1/\varepsilon}\varphi_\varepsilon^{p+r+1}(\eta)\,d\eta\Big(\int_{-1/\varepsilon}^{1/\varepsilon} g_\delta\varphi_\varepsilon^r(\eta)d\eta\Big)^{-q-1} + O(\delta^2\varepsilon^2).$$

The asymptotic expansion of $\lambda_1(\varepsilon)$ will be calculated using the same technique as in [6] and [7]. We compute an approximate eigenfunction w, $\|w\| = 1$ of the operator L_ε and we show that

$$\langle L_\varepsilon w, w\rangle = \nu_\varepsilon(1 + O(\widetilde{R}_\varepsilon)) \quad\text{and}\quad \|L_\varepsilon w\|^2 = O(\widetilde{R}_\varepsilon\, R_\varepsilon)\,, \quad (3.6)$$

where $R_\varepsilon = o(1)$ and $\widetilde{R}_\varepsilon = o(1)$ for $\varepsilon \to 0$.

The generalized Fourier expansion of w in the true eigenfunctions $\{\psi_k : k = 0, 1, \dots\}$ of L_ε is

$$w = \sum_{k=0}^\infty c_k\psi_k \quad\text{with}\quad \sum_{k=0}^\infty |c_k|^2 = \|w\|^2 = 1\,.$$

Since all eigenvalues of L_ε except $\lambda_1(\varepsilon)$ are uniformly bounded away from $\lambda_1(0) = 0$ by a distance $d > 0$, we find from (3.6)

$$1 - |c_1|^2 = \sum_{k=0\,,\ k\neq 1}^\infty |c_k|^2 \ \le\ d^{-2}\sum_{k=0\,,\ k\neq 1}^\infty \lambda_k^2\,|c_k|^2 \ \le\ d^{-2}\,\|L_\varepsilon w\|^2 = O(\widetilde{R}_\varepsilon\, R_\varepsilon)\,,$$

implying that $|c_1|^2 = 1 + O(\widetilde{R}_\varepsilon\, R_\varepsilon)$. The estimate for the inner product in (3.6) now implies that

$$\langle L_\varepsilon w, w\rangle - \nu_\varepsilon = |c_1|^2\lambda_1(\varepsilon) - \nu_\varepsilon + \sum_{k=0,\ k\neq 1}^\infty \lambda_k|c_k|^2 = O\big(\widetilde{R}_\varepsilon(\nu_\varepsilon + R_\varepsilon)\big)$$

and hence that

$$\lambda_1(\varepsilon) = \nu_\varepsilon + O\big(\widetilde{R}_\varepsilon(\nu_\varepsilon + R_\varepsilon)\big).$$

Let $\psi_1(\cdot,\varepsilon)$ be the true eigenfunction of L_ε corresponding to $\lambda_1(\varepsilon)$. We look for an approximate eigenfunction of the form $\psi_1(\cdot,\varepsilon) \approx \dot\varphi_\varepsilon +$ boundary layer corrections.

Within the interval $[-1/\varepsilon, 1/\varepsilon]$ the tails of $\dot\varphi_\varepsilon$ are exponentially small by (2.48) and (2.4),

$$\dot\varphi_\varepsilon(\frac{\pm 1}{\varepsilon}) = 0 \quad \text{and} \quad \ddot\varphi_\varepsilon(\pm\frac{1}{\varepsilon}) = ae^{-1/\varepsilon}(1 + O(\varepsilon^\gamma)), \quad 0 < \varepsilon < \varepsilon_0(\delta),\ \gamma > 0, \quad (3.7)$$

where $a := 2\alpha(\int_{-\infty}^{\infty} g_\delta w_p^r(\eta)d\eta)^{-\alpha_r}$. .

We construct boundary layer terms at both endpoints by standard matched asymptotic expansions. Suitable boundary layer corrections at the right and left endpoints are

$$\begin{gathered} h(\xi) := -\ddot\varphi_\varepsilon(\tfrac{1}{\varepsilon})\varrho(\varepsilon\xi)\exp\big(\xi - \frac{1}{\varepsilon})\big), \\ k(\xi) := \ddot\varphi_\varepsilon(-\tfrac{1}{\varepsilon})\,\varrho(-\varepsilon\xi)\exp\big(-\xi - \frac{1}{\varepsilon})\big), \\ \widetilde\psi_1 := \dot\varphi_\varepsilon + h + k \end{gathered} \quad (3.8)$$

where ϱ is a monotonic $\mathcal{C}^\infty$ cut-off function satisfying $\varrho(x) = 1$ if $x \ge 3/4$ and $\varrho(x) = 0$ if $x \le 1/2$. From the definition it is clear that $\widetilde\psi_1$ satisfies the boundary conditions at $\xi = \pm 1/\varepsilon$ and

$$L_\varepsilon h = -p\,\varphi_\varepsilon^{p-1}h\Big(\int_{-1/\varepsilon}^{1/\varepsilon} G_\delta(\varepsilon\xi,\varepsilon\eta)\varphi_\varepsilon^r(\eta)d\eta\Big)^{-q} + \ddot\varphi_\varepsilon(\tfrac{1}{\varepsilon})\left(\varepsilon^2\varrho'' + 2\varepsilon\varrho'\right)\exp(\xi - \tfrac{1}{\varepsilon}).$$

For $p > 1$ we have

$$\|L_\varepsilon\widetilde\psi_1\|^2 = \delta^4\varepsilon^2 + O(e^{-(2+\gamma)/\varepsilon}), \quad 0 < \varepsilon < \varepsilon_0(\delta). \quad (3.9)$$

Further,

$$\langle L_\varepsilon\widetilde\psi_1, \widetilde\psi_1\rangle = \langle L_\varepsilon\dot\varphi_\varepsilon, \dot\varphi_\varepsilon\rangle - \langle L_\varepsilon(h+k), h+k\rangle - [(\dot h + \dot k)\widetilde\psi_1]_{-1/\varepsilon}^{1/\varepsilon}.$$

We can calculate the last two terms, hence

$$\langle L_\varepsilon\widetilde\psi_1, \widetilde\psi_1\rangle = \langle L_\varepsilon\dot\varphi_\varepsilon, \dot\varphi_\varepsilon\rangle - 2[\ddot\varphi_\varepsilon(1/\varepsilon)]^2 + e^{-(2+\gamma)/\varepsilon}O(1). \quad (3.10)$$

On the other hand,

$$\|\dot\varphi_\varepsilon\|^2 = \Big(\int_{-\infty}^{\infty} g_\delta w_p^r(\eta)d\eta\Big)^{-2\alpha_r}\int_{-\infty}^{\infty}(\dot w_p)^2 d\eta\,(1 + \varepsilon^\gamma\,O(1)).$$

Therefore, the above estimates show that

$$\lambda_1(\varepsilon) = -a(\delta)\varepsilon - a_1 e^{-2/\varepsilon} + (\delta^2\varepsilon^{1+\gamma} + e^{-(2+\gamma)/\varepsilon})\,O(1), \quad 0 < \varepsilon < \varepsilon_0(\delta), \quad (3.11)$$

where $a(\delta) > 0,\ a(\delta) = \delta^2\,O(1)$ and

$$a_1 = 8\alpha^2\Big(\int_{-\infty}^{\infty}(\dot w_p)^2 d\eta\Big)^{-1}. \quad (3.12)$$

In particular, for any fixed $\delta > 0$ we have the asymptotic

$$\lambda_1(\varepsilon) = -a(\delta)\varepsilon + \varepsilon^{1+\gamma}\,O(1), \quad 0 < \varepsilon < \varepsilon_0(\delta).$$

Thus the small eigenvalue $\lambda_1(\varepsilon)$ of the differential operator L_ε is always negative. In contrast, in the next section we shall prove that the small eigenvalue λ_ε of the perturbed operator A_ε is positive for any fixed $\delta > 0$ if $0 < \varepsilon < \varepsilon_0(\delta)$. If we allow dependence of δ on ε, then λ_ε is positive for all $\delta > \delta(\varepsilon)$, where $\delta(\varepsilon)$ is exponentially small w.r.t. $\varepsilon \in (0, \varepsilon_0)$. To prove these facts, we need two type of estimates: for any fixed $\delta > 0$ or for all small δ.

4. Perturbation of the spectrum by the non-local term

In this section we consider how the nonlocal operator B_ε perturbs the eigenvalues of L_ε. Since $\|B_\varepsilon\| = O(1)$ it follows that the spectrum of A_ε lies in a strip around the real axis. Hence this is an operator with compact resolvent and according to Kato, p. 237 [10], its spectrum consists of eigenvalues with finite multiplicity.

Our goal is to find conditions on the parameters p, q, r so that the spectrum of A_ε lies in the right half-plane. We shall prove that this is true under the same conditions on the parameters p, q, r as in the shadow case, cf. [8].

4.1. **Perturbation of the small eigenvalue by the non-local term.** In this subsection we consider how the non-local operator B_ε perturbs the small eigenvalue $\lambda_1(\varepsilon)$ of L_ε. Both the operators L_ε and B_ε are invariant under the change of sign $\xi \mapsto -\xi$ and hence leave the subspaces of even and odd functions invariant. Hence in this subsection we can consider the operator $A_\varepsilon = L_\varepsilon + B_\varepsilon$ on the subspace of odd functions only. Then A_ε is a small perturbation of the selfadjoint operator L_ε. Indeed, since

$$B_\varepsilon = B_{0\varepsilon} + \delta^2\varepsilon(1+|\xi|)\varphi_\varepsilon^p O(1),$$

where

$$B_{0\varepsilon} v = q\, r\, \varphi_\varepsilon^p \Big(\int_{-1/\varepsilon}^{1/\varepsilon} g_\delta \varphi_\varepsilon^r(\eta) d\eta \Big)^{-q-1} \int_{-1/\varepsilon}^{1/\varepsilon} g_\delta \varphi_\varepsilon^{r-1}(\eta) v(\eta) d\eta\,,$$

and $B_{0\varepsilon} = 0$ on odd functions, it follows

$$\|B_\varepsilon\| = \delta^2\varepsilon O(1), \quad 0 < \varepsilon < \varepsilon_0(\delta).$$

Hence by Kato [10, p. 364] the spectrum of A_ε (on odd functions) consists of one simple small eigenvalue λ_ε (see (3.11)),

$$\lambda_\varepsilon = (\delta^2\varepsilon + e^{-2/\varepsilon})\, O(1), \tag{4.1}$$

and eigenvalues close to the real axis and in the half plane $Re\lambda > 1/2$.

Thus the problem is reduced to determine the sign of $Re\lambda_\varepsilon$. To this end we shall find its asymptotic behaviour. This will be done in two steps. In the first step we use the a priori estimate (4.1) and derive a better estimate for λ_ε (see (4.8) below). To this end we use the same technique as for the selfadjoint operator L_ε, exploiting the fact that the non-selfadjoint operator A_ε is a sufficiently small perturbation of L_ε.

Let

$$A_\varepsilon\psi_\varepsilon = \lambda_\varepsilon\psi_\varepsilon, \quad \|\psi_\varepsilon\| = 1$$

(the eigenfunction being odd one). As an approximate eigenfunction we use the same function $\widetilde{\psi}$ as before: $\widetilde{\psi} = \dot{\varphi}_\varepsilon + h + k$. Note that this is also an odd function. Let

$$\widetilde{\psi} = c\psi_\varepsilon + dg, \quad \|g\| = 1, \text{ with } g \text{ orthogonal to } \psi_\varepsilon.$$

Then

$$\|L_\varepsilon\widetilde{\psi}\|^2 = |c|^2\|L_\varepsilon\psi_\varepsilon\|^2 + |d|^2\|L_\varepsilon g\|^2 \tag{4.2}$$

and $L_\varepsilon\psi_\varepsilon = \lambda_\varepsilon\psi_\varepsilon + \delta^2\varepsilon\, O(1)$, hence

$$\|L_\varepsilon\psi_\varepsilon\| = (\delta^2\varepsilon + e^{-2/\varepsilon})\, O(1).$$

On the other hand we already know that (see (3.9)),

$$\|L_\varepsilon\widetilde{\psi}\| = (\delta^2\varepsilon + e^{-(1+\gamma)/\varepsilon})\, O(1), \quad p > 1. \tag{4.3}$$

Now we need the uniform estimate $\|L_\varepsilon g\| \geq C$. Suppose on the contrary that $\|L_\varepsilon g\| = o(1)$ as $\varepsilon \to 0$. Let $L_\varepsilon \omega_1 = \lambda_1 \omega_1$, $\|\omega_1\| = 1$. If $g = c_1\omega_1 + d_1 h_1$, $|c_1|^2 + |d_1|^2 = 1$ is the orthogonal decomposition, we find that $g = \omega_1 + o(1)$. On the other hand, if $\psi_\varepsilon = c_2\omega_1 + d_2 h_2$, $|c_2|^2 + |d_2|^2 = 1$ is the orthogonal decomposition of ψ_ε, then since $\|L_\varepsilon \psi_\varepsilon\| = O(\lambda_1)$ we find that $\psi_\varepsilon = \omega_1 + O(\lambda_1)$. Then $\langle g, \psi_\varepsilon\rangle = 1 + o(1)$, what contradicts orthogonality of g and ψ_ε. Hence

$$|d| = (\delta^2\varepsilon + e^{-(1+\gamma)/\varepsilon})\, O(1). \tag{4.4}$$

Further,

$$\langle A_\varepsilon \widetilde{\psi}, \widetilde{\psi}\rangle = |c|^2\lambda_\varepsilon + d\bar{c}\langle A_\varepsilon g, \psi_\varepsilon\rangle + |d|^2\langle A_\varepsilon g, g\rangle.$$

Since

$$\langle A_\varepsilon g, \psi_\varepsilon\rangle = \langle L_\varepsilon g, \psi_\varepsilon\rangle + \delta^2\varepsilon\, O(1)$$

it follows

$$\langle A_\varepsilon g, \psi_\varepsilon\rangle = (\delta^2\varepsilon + e^{-2/\varepsilon})\, O(1).$$

On the other hand, (4.2), (4.3) imply

$$|d|\|A_\varepsilon g\| = (\delta^2\varepsilon + e^{-(1+\gamma)/\varepsilon}) O(1).$$

Therefore,

$$\langle A_\varepsilon \widetilde{\psi}, \widetilde{\psi}\rangle = \lambda_\varepsilon \|\widetilde{\psi}\|^2 + (\delta^4\varepsilon^2 + e^{-(2+\gamma)/\varepsilon})\, O(1).$$

To evaluate the quadratic form we write

$$\langle A_\varepsilon \widetilde{\psi}, \widetilde{\psi}\rangle = \langle A_\varepsilon \dot{\varphi}_\varepsilon, \dot{\varphi}_\varepsilon\rangle - [(\dot{h} + \dot{k})\widetilde{\psi}]_{-1/\varepsilon}^{1/\varepsilon} - \langle L_\varepsilon(h+k), h+k\rangle + \langle B_\varepsilon \dot{\varphi}_\varepsilon, h+k\rangle.$$

Since

$$\langle B_\varepsilon \dot{\varphi}_\varepsilon, h+k\rangle = \delta^2 e^{-2/\varepsilon}\, O(1), \quad p > 1,$$

we get from above estimates

$$\lambda_\varepsilon = \frac{\langle A_\varepsilon \dot{\varphi}_\varepsilon, \dot{\varphi}_\varepsilon\rangle}{\|\dot{\varphi}_\varepsilon\|^2} - \frac{2[\ddot{\varphi}_\varepsilon(1/\varepsilon)]^2}{\|\dot{\varphi}_\varepsilon\|^2} + (\delta^4\varepsilon^2 + e^{-(2+\gamma)/\varepsilon} + \delta^2 e^{-2/\varepsilon})\, O(1), \tag{4.5}$$

where $0 < \varepsilon < \varepsilon_0(\delta)$, $p > 1$, and it remains to evaluate $\langle A_\varepsilon \dot{\varphi}_\varepsilon, \dot{\varphi}_\varepsilon\rangle$. We have

$$L_\varepsilon \dot{\varphi}_\varepsilon = -\, q\, \varepsilon\, \varphi_\varepsilon^p \Big(\int_{-1/\varepsilon}^{1/\varepsilon} G_\delta(\varepsilon\xi, \varepsilon\eta)\varphi_\varepsilon^r(\eta)d\eta\Big)^{-q-1} \int_{-1/\varepsilon}^{1/\varepsilon} \partial_x G_\delta(\varepsilon\xi, \varepsilon\eta)\varphi_\varepsilon^r(\eta)d\eta$$

and

$$B_\varepsilon \dot{\varphi}_\varepsilon = q\, \varphi_\varepsilon^p \Big(\int_{-1/\varepsilon}^{1/\varepsilon} G_\delta(\varepsilon\xi, \varepsilon\eta)\varphi_\varepsilon^r(\eta)d\eta\Big)^{-q-1} \int_{-1/\varepsilon}^{1/\varepsilon} G_\delta(\varepsilon\xi, \varepsilon\eta)d\varphi_\varepsilon^r(\eta).$$

Since

$$\int_{-1/\varepsilon}^{1/\varepsilon} G_\delta(\varepsilon\xi, \varepsilon\eta)d\varphi_\varepsilon^r(\eta) = \\ = \big[G_\delta(\varepsilon\xi, 1) - G_\delta(\varepsilon\xi, -1)\big]\varphi_\varepsilon^r(1/\varepsilon) - \varepsilon\int_{-1/\varepsilon}^{1/\varepsilon} \partial_y G_\delta(\varepsilon\xi, \varepsilon\eta)\varphi_\varepsilon^r(\eta)d\eta,$$

$$\partial_y G_\delta(\varepsilon\xi, \varepsilon\eta) + \partial_x G_\delta(\varepsilon\xi, \varepsilon\eta) = \delta c_\delta \sinh\delta\varepsilon(\xi + \eta)\,, \quad \text{where } c_\delta := \frac{\delta}{\mu\sinh 2\delta}$$

$$G_\delta(\varepsilon\xi, 1) - G_\delta(\varepsilon\xi, -1) = \delta^2\, O(1)$$

we find

$$\begin{aligned} A_\varepsilon \dot\varphi_\varepsilon = & -q\delta c_\delta \varepsilon \varphi_\varepsilon^p \int_{-1/\varepsilon}^{1/\varepsilon} \sinh \delta\varepsilon(\xi+\eta)\varphi_\varepsilon^r(\eta)d\eta \Big(\int_{-1/\varepsilon}^{1/\varepsilon} G_\delta(\varepsilon\xi,\varepsilon\eta)\varphi_\varepsilon^r(\eta)d\eta\Big)^{-q-1} \\ & + O\Big(\delta^2\varphi_\varepsilon^r(1/\varepsilon)\varphi_\varepsilon^p \Big(\int_{-1/\varepsilon}^{1/\varepsilon} G_\delta(\varepsilon\xi,\varepsilon\eta)\varphi_\varepsilon^r(\eta)d\eta\Big)^{-q-1}\Big). \end{aligned}$$

We simplify this expression as follows. Since

$$\sinh \delta\varepsilon(\xi+\eta) = \delta\varepsilon(\xi+\eta) + O(\delta^3\varepsilon^3(|\xi|^3+|\eta|^3)),$$

it follows

$$\int_{-1/\varepsilon}^{1/\varepsilon} \sinh \delta\varepsilon(\xi+\eta)\varphi_\varepsilon^r(\eta)d\eta = \delta\varepsilon\xi \int_{-1/\varepsilon}^{1/\varepsilon} \varphi_\varepsilon^r(\eta)d\eta + O(\delta^3\varepsilon^3(1+|\xi|^3)).$$

Using also (3.5) we get

$$\begin{aligned} &\langle A_\varepsilon\dot\varphi_\varepsilon, \dot\varphi_\varepsilon\rangle = \\ &= -q\delta^2\varepsilon^2 c_\delta \int_{-1/\varepsilon}^{1/\varepsilon} \varphi_\varepsilon^r(\eta)d\eta \int_{-1/\varepsilon}^{1/\varepsilon} \xi\varphi_\varepsilon^p(\eta)d\varphi_\varepsilon \Big(\int_{-1/\varepsilon}^{1/\varepsilon} g_\delta\varphi_\varepsilon^r(\eta)d\eta\Big)^{-q-1} + O(\delta^2\varepsilon^3) \\ &= \frac{q}{p+1}\delta^2\varepsilon^2 c_\delta \int_{-1/\varepsilon}^{1/\varepsilon} \varphi_\varepsilon^r(\eta)d\eta \int_{-1/\varepsilon}^{1/\varepsilon} \varphi_\varepsilon^{p+1}(\eta)d\eta \Big(\int_{-1/\varepsilon}^{1/\varepsilon} g_\delta\varphi_\varepsilon^r(\eta)d\eta\Big)^{-q-1} + O(\delta^2\varepsilon^3). \end{aligned}$$

In this expression we can replace as before φ_ε by $\widetilde{\varphi}_\varepsilon$ and then by w_p. As a result we get

$$\frac{\langle A_\varepsilon\dot\varphi_\varepsilon, \dot\varphi_\varepsilon\rangle}{\|\dot\varphi_\varepsilon\|^2} = \frac{q\,\delta^2\varepsilon^2 \int_{-\infty}^{\infty} w_p^{p+1}(\eta)d\eta}{(p+1)\cosh^2\delta \int_{-\infty}^{\infty}(\dot w_p)^2 d\eta} + O(\delta^2\varepsilon^{5/2}). \tag{4.6}$$

Finally, (4.5), (4.6) and (3.7) give

$$\frac{q\,\delta^2\varepsilon^2 \int_{-\infty}^{\infty} w_p^{p+1}(\eta)d\eta}{(p+1)\cosh^2\delta \int_{-\infty}^{\infty}(\dot w_p)^2 d\eta} - \frac{8\alpha^2 e^{-2/\varepsilon}}{\int_{-\infty}^{\infty}(\dot w_p)^2 d\eta} + (\delta^2\varepsilon^{5/2} + \delta^4\varepsilon^2 + O(e^{-(2+\gamma)/\varepsilon})) \tag{4.7}$$

if $0<\varepsilon<\varepsilon_0(\delta)$. In particular,

$$\lambda_\varepsilon = O(\delta^2\varepsilon^2 + e^{-2/\varepsilon}), \qquad 0<\varepsilon<\varepsilon_0(\delta). \tag{4.8}$$

To find the asymptotic behaviour of λ_ε, we notice that using (4.8) we can improve the bound for d (cf. (4.4)):

$$|d| = O(\delta^2\varepsilon^2 + e^{-(1+\gamma)/\varepsilon}). \tag{4.9}$$

Indeed, since $A_\varepsilon\widetilde{\psi} = c\lambda_\varepsilon\psi_\varepsilon + dA_\varepsilon g$ and $A_\varepsilon g = L_\varepsilon g + \delta^2\varepsilon\, O(1)$, hence $\|A_\varepsilon g\| \geq C > 0$, it follows $|d| = (\|A_\varepsilon\widetilde{\psi}\| + \delta^2\varepsilon^2 + e^{-2/\varepsilon})$. Using $\|A_\varepsilon\dot\varphi_\varepsilon\| = \delta^2\varepsilon^2\, O(1)$ and $\|A_\varepsilon h\| = e^{-(1+\gamma)/\varepsilon}\, O(1)$ we get (4.9).

Now, having the better estimate (4.9) we can repeat the above arguments and show that instead of (4.7) we have for $0<\varepsilon<\varepsilon_0(\delta)$:

$$\lambda_\varepsilon = \frac{q\delta^2\varepsilon^2 \int_{-\infty}^{\infty} w_p^{p+1}(\eta)d\eta}{(p+1)\cosh^2\delta \int_{-\infty}^{\infty}(\dot w_p)^2 d\eta} - \frac{8\alpha^2 e^{-2/\varepsilon}}{\int_{-\infty}^{\infty}(\dot w_p)^2 d\eta} + (\delta^2\varepsilon^{5/2} + e^{-(2+\gamma)/\varepsilon})\, O(1). \tag{4.10}$$

In particular, for any fixed $\delta>0$ we have the asymptotic

$$\lambda_\varepsilon = \frac{q\delta^2\varepsilon^2 \int_{-\infty}^{\infty} w_p^{p+1}(\eta)d\eta}{(p+1)\cosh^2\delta \int_{-\infty}^{\infty}(\dot w_p)^2 d\eta} + \varepsilon^{5/2}\, O(1),\ 0<\varepsilon<\varepsilon_0(\delta).$$

If δ is not fixed and we allow $\delta \to 0$ as $\varepsilon \to 0$, then $Re\lambda_\varepsilon$ changes sign around the point $\delta(\varepsilon)$ given as a solution to the equation

$$\frac{q}{p+1}\,\frac{\mu}{8\alpha^2}\,\varepsilon^2\delta^2 e^{2/\varepsilon}\int_{-\infty}^{\infty} w_p^{p+1}(\eta)d\eta = 1.$$

Note that the same expression is obtained in [17] using formal asymptotic methods.

4.2. **Perturbation of the non-small eigenvalues and uniform estimates of the resolvent.** According to (3.3), (3.4) and Kato [10, p. 364], the eigenvalues of A_ε lie in a $O(\varepsilon^\gamma)$ neighbourhood of the eigenvalues of the shadow operator $\widetilde{A}_\varepsilon$. Hence, under the same conditions on the parameters p, q, r as in [8], the spectrum of A_ε lies in the right-half plane for all $0 < \varepsilon < \varepsilon(D)$. In particular, there exists an angle $\chi_D \in (0, \pi/2)$, such that the resolvent set of A_ε contains the sector

$$\Lambda := \{\lambda \in \mathbb{C} : \chi_D \le |\arg(\lambda - \mu_\varepsilon)| \le \pi\},$$

where $\mu_\varepsilon = \frac{1}{2}\operatorname{Re}\lambda_\varepsilon$. Moreover, in this sector the resolvent satisfies for some constant $M_{\varepsilon,D}$, for all $0 < \varepsilon < \varepsilon(D)$, the estimate

$$\|(A_\varepsilon - \lambda)^{-1}\| \le \frac{M_{\varepsilon,D}}{|\lambda - \mu_\varepsilon|} \quad \text{for all } \lambda \in \Lambda. \tag{4.11}$$

To prove this estimate, we use the formula

$$(A_\varepsilon - \lambda)^{-1} = (1 + (L_\varepsilon - \lambda)^{-1}B_\varepsilon)^{-1}(L_\varepsilon - \lambda)^{-1},$$
$$\|(L_\varepsilon - \lambda)^{-1}\| \le 1/\operatorname{dist}(\lambda, \sigma(L_\varepsilon)).$$

Since

$$\|(L_\varepsilon - \lambda)^{-1}\| \le \frac{C}{|\lambda|}, \quad \|(L_\varepsilon - \lambda)^{-1}B_\varepsilon\| \le \frac{1}{2},$$

uniformly for all $|\lambda| > N_D$, $\lambda \in \Lambda$, $0 < \varepsilon < \varepsilon(D)$ for some N_D large enough, it follows

$$\|(A_\varepsilon - \lambda)^{-1}\| \le \frac{C}{|\lambda - \mu_\varepsilon|},$$

uniformly for all $|\lambda| > N_D$, $\lambda \in \Lambda$, and $0 < \varepsilon < \varepsilon(D)$. If $|\lambda| \le N_D$, $\lambda \in \Lambda$, then

$$\|(\widetilde{A}_\varepsilon - \lambda)^{-1}\| \le C_{\varepsilon,D}.$$

5. Contraction around the steady state $S(x, \varepsilon)$

In this section we study stability of the spike solution S of (1.3) as given in (2.3). We assume that the parameters $(p, q, r, \mu, \varepsilon)$ are such that all eigenvalues of A_ε are located in the right half plane.

Besides the standard L^2-norm for functions on the interval $[-1, 1]$ denoted by $\|\cdot\|$, we use in this section the "energy norm" $\|\cdot\|_1$, which is associated naturally to a problem with a small parameter like (2.54) and is defined by: $\|u\|_1^2 := \|u\|^2 + \|\varepsilon\, u'\|^2$.

For fixed positive a large enough and uniformly for all $\varepsilon \in (0, \varepsilon_0]$ this norm satisfies the equivalences

$$\langle (A + a)u, u\rangle^{1/2} \asymp \|u\|_1.$$

We study perturbations around the steady state spike solution S, using the contraction method as in [8] The perturbation satisfies equation (2.55), which reads:

$$v_t + Av = f[v]\,, \quad v(x, 0) = v_0(x),$$

where the quadratic term f is given by (2.56) and the (linear) operator A is defined by (2.57). Obviously, this operator A has the same spectral properties as its (stretched) cousin A_ε has in sections 3 and 4. Hence, under the positivity condition, stated above, A is a sectorial operator, see [9], it satisfies the estimate (4.11).

Associated to A is the semigroup

$$e^{-At} := \frac{1}{2\pi i}\int_\Gamma (A-\lambda)^{-1}e^{-\lambda t}d\lambda\,, \quad t>0\,,$$

where Γ is a suitable contour in the resolvent set Λ. As in [8] we prove the following statement.

Lemma 5.1. *For all $t>0$, all $\varepsilon \in (0,\varepsilon(D)]$ and for some constant $C_{\varepsilon,D}$, not depending on t, this semigroup satisfies:*

$$\|e^{-At}\| \le C_{\varepsilon,D}e^{-\mu_\varepsilon t}\,, \tag{5.1}$$

$$\|e^{-At}\|_1 \le C_{\varepsilon,D}e^{-\mu_\varepsilon t}\begin{cases}\|u\|_1\,,\\ (1+t^{-1/2})\|u\|\end{cases} \tag{5.2}$$

Hence we can apply the contraction method as in [9], [8], and prove the local stability of the single internal spike solution $S(x,\varepsilon)$ in the Sobolev norm $\|\cdot\|_1$.

Theorem 5.2 (Local stability of the single internal spike for $0<\varepsilon<\varepsilon(D)$)**.** *There exist positive constants $C(D), C_\varepsilon(D)$ and $\varepsilon(D)$, depending also on p, q, r and μ, and small $\varrho_\varepsilon(D)$ such that the solution (U,H) of the system (1.3) exists for all times $t>0$ and satisfies*

$$\|U(\cdot,t)-S\|_1 \le \varrho e^{-\mu_\varepsilon t},$$
$$\|H(\cdot,t)-H\|_1 \le C(D)\varrho\varepsilon^{-1}e^{-\mu_\varepsilon t},$$

for all ε and ϱ satisfying

$$0<\varepsilon<\varepsilon(D), \quad 0<\varrho<\varrho_\varepsilon(D),$$

for all initial conditions $U_0 \in H^1(-1,1)$ in the vicinity of S, that satisfy the compatibility conditions $U_0'(-1)=U_0'(1)=0$ and satisfy the bound

$$\|U_0-S\|_1 \le C_\varepsilon(D)\varrho.$$

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