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ON THE Ψ-STABILITY OF A NONLINEAR VOLTERRA INTEGRO-DIFFERENTIAL SYSTEM

AUREL DIAMANDESCU

Abstract. In this paper we prove sufficient conditions for Ψ-stability of the zero solution of a nonlinear Volterra integro-differential system.

1. Introduction

Akinyele [1] introduced the notion of Ψ-stability of degree k with respect to a function $\Psi \in C(\mathbb{R}_+, \mathbb{R}_+)$, increasing and differentiable on $\mathbb{R}_+$ and such that $\Psi(t) \geq 1$ for $t \geq 0$ and $\lim_{t\to\infty} \Psi(t) = b$, $b \in [1, \infty)$. The fact that the function Ψ is bounded does not enable a deeper analysis, of the asymptotic properties of the solutions of a differential equations, than the notion of stability in sense Lyapunov.

Constantin [5] introduced the notions of degree of stability and degree of boundedness of solutions of an ordinary differential equation, with respect to a continuous positive and nondecreasing function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$. Some criteria for these notions are proved there too.

Morchalo [13] introduced the notions of Ψ-stability, Ψ-uniform stability, and Ψ-asymptotic stability of trivial solution of the nonlinear system $x' = f(t, x)$. Several new and sufficient conditions for mentioned types of stability are proved for the linear system $x' = A(t)x$. Furthermore, sufficient conditions are given for the uniform Lipschitz stability of the system $x' = f(t, x) + g(t, x)$. In this paper, the function Ψ is a scalar continuous function.

The purpose of our paper is to prove sufficient conditions for Ψ-(uniform) stability of trivial solution of the nonlinear Volterra integro-differential system

$$x' = A(t)x + \int_0^t F(t, s, x(s))\, ds \tag{1.1}$$

which can be seen as a perturbed system of

$$y' = A(t)y \tag{1.2}$$

We investigate conditions on the fundamental matrix $Y(t)$ for the linear system (1.2) and on the function $F(t, s, x)$ under which the trivial solution of (1.1) or (1.2) is Ψ-(uniformly) stable on $\mathbb{R}_+$. Here, Ψ is a matrix function whose introduction permits us obtaining a mixed asymptotic behavior for the components of solutions.

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Recent works for stability of solutions of (1.1) have been given by Mahfoud [12] who used Lyapunov functionals; Lakshmikantham and Rama Mohana Rao [11] who used the comparison method; Hara, Yoneyama and Itoh [10] who used "variation of parameters" formula; in other words, the solution of equation (1.1) with the initial function φ on $[0, t_0]$ - namely $x(t) = \varphi(t)$ for t $\in [0, t_0]$ - is written

$$x(t; t_0, \varphi) = Y(t)Y^{-1}(t_0)\varphi(t_0) + \int_0^t Y(t)Y^{-1}(s) \int_0^s F(s, u, x(u; t_0, \varphi))\, du\, ds\,;$$

and by Avramescu [2] who used the method of admissibility of a pair of subspaces with respect to an operator.

2. Definitions, notation and hypotheses

Let $\mathbb{R}^n$ denote the Euclidean n-space. For $x = (x_1, x_2, x_3, \dots, x_n)^T$ in $\mathbb{R}^n$, let $\|x\| = \max\{|x_1|, |x_2|, \dots, |x_n|\}$ be the norm of x. For an $n \times n$ matrix $A = (a_{ij})$, we define the norm $|A| = \sup_{\|x\| \le 1} \|Ax\|$.

In the system (1.1) we assume that A is a continuous $n \times n$ matrix on $\mathbb{R}_+ = [0, \infty)$ and $F : D \times \mathbb{R}^n \to \mathbb{R}^n$, $D = \{(t, s) \in \mathbb{R}^2 : 0 \le s \le t < \infty\}$, is a continuous n-vector such that $F(t, s, 0) = 0$ for $(t, s) \in D$.

Let $\Psi_i : \mathbb{R}_+ \to (0, \infty)$, $i = 1, 2 \dots n$, be continuous functions and

$$\Psi = \operatorname{diag}[\Psi_1, \Psi_2, \dots \Psi_n].$$

Now, we give definitions of various types of Ψ-stability.

Definitions. The trivial solution of (1.1) is said to be Ψ-stable on $\mathbb{R}_+$ if for every $\varepsilon > 0$ and every t_0 in $\mathbb{R}_+$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that any solution $x(t)$ of (1.1) which satisfies the inequality $\|\Psi(t_0)x(t_0)\| < \delta$, also satisfies the inequality $\|\Psi(t)x(t)\| < \varepsilon$ for all $t \ge t_0$.

The trivial solution of (1.1) is said to be Ψ-uniformly stable on $\mathbb{R}_+$ if it is Ψ-stable on $\mathbb{R}_+$ and the above δ is independent of t_0.

Remarks. 1. For $\Psi_i = 1$, $i = 1, 2 \dots n$, we obtain the notions of classical stability and uniform-stability.

2. If in the definitions above, we replace Ψ with Ψ^k, $k \in \mathbb{Z} \setminus \{0, 1\}$, we obtain stability and uniform-stability of degree k with respect to a scalar function Ψ [5].

3. Ψ-stability of linear systems

The purpose of this section is to study conditions for Ψ-(uniform) stability of trivial solution of linear systems. These conditions can be expressed in terms of a fundamental matrix for (1.2).

Theorem 3.1. *Let $Y(t)$ be a fundamental matrix for (1.2). Then*

(a) *The trivial solution of (1.2) is Ψ-stable on $\mathbb{R}_+$ if and only if there exists a positive constant K such that $|\Psi(t)Y(t)| \le K$ for all $t \ge 0$.*
(b) *The trivial solution of (1.2) is Ψ-uniformly stable on $\mathbb{R}_+$ if and only if there exists a positive constant K such that $|\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)| \le K$ for all $0 \le s \le t < \infty$.*

Proof. The solution of (1.2) which takes the value y in $\mathbb{R}^n$ at $a \geq 0$ is $y(t) = Y(t)Y^{-1}(a)y$ for $t \geq 0$.

Suppose first that the trivial solution of (1.2) is Ψ-stable on $\mathbb{R}_+$. Then, for $\varepsilon = 1$ and $t_0 = 0$, there exists $\delta > 0$ such that any solution $y(t)$ of (1.2) which satisfies the inequality $\|\Psi(0)y(0)\| < \delta$, there exists and satisfies the inequality

$$\|\Psi(t)Y(t)(\Psi(0)Y(0))^{-1}\Psi(0)y(0)\| < 1 \quad \text{for } t \geq 0.$$

Let u $\in \mathbb{R}^n$ be such that $\|u\| \leq 1$. If we take $y(0) = \frac{\delta}{2}\Psi^{-1}(0)u$, then we have $\|\Psi(0)y(0)\| < \delta$. Hence, $\|\Psi(t)Y(t)(\Psi(0)Y(0))^{-1}\frac{\delta}{2}u\| < 1$ for $t \geq 0$. Therefore, $|\Psi(t)Y(t)(\Psi(0)Y(0))^{-1}| \leq 2/\delta$ for $t \geq 0$. Hence, $|\Psi(t)Y(t)| \leq K$, a constant, for $t \geq 0$.

Suppose next that $|\Psi(t)Y(t)| \leq K$ for $t \geq 0$. For $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, let $\delta(\varepsilon, t_0) = \varepsilon K^{-1}|(\Psi(t_0)Y(t_0))^{-1}|^{-1}$. For $\|\Psi(t_0)y(t_0)\| < \delta$ and $t \geq t_0$, we have

$$\|\Psi(t)y(t)\| = \|\Psi(t)Y(t)(\Psi(t_0)Y(t_0)^{-1}\Psi(t_0)y(t_0)\| < \varepsilon.$$

Thus, the trivial solution of (1.2) is Ψ-stable on $\mathbb{R}_+$.

Part (b) is proved similarly and omit its proof. The proof is complete. □

Remarks. 1. It is easy to see that if $|\Psi(t)|$ and $|\Psi^{-1}(t)|$ are bounded on $\mathbb{R}_+$, then the Ψ-stability is equivalent with the classical stability.
2. Theorem 3.1 generalizes a similar result for classical stability [7].
3. In the same manner as in classical stability, we can speak about Ψ-(uniform) stability of a linear system (1.2).

Example 3.2. Consider the linear system (1.2) with

$$A(t) = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Then

$$Y(t) = \begin{pmatrix} e^t \sin t & e^t \cos t & 0 \\ -e^t \cos t & e^t \sin t & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}$$

is a fundamental matrix for the system (1.2). Because $Y(t)$ is unbounded on $\mathbb{R}_+$, it follows that the system (1.2) is not stable on $\mathbb{R}_+$. Consider

$$\Psi(t) = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix}.$$

Then, for all $0 \leq s \leq t < \infty$, we have

$$\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s) = \begin{pmatrix} \cos(t-s) & -\sin(t-s) & 0 \\ \sin(t-s) & \cos(t-s) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, the system (1.2) is Ψ-uniformly stable on $\mathbb{R}_+$.

Remark. The introduction of the matrix function Ψ permits us obtain a mixed asymptotic behavior of the components of the solutions.

Theorem 3.3. *Let $Y(t)$ be a fundamental matrix for (1.2). If there exist a continuous function $\varphi : \mathbb{R}_+ \to (0,\infty)$ and the constants $p \geq 1$ and $M > 0$ which fulfil one of the following conditions:*

(i) $\int_0^t \varphi(s)|\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)|^p\,ds \leq M$, *for all* $t \geq 0$

(ii) $\int_0^t \varphi(s)|Y^{-1}(s)\Psi^{-1}(s)\Psi(t)Y(t)|^p\,ds \leq M$, *for all* $t \geq 0$,

then, the system (1.2) is Ψ-stable on $\mathbb{R}_+$.

Proof. For the case (i), first, we consider $p = 1$. Let q(t) = $|\Psi(t)Y(t)|^{-1}$ for $t \geq 0$. From the identity

$$\Big(\int_0^t \varphi(s)q(s)\,ds\Big)\Psi(t)Y(t) = \int_0^t \varphi(s)\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)\Psi(s)Y(s)q(s)\,ds,$$

it follows that

$$\begin{aligned}&\Big(\int_0^t \varphi(s)q(s)\,ds\Big)|\Psi(t)Y(t)|\\ &\leq \int_0^t \varphi(s)|\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)||\Psi(s)Y(s)|q(s)\,ds.\end{aligned}$$

Thus, the scalar function $h(t) = \int_0^t \varphi(s)q(s)\,ds$ satisfies the inequality

$$h(t)q^{-1}(t) \leq M, for t \geq 0\,.$$

We have $h'(t) = \varphi(t)q(t) \geq M^{-1}\varphi(t)h(t)$ for $t \geq 0$. It follows that

$$h(t) \geq h(t_1)e^{M^{-1}\int_{t_1}^t \varphi(s)\,ds}, \quad \text{for } t \geq t_1 > 0$$

and hence

$$|\Psi(t)Y(t)| = q^{-1}(t) \leq Mh^{-1}(t_1)e^{-M^{-1}\int_{t_1}^t \varphi(s)\,ds}, \quad \text{for } t \geq t_1 > 0\,.$$

Because $|\Psi(t)Y(t)|$ is a continuous function on $[0,t_1]$, it follows that there exists a positive constant K such that $|\Psi(t)Y(t)| \leq K$ for $t \geq 0$. Hence, the theorem follows immediately from the Theorem 3.1.

Next, suppose that $p > 1$. Let $r(t) = |\Psi(t)Y(t)|^{-p}$ for $t \geq 0$. In the same manner as above, we have

$$\Big(\int_0^t \varphi(s)r(s)\,ds\Big)|\Psi(t)Y(t)| \leq \int_0^t \varphi(s)|\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)||\Psi(s)Y(s)|r(s)\,ds.$$

Because $\varphi(s)|\Psi(s)Y(s)|r(s) = (\varphi(s))^{1/p}(\varphi(s)r(s))^{1/q}$, where $\frac{1}{p}+\frac{1}{q} = 1$, we have

$$\begin{aligned}&\Big(\int_0^t \varphi(s)r(s)\,ds\Big)|\Psi(t)Y(t)|\\ &\leq \int_0^t (\varphi(s))^{1/p}|\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)|(\varphi(s)r(s))^{1/q}\,ds\,.\end{aligned}$$

Using the Hölder inequality, we obtain

$$\begin{aligned}&\Big(\int_0^t \varphi(s)r(s)\,ds\Big)|\Psi(t)Y(t)|\\ &\leq \Big(\int_0^t \varphi(s)|\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)|^p\,ds\Big)^{1/p}\Big(\int_0^t \varphi(s)r(s)\,ds\Big)^{1/q}, \quad t \geq 0;\end{aligned}$$

or

$$\Big(\int_0^t \varphi(s)r(s)\,ds\Big)|\Psi(t)Y(t)| \le M^{1/p}\Big(\int_0^t \varphi(s)r(s)\,ds\Big)^{1/q}, \quad t \ge 0\,.$$

Thus, the matrix Ψ(t)Y(t) satisfies the inequality

$$|\Psi(t)Y(t)| \le M^{1/p}\Big(\int_0^t \varphi(s)r(s)\,ds\Big)^{-1/p}, \quad \forall t \ge 0\,.$$

Denoting $Q(t) = \int_0^t \varphi(s)r(s)\,ds$ for $t \ge 0$, we obtain

$$|\Psi(t)Y(t)| \le M^{\frac{1}{p}}(Q(t))^{-1/p}, \quad \forall t \ge 0.$$

Because $Q'(t) = \varphi(t)r(t) = \varphi(t)|\Psi(t)Y(t)|^{-p} \ge M^{-1}\varphi(t)Q(t)$, we have

$$Q(t) \ge Q(1)e^{M^{-1}\int_1^t \varphi(s)\,ds}, \quad t \ge 1\,.$$

It follows that

$$|\Psi(t)Y(t)| \le M^{1/p}(Q(1))^{-1/p}e^{-p^{-1}M^{-1}\int_1^t \varphi(s)\,ds}, \quad t \ge 1.$$

Because $|\Psi(t)Y(t)|$ is a continuous function on $[0,1]$, it follows that there exists a positive constant K such that $|\Psi(t)Y(t)| \le K$ for $t \ge 0$. Hence, the theorem follows immediately from the Theorem 3.1.

For case (ii), the proof is similar and we omit it. The proof is complete. □

Remarks. 1. The function φ can serve to weaken the required hypotheses on the fundamental matrix Y.
2. Theorem 3.3 generalizes a result of Dannan and Elaydi [8].
3. In the conditions of the Theorem, the linear system (1.2) can not be Ψ-uniformly stable on $\mathbb{R}_+$. This is shown in [9, Example 2].

Finally, we consider various Ψ-stability problems connected with the linear system

$$x' = (A(t) + B(t))x \tag{3.1}$$

as a perturbed system of (1.2). We seek conditions under which the Ψ-(uniform) stability of (1.2) implies the Ψ-(uniform) stability of (3.1).

Theorem 3.4. *Suppose that B is a continuous $n \times n$ matrix function for $t \ge 0$. If the linear system* (1.2) *is Ψ-uniformly stable on $\mathbb{R}_+$ and*

$$\int_0^\infty |\Psi(t)B(t)\Psi^{-1}(t)|\,dt < +\infty,$$

then the linear system (3.1) *is also Ψ-uniformly stable on $\mathbb{R}_+$.*

Proof. Let $Y(t)$ be a fundamental matrix for the homogeneous system (1.2). Because the system (1.2) is Ψ-uniformly stable on $\mathbb{R}_+$, there exists a positive constant K such that

$$|\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)| \le K \quad \text{for } 0 \le s \le t < +\infty.$$

The solution of (3.1) with initial condition $x(t_0) = x_0$ is unique and defined for all $t \ge 0$. Then it is also a solution of the problem

$$x' = A(t)x + B(t)x, x(t_0) = x_0.$$

Therefore, by the variation of constants formula,

$$x(t) = Y(t)Y^{-1}(t_0)x_0 + \int_{t_0}^t Y(t)Y^{-1}(s)B(s)x(s)\,ds$$

or, for $t, t_0 \geq 0$,

$$\begin{aligned}\Psi(t)x(t) &= \Psi(t)Y(t)Y^{-1}(t_0)\Psi^{-1}(t_0)\Psi(t_0)x_0 \\ &\quad + \int_{t_0}^{t} \Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)\Psi(s)B(s)\Psi^{-1}(s)\Psi(s)x(s)\,ds\,.\end{aligned}$$

From the above conditions, it results that

$$\|\Psi(t)x(t)\| \leq K\|\Psi(t_0)x(t_0)\| + K\int_{t_0}^{t} |\Psi(s)B(s)\Psi^{-1}(s)|\|\Psi(s)x(s)\|\,ds,$$

for $t \geq t_0 \geq 0$. Therefore, by Gronwall's inequality,

$$\|\Psi(t)x(t)\| \leq K\|\Psi(t_0)x(t_0)\|e^{K\int_{t_0}^{t}|\Psi(s)B(s)\Psi^{-1}(s)|\,ds}, \quad \text{for } t \geq t_0\,.$$

Thus, putting $L = \int_0^\infty |\Psi(t)B(t)\Psi^{-1}(t)|\,dt$, we have

$$\|\Psi(t)x(t)\| \leq K\|\Psi(t_0)x(t_0)\|e^{KL}, \quad \text{for all } t \geq t_0 \geq 0.$$

This inequality shows that the system (3.1) is Ψ-uniformly stable on $\mathbb{R}_+$. The proof is complete. □

Remark. The above theorem generalizes a results of Caligo [3], Conti [6] in connection with uniform stability.

If the linear system (1.2) is only Ψ-stable, then the linear system (3.1) can not be Ψ-stable. This is shown by the next example transformed after an example due to Perron [14].

Example 3.5. Let $a \in \mathbb{R}$ be such that $1 \leq 2a < 1 + e^{-\pi}$ and let

$$A(t) = \begin{pmatrix} -a & 0 \\ 0 & \sin\ln(t+1) + \cos\ln(t+1) - 2a \end{pmatrix}$$

Then

$$Y(t) = \begin{pmatrix} e^{-a(t+1)} & 0 \\ 0 & e^{(t+1)[\sin\ln(t+1) - 2a]} \end{pmatrix}.$$

is a fundamental matrix for the homogeneous system (1.2).

Let $\Psi(t) = \begin{pmatrix} e^{a(t+1)} & 0 \\ 0 & 1 \end{pmatrix}$. We have

$$\Psi(t)Y(t) = \begin{pmatrix} 1 & 0 \\ 0 & e^{(t+1)[\sin\ln(t+1) - 2a]} \end{pmatrix}.$$

Because $|\Psi(t)Y(t)|$ is bounded on $\mathbb{R}_+$, it follows that the system (1.2) is Ψ-stable on $\mathbb{R}_+$. For 0 ≤ s ≤ t ¡ ∞, we have

$$|\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)| = \begin{pmatrix} 1 & 0 \\ 0 & e^{f(t)-f(s)} \end{pmatrix},$$

where $f(t) = (t+1)\sin\ln(t+1) - 2at$.

It is easy to see that $\lim_{n\to\infty}[f(t_n e^{\alpha} - 1) - f(t_n - 1)] = \infty$, where $t_n = e^{(8n+1)\frac{\pi}{4}}$ and $\alpha = \arccos\frac{1+e^{-\pi}}{\sqrt{2}}$. Thus, $|\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)|$ is not bounded for $0 \leq s \leq t < \infty$. From Theorem 1, it follows that the system (1.2) is not Ψ-uniformly stable on $\mathbb{R}_+$.

If we take
$$B(t) = \begin{pmatrix} 0 & 0 \\ e^{-a(t+1)} & 0 \end{pmatrix},$$
then
$$Y_1(t) = \begin{pmatrix} e^{-a(t+1)} & 0 \\ e^{(t+1)[\sin\ln(t+1)-2a]}\int_1^{t+1} e^{-s\sin\ln s}\,ds & e^{(t+1)[\sin\ln(t+1)-2a]} \end{pmatrix}$$
is a fundamental matrix for the perturbed system (3.1). We have
$$\Psi(t)Y_1(t) = \begin{pmatrix} 1 & 0 \\ e^{(t+1)[\sin\ln(t+1)-2a]}\int_1^{t+1} e^{-s\sin\ln s}\,ds & e^{(t+1)[\sin\ln(t+1)-2a]} \end{pmatrix}.$$
Let $\alpha \in (0, \pi/2)$ be such that $\cos\alpha > (2a-1)e^{\pi}$. Let $t_n = e^{(2n-\frac{1}{2})\pi}$ for $n = 1, 2 \dots$. For $t_n \le s \le t_n e^{\alpha}$ we have $s\cos\alpha \le -s\sin\ln s \le s$ and hence
$$\begin{aligned}
&e^{t_n e^{\pi}(\sin\ln t_n e^{\pi}-2a)}\int_1^{t_n e^{\pi}} e^{-s\sin\ln s}\,ds \\
&> e^{t_n e^{\pi}(\sin\ln t_n e^{\pi}-2a)}\int_{t_n}^{t_n e^{\alpha}} e^{-s\sin\ln s}\,ds \\
&> e^{t_n e^{\pi}(1-2a)}\int_{t_n}^{t_n e^{\alpha}} e^{s\cos\alpha}\,ds \\
&= e^{t_n[(1-2a)e^{\pi}+\cos\alpha]}\big(e^{t_n(e^{\alpha}-1)\cos\alpha} - 1\big)\cos^{-1}\alpha \to \infty
\end{aligned}$$
This shows that $|\Psi(t)Y_1(t)|$ is unbounded on $\mathbb{R}_+$. It follows that the equation (3.1) is not Ψ-stable on $\mathbb{R}_+$. Finally, we have $\int_0^{\infty} |\Psi(s)B(s)\Psi^{-1}(s)|\,ds < +\infty$.

Also, the Theorem 3 is no longer true if we require that $\Psi(t)B(t)\Psi^{-1}(t) \to 0$ as $t \to \infty$, instead of the condition
$$\int_0^{\infty} |\Psi(s)B(s)\Psi^{-1}(s)|\,ds < +\infty.$$
This is shown by the next example, adapted from an example in Cesari [4].

Example 3.6. Consider the system (1.2) with
$$A(t) = \begin{pmatrix} 0 & 1 \\ -1 & -\frac{2}{t+1} \end{pmatrix}.$$
Then
$$Y(t) = \begin{pmatrix} \frac{\sin(t+1)}{t+1} & \frac{\cos(t+1)}{t+1} \\ \frac{(t+1)\cos(t+1)-\sin(t+1)}{(t+1)^2} & -\frac{(t+1)\sin(t+1)+\cos(t+1)}{(t+1)^2} \end{pmatrix}.$$
is a fundamental matrix for the homogeneous system (1.2).

Let $\Psi(t) = \begin{pmatrix} t+1 & 0 \\ 0 & t+1 \end{pmatrix}$. We have
$$\begin{aligned}
&\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s) \\
&= \begin{pmatrix} \frac{(s+1)\cos(t-s)+\sin(t-s)}{s+1} & \sin(t-s) \\ \frac{(t-s)\cos(t-s)-(ts+t+s+2)\sin(t-s)}{(t+1)(s+1)} & \frac{(t+1)\cos(t-s)-\sin(t-s)}{t+1} \end{pmatrix},
\end{aligned}$$
for $0 \le s \le t < \infty$. It is easy to see that the system (1.2) is Ψ-uniformly stable on $\mathbb{R}_+$.

Now, we consider the system (3.1) with

$$B(t) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{2}{t+1} \end{pmatrix}.$$

Then

$$\widetilde{Y}(t) = \begin{pmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{pmatrix}.$$

is a fundamental matrix for the perturbed system (3.1). We have

$$\Psi(t)\widetilde{Y}(t) = (t+1)\begin{pmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{pmatrix}.$$

It follows that the system (3.1) is not Ψ-(uniformly) stable on $\mathbb{R}_+$. Finally, we have

$$\int_0^\infty |\Psi(s)B(s)\Psi^{-1}(s)|\,ds = +\infty \quad \text{and} \quad \lim_{t\to\infty} |\Psi(t)B(t)\Psi^{-1}(t)| = 0.$$

Theorem 3.7. *Suppose that:*

1. *There exist a continuous function $\varphi : \mathbb{R}_+ \to (0,\infty)$ and a positive constant M such that the fundamental matrix $Y(t)$ of the system* (1.2) *satisfies the condition*
$$\int_0^t \varphi(s)|\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)|\,ds \le M, \quad \forall t \ge 0$$
2. *$B(t)$ is a continuous $n \times n$ matrix function on $\mathbb{R}_+$ such that*
$$\sup_{t\ge 0} \varphi^{-1}(t)|\Psi(t)B(t)\Psi^{-1}(t)|$$
is a sufficiently small number.

Then the linear system (3.1) *is Ψ-stable on $\mathbb{R}_+$.*

Proof. From the first assumption of theorem it follows that there exists a positive constant N such that

$$|\Psi(t)Y(t)| \le N, \quad \forall t \ge 0.$$

The solution of (3.1) with initial condition $x(t_0) = x_0$ is unique and defined for all $t \ge 0$. Then it is also a solution of the problem

$$x' = A(t)x + B(t)x, \quad x(t_0) = x_0.$$

Therefore, by the variation of constants formula,

$$x(t) = Y(t)Y^{-1}(t_0)x_0 + \int_{t_0}^t Y(t)Y^{-1}(s)B(s)x(s)\,ds, \quad t \ge 0\,.$$

Hence,

$$\begin{aligned} \|\Psi(t)x(t)\| &\le \|\Psi(t)Y(t)Y^{-1}(t_0)\Psi^{-1}(t_0)\Psi(t_0)x_0\| \\ &\quad + \int_{t_0}^t \|\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)\Psi(s)B(s)\Psi^{-1}(s)\Psi(s)x(s)\|\,ds, \end{aligned}$$

for all $t \ge t_0$. If we put

$$b = \sup_{t\ge 0} \varphi^{-1}(t)|\Psi(t)B(t)\Psi^{-1}(t)| < M^{-1},$$

then, for $T > t_0$ and $t \in [t_0, T]$, we have

$$\|\Psi(t)x(t)\| \le |\Psi(t)Y(t)||Y^{-1}(t_0)\Psi^{-1}(t_0)|\|\Psi(t_0)x_0\| + Mb \sup_{t_0\le t\le T} \|\Psi(t)x(t)\|.$$

Therefore,

$$\sup_{t_0 \le t \le T} \|\Psi(t)x(t)\| \le (1 - Mb)^{-1} N |Y^{-1}(t_0)\Psi^{-1}(t_0)| \|\Psi(t_0)x_0\|.$$

It follows that the system (3.1) is Ψ-stable on $\mathbb{R}_+$. The proof is complete. □

Remark. We can show that the conclusion of Theorem 4 is valid if the condition

$$\sup_{t \ge 0} \varphi^{-1}(t)|\Psi(t)B(t)\Psi^{-1}(t)| < M^{-1}$$

is replaced with the condition

$$\lim_{t \to \infty} \varphi^{-1}(t)|\Psi(t)B(t)\Psi^{-1}(t)| = 0\,.$$

Theorem 3.7 is no longer true if we require that the system (1.2) be Ψ-(uniformly) stable on $\mathbb{R}_+$ instead of the condition

$$\int_0^t \varphi(s)|\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)|\,ds \le M, \quad \forall t \ge 0\,.$$

This is shown by the next example.

Example 3.8. Consider the system (1.2) with $A(t) = O_2$. Then, a fundamental matrix for the system (1.2) is $Y(t) = I_2$. Consider

$$\Psi(t) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{t+1} \end{pmatrix}.$$

Because

$$\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{s+1}{t+1} \end{pmatrix}$$

is bounded for $0 \le s \le t < +\infty$, it follows that the system (1.2) is Ψ-uniformly stable on $\mathbb{R}_+$. If we take

$$B(t) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{a}{\sqrt{t+1}} \end{pmatrix},$$

where $a > 0$, then

$$\widetilde{Y}(t) = \begin{pmatrix} 1 & 0 \\ 0 & e^{2a\sqrt{t+1}} \end{pmatrix}.$$

is a fundamental matrix for the perturbed system (3.1). Because

$$\Psi(t)\widetilde{Y}(t) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{e^{2a\sqrt{t+1}}}{t+1} \end{pmatrix}$$

is unbounded on $\mathbb{R}_+$, it follows that the perturbed system (3.1) is not Ψ-stable on $\mathbb{R}_+$.

Finally, we have $\sup_{t \ge 0} |\Psi(t)B(t)\Psi^{-1}(t)| = a$ and $\lim_{t \to \infty} |\Psi(t)B(t)\Psi^{-1}(t)| = 0$.

4. Ψ-stability of the nonlinear system (1.1)

The purpose of this section is to study the Ψ-(uniform) stability of trivial solution of (1.1). Now, we state a hypothesis which we shall use in various places.

(H0) For all $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$ and $\rho > 0$, if $\|\Psi(t_0)x_0\| < \rho$, then there exists a unique solution $x(t)$ on $\mathbb{R}_+$ of (1.1) such that $x(t_0) = x_0$ and $\|\Psi(t)x(t)\| \leq \rho$ for all t in $[0, t_0]$.

This is a natural hypothesis in studying Ψ-stability of system (1.1). In [10], this hypothesis is tacitly used in particular case $\Psi = I_n$.

Theorem 4.1. *Assume that Hypothesis (H0) is satisfied. Assume that there exist a continuous function $\varphi : \mathbb{R}_+ \to (0, \infty)$ and a positive constant M such that the fundamental matrix $Y(t)$ of the system (1.2) satisfies the condition*

$$\int_0^t \varphi(s)|\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)|\,ds \leq M, \quad \forall t \geq 0.$$

Also assume that function F satisfies the condition

$$\|\Psi(t)F(t,s,x)\| \leq f(t,s)\|\Psi(s)x\|,$$

for $0 \leq s \leq t < \infty$ and for all x in $\mathbb{R}^n$, where f is a continuous nonnegative function on D such that

$$\sup_{t\geq 0} \int_0^t \frac{f(t,s)}{\varphi(t)}\,ds < \frac{1}{M}.$$

Then, the trivial solution of the system (1.1) is Ψ-stable on $\mathbb{R}_+$.

Proof. From the second assumption of the theorem, it follows that there exists a positive constant N such that

$$|\Psi(t)Y(t)| \leq N, \quad \text{for all } t \geq 0\,.$$

From the third assumption of the theorem, there exists q such that

$$\int_0^t \frac{f(t,s)}{\varphi(t)}\,ds \leq q < \frac{1}{M}, \quad \text{for all } t \geq 0\,.$$

For a given $\varepsilon > 0$ and $t_0 \geq 0$, we choose

$$\delta = \min\{\frac{\varepsilon}{2}, \frac{(1-qM)\varepsilon}{2N|Y^{-1}(t_0)\Psi^{-1}(t_0)|}\}.$$

Let $x_0 \in \mathbb{R}^n$ be such that $\|\Psi(t_0)x_0\| < \delta$.

From the first assumption of the theorem , there exists a unique solution $x(t)$ on $\mathbb{R}_+$ of the system (1.1) such that $x(t_0) = x_0$ and $\|\Psi(t)x(t)\| \leq \delta$ for all $t \in [0, t_0]$. Suppose that there exists $\tau > t_0$ such that

$$\|\Psi(\tau)x(\tau)\| = \varepsilon \quad \text{and} \quad \|\Psi(t)x(t)\| < \varepsilon \quad \text{for } t \in [t_0, \tau).$$

By the classical formula of variation of constants, we have

$$
\begin{aligned}
\|\Psi(\tau)x(\tau)\| &\leq \|\Psi(\tau)Y(\tau)Y^{-1}(t_0)\Psi^{-1}(t_0)\Psi(t_0)x_0\| \\
&\quad + \int_{t_0}^{\tau} |\Psi(\tau)Y(\tau)Y^{-1}(s)\Psi^{-1}(s)| \int_0^s \|\Psi(s)F(s,u,x(u))\|\,du\,ds \\
&\leq N|Y^{-1}(t_0)\Psi^{-1}(t_0)|\delta \\
&\quad + \int_{t_0}^{\tau} \varphi(s)|\Psi(\tau)Y(\tau)Y^{-1}(s)\Psi^{-1}(s)| \int_0^s \frac{f(s,u)}{\varphi(s)}\|\Psi(u)x(u)\|\,du\,ds \\
&\leq N|Y^{-1}(t_0)\Psi^{-1}(t_0)|\delta \\
&\quad + \varepsilon \int_{t_0}^{\tau} \varphi(s)|\Psi(\tau)Y(\tau)Y^{-1}(s)\Psi^{-1}(s)| \int_0^s \frac{f(s,u)}{\varphi(s)}\,du\,ds \\
&\leq N|Y^{-1}(t_0)\Psi^{-1}(t_0)|\delta + \varepsilon q \int_{t_0}^{\tau} \varphi(s)|\Psi(\tau)Y(\tau)Y^{-1}(s)\Psi^{-1}(s)|\,ds \\
&\leq N|Y^{-1}(t_0)\Psi^{-1}(t_0)|\delta + \varepsilon qM \\
&< \varepsilon(1-qM)+\varepsilon qM = \varepsilon,
\end{aligned}
$$

which is a contradiction. Therefore, the trivial solution of system (1.1) is Ψ-stable on $\mathbb{R}_+$. The proof is complete. □

Corollary 4.2. *Suppose that g and h are continuous nonnegative functions on $\mathbb{R}_+$ such that*

$$\sup_{t\geq 0} \frac{g(t)}{\varphi(t)} \int_0^t h(s)\,ds < \frac{1}{M}.$$

Then in Theorem 4.1 we can consider $f(t,s) = g(t)h(s)$.

Corollary 4.3. *Suppose that k is a continuous nonnegative function on $\mathbb{R}_+$ such that*

$$\sup_{t\geq 0} \frac{1}{\varphi(t)} \int_0^t k(u)\,du < \frac{1}{M}.$$

Then in Theorem 4.1 we can consider $f(t,s) = k(t-s)$.

Corollary 4.4. *If in Theorem 4.1, the third condition is replaced by the condition: The function F satisfies: For all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for all $x in$*

$$B_{\delta(\varepsilon)} = \{x \in C_c : \sup_{t\geq 0} \|\Psi(t)x(t)\| \leq \delta(\varepsilon)\}$$

we have

$$\|\Psi(t)F(t,s,x(s))\| \leq \varepsilon f(t,s)\|\Psi(s)x(s)\| \quad for 0 \leq s \leq t < +\infty,$$

where f is a continuous nonnegative function on D such that

$$\sup_{t\geq 0} \int_0^t \frac{f(t,s)}{\varphi(t)}\,ds < +\infty,$$

then the trivial solution of system (1.1) is Ψ-stable on $\mathbb{R}_+$.

The proof of the above corollary is similar to that of Theorem 4.1.

Theorem 4.5. *Assume hypothesis (H0) is satisfied. Assume the function F satisfies*

$$\|\Psi(t)F(t,s,x)\| \leq f(t,s)\|\Psi(s)x\|, \quad \text{for } 0 \leq s \leq t < \infty$$

and for every $x \in \mathbb{R}^n$, where f is a continuous nonnegative function on D such that

$$M = \int_0^\infty \int_0^t f(t,s)\,ds\,dt < \infty.$$

Also assume the fundamental matrix $Y(t)$ of the system (1.2) *is such that*

$$|\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)| \leq K$$

for all $0 \leq s \leq t < +\infty$, where K is a positive constant. Then, the trivial solution of (1.1) *is Ψ-uniformly stable on $\mathbb{R}_+$.*

Proof. Let $\varepsilon > 0$ and $\delta(\varepsilon) = 0.5\varepsilon K^{-1}(1+M)^{-1}e^{-KM}$. Let $t_0 \geq 0$ and $x_0 \in \mathbb{R}^n$ be such that $\|\Psi(t_0)x_0\| < \delta(\varepsilon)$. There exists a unique solution $x(t)$ on $\mathbb{R}_+$ of (1.1) such that $x(t_0) = x_0$ and $\|\Psi(t)x(t)\| \leq \delta(\varepsilon)$ for all $t \in [0, t_0]$. For $t \geq t_0$, we have

$$\begin{aligned}
&\|\Psi(t)x(t)\| \\
&= \|\Psi(t)Y(t)Y^{-1}(t_0)\Psi^{-1}(t_0)\Psi(t_0)x_0 \\
&\quad + \int_{t_0}^t \Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)\int_0^s \Psi(s)F(s,u,x(u))\,du\,ds\| \\
&\leq K\|\Psi(t_0)x_0\| + K\int_{t_0}^t\int_0^s f(s,u)\|\Psi(u)x(u)\|\,du\,ds = K\|\Psi(t_0)x_0\| \\
&\quad + K\int_{t_0}^t\int_0^{t_0} f(s,u)\|\Psi(u)x(u)\|\,du\,ds + K\int_{t_0}^t\int_{t_0}^s f(s,u)\|\Psi(u)x(u)\|\,du\,ds \\
&\leq K\delta(\varepsilon)(1+M) + K\int_{t_0}^t\int_{t_0}^s f(s,u)\|\Psi(u)x(u)\|\,du\,ds.
\end{aligned}$$

It is easy to see that the function $Q(t) = \int_{t_0}^t\int_{t_0}^s f(s,u)\|\Psi(u)x(u)\|\,du\,ds$ is continuously differentiable and increasing on $[t_0, \infty)$. For $t \geq t_0$, we have

$$\begin{aligned}
Q'(t) &= \int_{t_0}^t f(t,u)\|\Psi(u)x(u)\|\,du \\
&\leq \int_{t_0}^t f(t,u)[K\delta(\varepsilon)(1+M) + KQ(u)]\,du \\
&= K\delta(\varepsilon)(1+M)\int_{t_0}^t f(t,u)\,du + K\int_{t_0}^t f(t,u)Q(u)\,du.
\end{aligned}$$

Then

$$\begin{aligned}
&\big[Q(t)\exp\big(-K\int_{t_0}^{t}\int_{t_0}^{s} f(s,u)\,du\,ds\big)\big]' \\
&= \exp\big(-K\int_{t_0}^{t}\int_{t_0}^{s} f(s,u)\,du\,ds\big)\big[Q'(t)-KQ(t)\int_{t_0}^{t} f(t,u)\,du\big] \\
&\le \exp\big(-K\int_{t_0}^{t}\int_{t_0}^{s} f(s,u)\,du\,ds\big) \\
&\quad\times\big[K\delta(\varepsilon)(1+M)\int_{t_0}^{t} f(t,u)\,du + K\int_{t_0}^{t} f(t,u)(Q(u)-Q(t))\,du\big] \\
&\le \exp\big(-K\int_{t_0}^{t}\int_{t_0}^{s} f(s,u)\,du\,ds\big)\big[K\delta(\varepsilon)(1+M)\int_{t_0}^{t} f(t,u)\,du\big] \\
&= \big[-\delta(\varepsilon)(1+M)e^{-K\int_{t_0}^{t}\int_{t_0}^{s} f(s,u)\,du\,ds}\big]'.
\end{aligned}$$

Integrating from t_0 to t ($t \ge t_0$), we have

$$Q(t)e^{-K\int_{t_0}^{t}\int_{t_0}^{s} f(s,u)\,du\,ds} \le \delta(\varepsilon)(1+M)\left[1-e^{-K\int_{t_0}^{t}\int_{t_0}^{s} f(s,u)\,du\,ds}\right].$$

We deduce that

$$\|\Psi(t)x(t)\| \le \delta(\varepsilon)K(1+M)e^{KM} < \varepsilon, \quad \text{for all } t \ge t_0.$$

This proves that the trivial solution of (1.1) is Ψ-uniformly stable on R_+. The proof is complete. □

Corollary 4.6. *Suppose that g and h are continuous nonnegative functions on R_+ such that*

$$\int_0^\infty g(t)\int_0^t h(s)\,ds\,dt < +\infty.$$

Then in Theorem 4.5 we can consider $f(t,s) = g(t)h(s)$.

Remark. Theorem 4.5 generalizes a result of Hara, Yoneyama and Itoh [10].

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Aurel Diamandescu

University of Craiova, Department of Applied Mathematics, 13, "Al. I. Cuza" st., 200585, Craiova, Romania

E-mail address: `adiamandescu@central.ucv.ro`