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CONE-VALUED IMPULSIVE DIFFERENTIAL AND INTEGRODIFFERENTIAL INEQUALITIES

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Abstract. In this paper, we present impulsive analogues of the Gronwall-Bellman inequalities. Conditions for the existence of maximal solutions of some integrodifferential equations are obtained by finding upper bounds for these inequalities. Using monotone iterative techniques and a fixed point theorem, we obtained a priori estimates for the inequalities.

1. introduction

Integral inequalities play crucial roles in the study of qualitative properties of systems particularly in the process of obtaining results involving the existence, uniqueness, boundedness and stability and comparison equations for the solution of systems of differential and integral equations. The most widely encountered inequalities are the Gronwall-Bellman and Pachpatte families and their varieties; such inequalities have found applications in ordinary differential equations (ODEs) (Akinyele [1], Akinyele and Akpan [2], Hale [11]).

In the study of impulsive differential equations inequalities play the same crucial role just like the traditional ordinary differential equations (ODEs) ones. Hence, in the last few years, series of impulsive analogues of the Gronwall-Bellman inequalities have been evolved to study quantitative and qualitative properties of impulsive differential equations (Oyelami [17, 18], Oyelami et al. [15, 16], Bainov and Svetla [7], Bainov and Stamova [6])

In this paper, some new inequalities are proposed with potential applications in impulsive ordinary differential equations (IODEs), impulsive control systems (ICS) and impulsive partial differential equations (IPDEs) which are still in cradle of development. The existing inequalities in (Bainov and Svetla [7]; Lakshimikantham et.al. citel1) are special cases of these inequalities.

Furthermore, by means of monotone iterative technique couple with a fixed point theorem of Guo and Lakshimikantham [9], we obtain results on existence of the maxima solutions of the impulsive equation for the solution which gives the upper bound for the inequality .Some special cases of the inequalities were used in Ale et al. [4] to obtain some biological policies on normal-malignant cancer model using the Gronwall-Bellman's kind of impulsive inequality.

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2. Preliminaries, Notation and Definitions

Let $C(\mathbb{R}^+,\mathbb{R}^n)$ be the space of continuous functions in $\mathbb{R}^+ = [0,+\infty)$ and taking values in $\mathbb{R}^n$. Let $C_0^1(\mathbb{R}^+,\mathbb{R}^n)$ be the space of continuously differentiable and bounded functions on $\mathbb{R}^+$ taking values in the n-dimensional Euclidean space $\mathbb{R}^n$. Let

$$PC(\mathbb{R}^+,\mathbb{R}^n) = \big\{y(t) : y(t) \in C(\mathbb{R}^+ \setminus \{t_k\},\mathbb{R}^n),\ k = 1,2\dots, \\ \lim_{t\to t_k+0} y(t)\text{exists and equals } y(t_k)\big\}\,.$$

Definition (cones). Let $\mathbb{R}^n$ be n-dimensional Euclidean space. A non-empty set $E \subset \mathbb{R}^n$ is said to be a cone if and only if it satisfies the Following conditions:

1. If there exist sequences $(x_n, y_n) \subset E$, $n \in N = \{1, 2, \dots\}$ such that $x_n \to x$, $y_n \to y$ as $n \to \infty$. Then $\alpha x + \beta y \in E$, where α and β are constants;
2. If $x \in E$ then $\alpha x \in E$ for all $\alpha \geq 0$;
3. If $x, -x \in E$ then $\{x\} \cap \{-x\} = \{\phi\}$, where ϕ is the zero element of the cone E.

Let the specializing ordering on E be $x \leq_E y$ if $x - y$ is in E; which reads y weakly specializes x. Also let $x \leq_0 y$ if $y - x$ is in $\operatorname{int} E = E \setminus \{\phi\}$; which reads y strongly specializes x.

Let $M_n(E)$ be nxn symmetric matrices define on the cone and let $M_n^*(E)$ denote its transpose. We introduce the generalized inner product on E as follows:

Definition (inner products on cones). For $X(t) \in M_n'(PC(R, E))$ and $B(t) \in M_n'(E)$, let

$$\langle B(t), X(t)\rangle_E = \int_{t_0}^{t} B^*(s)X(s)ds\,.$$

For the impulse set $Q_k = \{t_k \in \mathbb{R}^+ : t_0 < t_k < t,\ t \in \mathbb{R}^+,\ k = 1, 2, 3, \dots\}$, the inner product is

$$\langle B(t), X(t_k)\rangle_E = \sum_{t_0<t_k<t} B^*(t_k)X(t_k)$$

where $M_n(PC(\mathbb{R}^+, E))$ is the set of nxn symmetric matrices whose elements are in $PC(\mathbb{R}^+, E)$ and $*$ denotes the transpose of the matrix.

Clearly, $\langle ., .\rangle_E$ satisfies the following properties:

1. $\langle x, y\rangle = \phi$ for any x, y in E. $\langle x, y\rangle = \phi$ if $x = y$, where ϕ is zero element of the cone E.
2. $\langle \lambda x + y, z\rangle_E = \lambda^* \langle x, y\rangle_E + \langle y, z\rangle_E$
3. $\langle x, \mu y + z\rangle_E = \mu\langle x, y\rangle_E + \langle x, z\rangle_E$,

where λ and μ are complex numbers and λ^* is the complex conjugate of λ.

Remark. If $x \in E$ then $\langle x, x\rangle_E = |x|_E$ defines the generalized norm on the cone E. Where

$$|x|_E = (|x_1|, |x_2|, |x_3|, \dots, |x_n|), \quad x = (x_1, x_2, x_{3,\dots,}x_n).$$

It must be emphasize that the classical norm is a real number, whereas, the generalized norm is a vector.

Definition (adjoint cone). A cone $E^* = \{y \in \mathbb{R}^n : \langle y, x\rangle \geq_E \phi, x \in E\} \subset \mathbb{R}^n$ is defined to be adjoint cone relative to the cone E.

The set $E^A = \{y \in E : \langle y, x\rangle = \phi, x \in E\}$ is an annihilator of E. While $\langle ., .\rangle$ is the generalized inner product on E.

Remark. A necessary and sufficient condition for a point to be an annihilator of E is that $x \in E$ for some $y \in \operatorname{int} E^A$, $\operatorname{int} E = E \setminus \{\phi\}$.

Definition (normal cone). A cone E is normal if there exists a constant $m > 0$ such that $|f| \leq_E m|g|$ for any $f, g \in E$ with $0 \leq_E f \leq_E g$.

Remark. It is not difficult to show that E is normal if and only if ll $\delta > 0$ such that $|f + g| > \delta$ for $f, g \in E$ with $|f| = |g| = 1$ where $|.|$ is the Euclidean norm inherited by the cone E.

Examples of Cones. The set

$$R^n_+ = \{u \in \mathbb{R}^n : u_i \geq 0, \quad i = 1, 2, \dots n, u = (u_{1,} u_2, \dots, u_n)\}$$

is a cone. The set of non-negative functions in $L_p(0, 1)$ with $p \geq 1$ is a cone. The set of non-negative definite matrices $M_n(R^+)$ is a cone. The set of monotone operators on any arbitrary Banach space is a cone. For further exposition on concept of abstract cones (see Huston and Pym [13], Guo and Lakshimikantham [9], Akinyele and Akpan [2], Guo and Liu [10], Akpan [3]).

Definition (order intervals). The order interval in the cone E can be define as

$$[U_0, V_0] = \{U(t) \in E : U_0 \leq U(t) \leq V_0(t), t \in \mathbb{R}^+\}.$$

For the rest of this paper we use the following notation: $M^+_n(E)$ is the set of $n \times n$ matrices defined on the cone E.
$PC(\mathbb{R}^+, E)$ is the subclass of $PC(\mathbb{R}^+, \mathbb{R}^n)$ where the values of $PC(\mathbb{R}^+, \mathbb{R}^n)$ is in the cone $E \subset \mathbb{R}^n$
$M(.)$ Denotes the measure of non-compactness of (.)
$L^1(\mathbb{R}^+ X \mathbb{R}^+ X E, E)$ is the space of absolutely integrable functions on $\mathbb{R}^+ X \mathbb{R}^+ X E$ and taking values in the cone E.

Definition. Let X be a Banach space. Denote by $CO\bar{\Omega}$ the convex hull of the set $\Omega \subset X$, $\bar{\Omega}$ is the closure of Ω and $\partial\Omega$ is the boundary of Ω. To each bounded set $Y \subset E \subseteq \Omega$, and associate the nonnegative number $\Psi(Y)$. The function defined this way is called a measure of non-compactness of the set Y if the following conditions are satisfied:

(a) $\Psi(\overline{COY}) = \Psi(Y)$
(b) If $Y_1 \subset Y_2 \in \Omega$ then $\Psi(Y) \leq \Psi(Y_2)$

Definition. The continuous and bounded operator A define on Ω is called Ψ-condensing if for a noncompact set $Y \subset \Omega$, $\Psi(Y_1) \leq \Psi(Y)$ for every $Y_1 \subset \Omega$.

Definition (set contractive). A map $A : Dom(A) \to R(A)$, from its domain $Dom(A)$ to its range$R(A)$, is said to be strict set contractive, if it is bounded, continuous and there exists a constant $\gamma > 0$ such that $M(A(Q)) \leq \gamma M(Q)$, where $M(.)$ denotes the Kuratowski's measure of non-compactness.

We introduce the concepts of measure of non-compactness and condensing maps due to Krasnose'skii, Zabreiko and Sadovskii (see Bainov and Kazakova [5]). Many types of measure of non-compactness have been defined by different academicians and employed to study qualitative properties of solutions of varieties of dynamical systems (see Hu et al[12]; Deimling [8], Rzezuchowski [21]; Guo and Liu [10]). Concepts of measure compactness of operator has a fundamental advantage of estimating a priori bonds without undergoing laborious estimation.

3. Statement of the problem

Consider the impulsive inequality

$$\begin{gathered} u(t) \leq f(t, u(t)) + W(t, \int_{t_0}^{t} G(t, s, u(s))ds), \quad t \neq t_k,\ k = 1, 2, 3, \dots \\ \Delta u(t = t_k) \leq I(u(t_k)) \\ u(\phi) \leq v_0 \,. \end{gathered} \tag{3.1}$$

For an increasing sequence of times, $0 < t_0 < t_2 < t_3 < \dots < t_k$, with $\lim_{t\to\infty} t_k = +\infty$. Where $f : \mathbb{R}^+ XPC(\mathbb{R}^+, E) \to E$, $W : \mathbb{R}^+ XL^1(\mathbb{R}^+ X\mathbb{R}^+ XE, E) \to E$ and $I : E \to E$.

Before the stage is set up for carrying out our investigations, we will assume that the following conditions:

- (A1) $f(t, u(t))$ is continuous on E and Lipschitzian with respect to the second variable.
- (A2) $W(t, \int_{t_0}^{t} G(t, s, u(s))ds)$ is a nonnegative definite matrix on E and Lipschizian with respect to the second argument. The function $G(t, s, u(t))$ is in $C_0(\mathbb{R}^+ X\mathbb{R}^+ XE, E)$ and there exists a constant $k > 0$ such that
$$|G(t, s, u_2(t)) - G(t, s, u_1(t)))| \leq_E k|u_2(t) - u_1(t)|$$
for $u_2(t), u_1(t) \in E$.
- (A3) $I(.)$ is continuous on $(.)$ and $I(\phi)$.

4. Main results

Consider the impulsive analogue of the Gronwall-Bellman inequalities defined on the cone $\mathbb{R}^+$:.

Lemma 4.1. *Let $u(t) \in PC(\mathbb{R}^+, \mathbb{R}^+)$, $\beta_k(t) \in PC(\mathbb{R}^+, \mathbb{R}^+)$ and $\gamma(t) \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $C \geq 0$ be a nonnegative constant such that*

$$u(t) \leq C + \int_{t_0}^{t} \gamma(s)u(s)ds + \sum_{t_0 < t_k < t} \beta_k(t_k)u(t_k), \quad k = 1, 2, \dots \tag{4.1}$$

Then

$$u(t) \leq C \prod_{j=i}^{k-1} (1 + \beta_j(t_j) \exp\big(\int_{t_{j-1}}^{t_j} \gamma(s)ds\big) \exp\big(\int_{t_j}^{t} \gamma(s)ds\big) \tag{4.2}$$

Proof. If $t \in (t_j t_{j+1})$, then the proof reduces to the classical continuous Gronwall-Bellman inequality (Hale [11]). If $t \notin (t_j t_{j+1})$, $j = 1, 2, \dots$ i.e. $t = t_j$, then

$$u(t_1) = u(t_1 + 0) \leq C + \int_{t_0+0}^{t_1} \gamma(s)u(s)ds \leq C \exp(\int_{t_0}^{t} \gamma(s)ds))$$

similarly,

$$\begin{aligned} u(t_2) &\leq C + \int_{t_0+0}^{t_2} \gamma(s)u(s)ds + \beta_1 u(t_1) \\ &\leq (1 + \beta_1(t_1))C \exp\big(\int_{t_0}^{t_1} \gamma(s)u(s)ds\big) \exp\big(\int_{t_1}^{t_2} \gamma(s)ds\big) \end{aligned}$$

and

$$\begin{aligned} u(t_3) &\le C + \int_{t_0}^{t} \gamma(s)u(s)ds + \beta_1(t_1)u(t_1) + \beta_2(t_2)u(t_2) \\ &\le (1+\beta_1(t_1))(1+\beta_2(t_2))C \exp\big(\int_{t_0}^{t}\gamma(s)ds\big) \\ &\quad \times \exp(\int_{t_1}^{t_2}\gamma(s)ds\big)\exp\big(\int_{t_1}^{t_2}\gamma(s)ds\big) \end{aligned}$$

Thus, by induction on $k \ge 4$,

$$u(t) \le \prod_{j=1}^{k-1}(1+\beta_k(t_k)C)\exp\big(\int_{t_j-1}^{t_j} gamma(s)ds\big)\exp\big(\int_{t_j}^{t}\gamma(s)ds\big)\,. \tag{4.3}$$

□

Now we consider another version of the above inequality in a generalized form:

Lemma 4.2. *Let the hypothesis in Lemma 4.1 be satisfied and let $\delta(t) \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $\tau_k(t_k) \in PC(\mathbb{R}^+, \mathbb{R}^+)$ be such that*

$$u(t) \le C(t) + \int_{t_0}^{t}\gamma(s)u(s)ds + \int_{t}^{\infty}\delta(s)u(s)ds + \sum_{t_0<t_k<t}(\beta_k(t_k)+\tau_k(t_k))u(t_k). \tag{4.4}$$

Then

$$u(t) \le \prod_{j=1}^{k-1}(1+\alpha_j(t_j))C(t_j)\exp\big(\int_{t_{j-1}}^{t_j}\gamma(s)ds\big)\exp\big(\int_{t_{j-1}}^{t_j}\gamma(s)ds\big)\exp\big(\int_{t}^{\infty}\delta(s)ds\big) \tag{4.5}$$

where $\alpha_k(t) := \tau_k(t) + \beta_k(t)$, $k = 1, 2, \dots$

Proof. By Lemma 4.1

$$\begin{aligned} u(t) &\le C(t) + \int_{t_0}^{t}\gamma(s)u(s)ds + \int_{t}^{\infty}\delta(s)u(s)ds + \sum_{t_0<t_k<t}(\alpha_k(t_k))u(t_k) \\ &\le A(t) + \int_{t}^{\infty}\delta(s)u(s)ds, \end{aligned}$$

where

$$A(t) = \prod_{j=1}^{k-1}(1+\alpha_j(t_j))C(t_j)\exp\big(\int_{t_{j-1}}^{t_j}\gamma(s)ds\big)\exp\big(\int_{t_{j-1}}^{t_j}\gamma(s)ds\big)\,.$$

Hence

$$u(t) \le A(t)\exp\big(\int_{t}^{\infty}\delta(s)ds\big). \tag{4.6}$$

□

Remark. Lemma 4.2 is a particular case of the inequality in (Lashimikamtham, et. al.,citel1) where $\delta_1 = C(t) = P(t) = 1$, $\gamma(t) = P(t)V(t)$, $\delta(t) = 0, \alpha(t) = \beta$, in which the inequality was stated without proof, whereas, Lemma 4.2 is a generalization of the same inequality when $\delta_1, = C(t) = P(t) = 1$, $\gamma(t) = P(t)V(t)$, $\delta(t) = 0$, $\alpha(t) = \beta =$ constant.

Remark. If $i(t_0,t)$ is the number of points of t_k present in the interval (t_0,t_{k+1}), $k=0,1,2,\dots$. Then Lemma 4.1 is the generalization a of the Bainov-Svetla's inequality [7, Lemma 2] whenever $\gamma=\gamma(t)$is a constant, $\beta=\beta(t)$ is also a constant.

For the next theorem, we set the following:

- (H1) $(\alpha L_1+L_2)\sum_{i=1}^n[\operatorname{diam}(\beta(t_i)a_i)<1$, $\alpha:=\max|t-t_0|, t\in\mathbb{R}^+, t\geq t_0$
- (H2) There exist constants $0<L$such that $\langle B(t_k),\eta(t_i)>\geq_E -L\eta(t_i)$ for some $\eta(t)\in E$.
- (H3) $G(t,s,.)$ is a nonnegative definite and monotonic nondecreasing function with respect to second variable such that exists a constant $p>0$ such that
$$W(t,\int_{t_0}^t G(s,t,\eta_1(s))ds)-W(t,\int_{t_0}^t G(s,t,\eta_2(s))ds)\geq -p(\eta_1(t)-\eta_2(t))$$
- (H4)
$$u(t)\leq H(t)+W(t,\int_{t_0}^t G(s,t,u(s))ds)+<B(t),X(t)>_{t=t_k}$$

Theorem 4.3. *Assume $u(t)\in PC(\mathbb{R}^+,E)$, $H(t)\in M_n'(E),\beta(t)\in M_n'(E)$ and $W\in C^1(\mathbb{R}^+XE,E)$ and that (H1)–(H4) are satisfied. Then*
$$u(t)\leq H(t)+W(t,u^*(t)) \tag{4.7}$$
where $u^*(t)$ is the maximal solution of the impulsive integral equation
$$u(t)=\int_{t_0}^t G(t,s,u(s))ds+\langle B(t),u(t)\rangle_{t=t_k},\quad k=1,2,\dots \tag{4.8}$$

Proof. The strategy is to show that the solution of the integral equation in (4.8) exists in a normal cone in an order interval containing the cone E. Moreover, $u^*(t)$ is the maximal solution of (4.8) and satisfies the inequality (4.7). Now define
$$A_1u(t)=\int_{t_0}^t G(t,s,u(s))ds$$
$$A_2u(t)=\langle\beta(t_i),u(t_i)\rangle_{i=1,2,3,\dots}.$$
Let $A=A_1+A_2$ be such that
$$A:\operatorname{Dom}(A_1)\cup\operatorname{Dom}(A_2)\supset[U_0,V_0]\to PC(\mathbb{R}^+,E)$$
and $M(A_1(Q))=\sup_{t\in\mathbb{R}^+}M(A_1(Q(t))$, where
$$Q(t)\in\operatorname{Dom}(A_1)=\{u(t)\in PC(\mathbb{R}^+,E):\Big|\int_{t_0}^t G(t,s,u(s))ds\Big|<\infty\}$$
similarly
$$Q(t_i)\in\operatorname{Dom}(A)=\{u(t_i):|\langle B(t_i),u(t_i)\rangle|<+\infty,\ i=1,2,\dots\}.$$
By [10, Lemma 1] it follows easily that there exist constants L_1 and L_2 such that
$$M(A_1(Q(t_i)))\leq\alpha L_1M(Q(t)),$$
$$M(A_2(Q(t_i)))\leq L_2M(Q(t))+\epsilon$$
For some arbitrary small positive number ϵ and since $t_0<t_i<t$ for $i=1,2,\dots$, it implies that
$$M(A_1(Q(t_i))\leq(\alpha L_1+L_2)M(Q(t))+L_2\epsilon$$

But

$$M(Q(t_i)) \le M(B(t_i))M(u(t_i)) \le \sum_{i=1}^{n} \operatorname{diam} B(t_i)M(u(t_i))a_i$$

For some constants a_i, $i = 1, 2, \dots$. Hence $M(A(u(t_i)) \le \gamma M(u(t_i))$. Since ϵ is arbitrarily small, A is strictly set contractive on $[U_0, V_0]$.

Step II Next it will be shown that A has a fixed point in $[U_0, V_0]$ which is in fact the maximal solution of equation (4.9) below. Let $u_n(t) \to u_+(t)$ as $n \to \infty$. Now define

$$Au_{n-1} = \int_{t_0}^{t} G(t, s, u_n(s))ds + \langle B(t_i), u(t_i)\rangle . \tag{4.9}$$

Suppose that η is the solution of equation (4.9). Then $A\eta = \eta$ which is a fixed point of A. For $u_0 \le \eta_1 \le \eta_2 \le V_0$, we have

$$\begin{aligned}
\eta &= \eta_2(t) - \eta_1(t) \\
&\ge W(t, \int_{t_0}^{t} G(t, s, \eta_2(s)ds) - W(t, \int_{t_0}^{t} G(t, s, \eta_1(s)ds) + \langle B(t_i), \eta_2(t_i) - \eta_2(t_i)\rangle \\
&\ge L\eta - L\eta = \phi.
\end{aligned}$$

Therefore, $A\eta_2 - A\eta_1 \ge_E \phi$ i.e. $A\eta_2 \ge_E A\eta_1$ for $\eta_2(t_i) \ge_E \eta_1(t_i)$ Hence A is nondecreasing and strictly set contractive from $[U_0, V_0] \to PC(\mathbb{R}^+, E)$. Since $u_0 \le u_0(t)$ and

$$Au_0 \le Au_0(t) = \int_{t_0}^{t} G(t, s, u_0(s))ds + \langle B(t_i), u_0(t_i)\rangle = u_0(t).$$

By [10, Theorem 1.2.1] there exists, a maximal fixed point $u(t)$ of A in $[U_0, V_0]$ such that $u_n(t) = Au_{n-1}(t)$ and satisfies the condition $u_0 \le u_1 \le \dots \le u_n \le u^*$, where

$$Au_{n-1}(t) = \int_{t_0}^{t} G(t, s, u_n(t)ds + \langle B(t_k), u_n(t_k)\rangle, \tag{4.10}$$

$u^*(t)$ is the maximal solution of the integral equation

$$u(t) = \int_{t_0}^{t} G(t, s, u(t)ds + \langle B(t_k)u(t_k)\rangle$$

existing in $[U_0, V_0]$. Hence $u(t) \le H(t) + W(t, u^*(t))$. $\square$

Corollary 4.4. *Under the conditions of Theorem 4.3, replace* (4.7) *by*

$$u(t) \le H(t) + W(t, \int_{t_0}^{t} G(t, s, u(s))ds) + \langle B(t_k), u(t_k)\rangle .$$

Then

$$u(t) \le H(t) + W(t, u^*(t)) + \langle B(t_k), u^*(t_k)\rangle, \tag{4.11}$$

where $u^*(t)$ is the maximal solution of the impulsive integral equation

$$u(t) = \int_{t_0}^{t} G(t, s, u(s)ds) + \langle B(t_k), u(t_k)\rangle \tag{4.12}$$

existing in $[U_0, V_0]$.

For the proof of this corollary, just set $\gamma = \alpha\_1 < 1$ and $P_1 = 0$, as in Theorem 4.3.
Remark. If $W = \int_{t_0}^t \gamma(s)u(s)ds$, H is constant, $B(t) = B_k(t)$ and $n = 1$, then Corollary 4 is a generalization of our Lemma 4.1 and Corollary 4.4 is a generalization of Lemma 4.2. On the other hand, if $\langle B(t_k), u(t_k)\rangle = \phi$. Then theorem 4.3 and Corollary 4.4 are generalizations of [1, Theorem 1].

Theorem 4.5. *Under the conditions of Theorem 4.3 assume that*

$$\frac{du(t)}{dt} \leq u(t)H(t) + W(t, \int_{t_0}^t G(t, s, u(s)ds)) + \langle B(t_k), u(t_k)\rangle\,. \tag{4.13}$$

Then $u(t) \leq A_*^{-1}(t)[u_0 + u^*(t)]$, *where*

$$A(t) = \exp\big(\int_{t_0}^t H(s)ds\big), \quad t \geq t_0,\ H(t) \in E$$

and $u^*(t)$ *is the maximal solution of the integral equation in equation* (4.13),

$$u(t) = \int_{t_0}^t A_*^{-1}(\tau)[W(\tau, \int_{t_0}^t G(\tau, s, u(s))ds) + Y(\tau, t_k)]dt$$

The constant γ *is replace by*

$$\gamma := A_0 u_{0'} + A_0\alpha L_2 + \sum_{i-1}^n k_i \operatorname{diam} B(t_i) < 1$$

where k_i *are arbitrary constants which are assumed to exist, and* $u_{0'} = \max|u_0|_E$.

of Theorem 4.5, assume that $F(t, u(t)) \in PC(\mathbb{R}^+ XE, E)$ and is measurable, begincorollary Under the condition monotonically nondecreasing and Lipschitz with respect to the second variable.

Also assume that $W_2 : \mathbb{R}^+ XC(\mathbb{R}^+, E)XL^1(\mathbb{R}^+ X\mathbb{R}^+ XE, E) \to E$ is measurable, monotonically nondecreasing and Lipshitz with respect to the third variable such that

$$\begin{aligned}
\frac{du(t)}{dt} &\leq A(t)H(t)u(t) + F(t, u(t)) \\
&\quad + \int_{t_0}^t d\tau H(\tau)A(\tau)W(t, F(\tau, u(\tau)), \int_{t_0}^t G(t, s, H(s)A(s)u(s))ds) \\
&\quad + \int_{t_0}^t d\tau H(\tau)A(\tau)W_2(t, F(\tau, u(\tau)), \int_{t_0}^t G(t, s, H(s)A(s)u(s))ds),
\end{aligned}$$

$u(t = t_k) \leq \langle B(t_k), u(t_k)\rangle_{k=1,2,3\dots}$. Then

$$u(t) \leq D(t)[u_0 + u^*(t) + F(t, u^*(t)] + \langle D(t_k)B(t_k), u^*(t_k)\rangle_{k=1,2,3\dots},$$

where

$$\begin{aligned}
u(t) = &\int_{t_0}^t d\tau Z(t,\tau)W(t, \int_{t_0}^t G(t, s, H(s)A(s)u(s))ds) \\
&+ \int_{t_0}^t d\tau Z(\tau)W_2(t, F(\tau, u(\tau)), \int_{t_0}^t G(t, s, H(s)A(s)u(s))ds)\,.
\end{aligned} \tag{4.14}$$

The function $Z(t,\tau)$ is define as

$$Z(t,\tau) = \begin{cases} D(t)H(t)A(\tau) & t \geq \tau \geq 0 \\ \phi & \tau < t < 0,\ t,\tau \in \mathbb{R}^+ \end{cases}$$

and has the properties that $Z(0,\tau) = H(\tau)A(\tau), Z(0,0) = I$,

$$D(t) = \exp\big(\int_{t_0}^{t} H(\tau)A(\tau)d\tau\big), \quad D(t) \in M_{n'}(E),$$

and I is identity matrix.

Proof. By contradiction, let $(u_n)_{n\in N}$ be a monotonically nondecreasing sequence in E such that $u_n \to u*$ as $n \to \infty$. Let $u^*(t)$ be the maximal solution of (4.14) such that

$$u(t) > D(t)[u_0 + u^*(t) + F(t,u^*(t))] + \langle D(t_k)B(t_k), u^*(t_k)\rangle] \,. \tag{4.15}$$

We show that there does not exist a function $u(t)$ in $[U_0, V_0]$ such that $u(t) > u^*(t) \geq u_*(t)$, otherwise $u^*(t)$ would cease to be maximal. Let

$$\begin{aligned} Au_n(t) = &\int_{t_0}^{t} d\tau Z(t,\tau)W(t, \int_{t_0}^{t} G(t,s,H(s)A_*(s)u_n(s))ds) \\ &+ \int_{t_0}^{t} d\tau Z(t,\tau)W_2(t, F(t,u_n(t), \int_{t_0}^{t} G(t,s,H(s)A_*(s)u_n(s))ds). \end{aligned} \tag{4.16}$$

Then $Au_0 \geq u_0 \geq u_0$, $Av_0 \leq v_0$ and the operator A is a set contraction if

$$\gamma = \alpha Z_0(h_0A_0L_1 + \alpha A_0h_0L_2) < 1, \quad Z_0 = \max_{\tau,t} \in \mathbb{R}^+|Z(t,\tau)|,$$
$$A_0 = \max_{\tau}|A_*(t)|, \quad h_0 = \max_{\tau}|H(\tau)|,$$

and L_1, L_2 are constants.

Hence by Theorem 4.3, there exists a maximal solution $u^*(t)$ of (4.16) in $[U_0, V_0]$ which is in fact the fixed point of A.

Now let

$$\delta_* = u_0 + F(t,u^*(t)) + \langle D(t_k)B(t_k)u^*(t_k)\rangle$$

then $\delta_*(t) \in E$ such that

$$u(t) > D(t)[u^*(t) + \delta_*(t)],$$
$$\psi(u(t)_*\delta_*(t)) > u^*(t),$$

where $\psi(u(t)_*\delta_*(t)) - (D^{-1}(t)u(t) - \delta_*(t)) \in [u_0 - \delta_{(*}, v_0 - \delta_*] \subseteq [U_0, V_0]$. But $u^*(t)$ is maximal. Hence there does not exist an element $\psi(u(t)_*\delta_*(t))$ in the other interval$[U_0, V_0]$ such that equation (4.15) is satisfied which is a contradiction. Hence,

$$u(t) \leq D(t)[u_0 + F(t,u^*(t)) + u^*(t)]$$

□

Theorem 4.6. *Under the conditions of Theorem 4.5, assume following conditions:*

$$\begin{aligned}\frac{du(t)}{dt} &\le A_*(t)H(t)u(t) + F(t,u(t))\\ &\quad + \int_{t_0}^t A_*(\tau)H(\tau)W(t, \int_{t_0}^t G(t,s,A(s)H(s)u(s)ds)d\tau)\\ &\quad + \int_{t_0}^t A_*(\tau)H(\tau)W_2(t,F(\tau,u(t))\int_{t_0}^t G(t,s,u(s)ds)d\tau)\end{aligned}$$

$\triangle u(t=t_k) \le \langle B(t_k), u(t_k)\rangle_{k=1,2,3\dots}$*; and the commutant satisfies*

$$[A(t),H(t)] = A_*^*(t)H.(t) - H^*(t)A*(t) = \phi,$$
$$\det(H(t)A_*^*(t)A_*(t)H(t) > 0\,.$$

Also assume that

$$F(t,u(t)) = O(|u(t)|_E), \tag{4.17}$$

$$\lim_{|x(t)|_E\to\phi} \frac{|W_2(.,.,x(t))|_E}{|x(t)|_E} = \phi, \tag{4.18}$$

$$\lim_{|y(t)|_E\to\phi} \frac{|W(.,Y(t))|_E}{|Y(t)|_E} = \phi \tag{4.19}$$

For $y(t) \in L^{'}(E)$*. Then*

$$u(t) < D(t)[u_0 + F(t,u^*(t) + u^*(t)] + \langle D(t_k)B(t_k), u^*(t_k)\rangle, \tag{4.20}$$

where $u^*(t)$ *is the maximal solution of the integral solution*

$$\begin{aligned}u(t) &= \int_{t_0}^t D(\tau)A_*(\tau)W(t, \int_{t_0}^t G(t,s,A_*(s))H(s)u(s)ds)d\tau\\ &\quad + \int_{t_0}^t D(\tau)A_*(\tau)W(t,F(\tau,u(\tau)), \int_{t_0}^t G(t,s,u(s))ds)\end{aligned} \tag{4.21}$$

and

$$D(\tau) = \exp\big(\int_{t_0}^t A_*(s)H(s)ds\big), \quad A_*(t) = \exp\big(\int_{t_0}^t H(s)ds\big)$$

The proof of the above theorem follows from Corollary 4 and Theorem 4.5. We will like to emphasize that the conditions (4.17)–(4.19) do not allow the quantity to be unbounded below and as a consequence of Theorem 4.5; we have the existence of a maximal solution to (4.21) in a normal cone $K \subseteq [U_0, V_0]$.

5. Applications

Example 5.1. Consider a nonlinear impulsive control system (NLICS)

$$\begin{gathered}\frac{dx(t)}{dt} = -xe^{-xy} + f(t,x(t),y(t),u_1(t)), \quad t \ne t_k,\ k = 0,1,2,\dots,\\ \frac{dy(t)}{dt} = y\sin(xy) + g(t,x(t),y(t),u_2(t)), \quad t \ne t_k,\ k = 0,1,2,\dots,\\ \Delta x = \frac{\beta_k^1 x^2(t_k)}{1+x(t_k)}, \quad t = t_k,\ k = 01,2,\dots,\\ \Delta y = \frac{\beta_k^1 y_k^2(t_k)}{1+y(t_k)}, \quad t = t_k,\ k = 01,2,\dots,\end{gathered}$$

where $0 < t_0 < t_2 < \dots < t_k$, $\lim u_{k\to\infty} t_k = +\infty$, $x(0) = x_0$, $y(0) = y_0$, $0 \le x \le \frac{\pi}{2}$ and $0 \le y \le \frac{\pi}{2}$. Also $f : \mathbb{R}^+ X\mathbb{R}^+ XC \to \mathbb{R}^+$, $g : \mathbb{R}^+ X\mathbb{R}^+ X\mathbb{R}^+ XC \to \mathbb{R}^+$, $u : \mathbb{R}^+ \to C$, $u(t)$ is the control variable belonging to the control space

$$C = \{u(t) = (u_1(t), u_2(t) : 0 \le 1, t \in \mathbb{R}^+\} \subset mathbbR^+.$$

To investigate the boundedness or stability property of the above nonlinear control inequality, we often use the comparison equation. In this particular problem, $e^{-xy} \le 1$ for all $x, y \in \mathbb{R}^+$ and $\sin(xy) \le 1$ for the given of x and $y\frac{z}{1+z} \le 1$ for every $z \ge 0$. Then the nonlinear control inequality

$$\begin{gathered}
\frac{dx}{dt} \le -x + f(t, x(t), y(t), u_1(t)), \quad t \ne t_k,\ k = 0, 1, 2, \dots \\
\frac{dx}{dt} \le -y + g(t, x(t), y(t), u_2(t)), \quad t \ne t_k,\ k = 0, 1, 2, \dots \\
\Delta x \le \beta_k^1 x(t_k) \\
\Delta x \le \beta_k^2 y(t_k) \\
0 < t_0 < t_1 < t_2 < \dots < t_k, \quad \lim_{k\to\infty} t_k = +\infty
\end{gathered}$$

serves as a basic comparison inequality for investigating (NLICE). The maxima solution (upper bound for the inequality) to NLICE can be found using standard results, see Lakshimikantham et al [14]. Thus

$$\begin{aligned}
x(t) \le& \prod_{t_0<t_k<t} (1 + \beta_k^1) e^{-(t-t_k)} x_0 \\
&+ \int_{t_0}^{t} \prod_{t_0<t_k<t} (1 + \beta_k^1) e^{-(t-s)} f(s, x(s), y(s), u_1(s)) ds, \\
y(t) \le& \prod_{t_0<t_k<t} (1 + \beta_k^2) e^{-(t-t_k)} y_0 \\
&+ \int_{t_0}^{t} \prod_{t_0<t_k<t} (1 + \beta_k^2) e^{-(t-s)} g(s, x(s), y(s), u_2(s)) ds
\end{aligned}$$

If

$$\begin{aligned}
f(t, x(t), y(t), u_1(t)) \le k_1(t) u_1(t) x(t) + \sum_{t_0<t_k<t} h^{(1)}(t_k) u_1(t_k + 0) x(t_k) \\
g(t, x(t), y(t), u_2(t)) \le k_2(t) u_2(t) y(t) + \sum_{t_0<t_k<t} h^{(2)}(t_k) u_2(t_k + 0) y(t_k),
\end{aligned}$$

where $k_1(t), h^{(i)}(t_k) \in \mathbb{R}^+$, $i = 1, 2$, $k = 0, 1, 2, \dots$,

$$\begin{aligned}
x(t) \le& \prod_{t_0<t_k<t} (1 + \beta_k^1) e^{-(t-t_k)} x_0 \\
&+ \int_{t_0}^{t} \prod_{t_0<t_k<t} (1 + \beta_k^1) e^{-(t-s)} k_1(s) u_1(s) x(s)) ds + \sum_{t_0<t_k<t} \phi_1(t_k, t) x(t_k) \\
y(t) \le& \prod_{t_0<t_k<t} (1 + \beta_k^2) e^{-(t-t_k)} y_0 \\
&+ \int_{t_0}^{t} \prod_{t_0<t_k<t} (1 + \beta_k^2) e^{-(t-s)} k_2(s) u_1(s) y(s)) ds + \sum_{t_0<t_k<t} \phi_2(t_k, t) y(t_k)
\end{aligned}$$

where

$$\phi_1(t_k,t) = h^{(1)}(t_k)u_1(t_k)\prod_{t_0<t_k<t}(1+\beta_k^1)e^{(t-t_k)}\int_{t_0}^t e^{-(t-s)}ds,$$

$$\phi_2(t_k,t) = h^{(1)}(t_k)u_2(t_k)\prod_{t_0<t_k<t}(1+\beta_k^2)e^{(t-t_k)}\int_{t_0}^t e^{-(t-s)}ds$$

Now let $z(t) = x(t)e^t$. Then

$$\begin{aligned} z(t) \le \prod_{t_0<t_k<t}&(1+\beta_k^1)e^{-(t-t_k)}z_0 \\ &+ \int_{t_0}^t \prod_{t_0<t_k<t}(1+\beta_k^1)e^{-(t-s)}k_1(s)u_1(s)z(s))ds + \sum_{t_0<t_k<t}\phi_1(t_k,t)z(t_k). \end{aligned}$$

Applying the lemma and substituting the value of $x(t)$, we get

$$\begin{aligned} x(t) \le &\Big(\prod_{t_0<t_k<t}(1+\beta_k^1)\Big)\Big(\prod_{t_0<t_k<t}(1+\phi_1(t_k,t)\Big) \\ &\times \exp\Big(\int_{t_k=t}(1+\beta_k)k_1(s)u_1(s)ds\Big)\exp\Big(-(t-t_k)x_0\Big). \end{aligned}$$

By a similar manipulation we obtain

$$\begin{aligned} y(t) \le &\Big(\prod_{t_0<t_k<t}(1+\beta_k^2)\Big)\Big(\prod_{t_0<t_k<t}(1+\phi_2(t_k,t)\Big) \\ &\times \exp\Big(\int_{t_k=t}(1+\beta_k^2)k_2(s)u_1(s)ds\Big)\exp\Big(-(t-t_k)y_0\Big). \end{aligned}$$

We have obtained the bounds for $x(t)$ and $y(t)$ under the conditions imposed on $k_i(t), h^{(i)}(t_k) \in \mathbb{R}^+$, $i = 1,2$, $k = 0,1,2,\dots$, when $u_i(t)$, $i = 1,2$ are bounded; that is, using control language, bounded input will give rise to bounded output. Many biological and physical control systems are of bounded input-bounded output types. Bounds on $x(t)$ and $y(t)$ can be used to make qualitative deductions about the control system.

Example 5.2. Consider the impulsive integrodifferential system (IIS)

$$\begin{gathered} \frac{du(t)}{dt} \le \operatorname{diag}[ae^{\alpha t}\ be^{\beta t}]u(t) + F(t,u(t)) \\ + \int_0^t d\tau z(t,\tau)w(t, \int_0^t G(t,s,H(s)A(s)u(s)ds), \quad t \ne t_k,\ k = 0,1,2,\dots \\ u(t=t_k)\sum_{t_0<t_k<t}\beta(t_k)u(t_k) \\ 0 < t_0 < t_1 < \dots < t_k, \lim_{k\to\infty} t_k = \infty\,, \end{gathered}$$

where $H(t) = \operatorname{diag}[a\ b]$, $A(t) = \operatorname{diag}[e^{\alpha t}e^{\beta t}]$, $\alpha > 0, \beta > 0$, $u(t) = (u_1(t), u_2(t))$,

$$G(t,s,H(s)A(s)u(s)) = \begin{cases} \frac{t-s}{h}\operatorname{diag}[ae^{\alpha t}\ be^{\beta t}]u(t) & t \ge s, \\ 0 & t < s\,, \end{cases}$$

$z(t,\tau) = D(t)H(t)A(t)$, and $w(\phi,\phi) = \phi$.

Assuming $\lim_{|v|\to 0} w(t,v)/|v| = \phi$, $\phi = (0,0) \in \mathbb{R}^+$, let $v = \int_0^t G(t,s,\dots)ds$ and $t - s = \theta$. Therefore,

$$\begin{pmatrix} \frac{-ae^{\alpha t}}{h} \int_\theta^{\theta+s} e^{-\alpha t} u_1(-\tau)d\tau \\ \frac{-be^{\alpha t}}{h} \int_\theta^{\theta+s} e^{-\beta t} u_2(-\tau)d\tau \end{pmatrix}.$$

Also $z(t,\tau) = \operatorname{diag}[\exp \frac{a}{\alpha}(e^{\alpha t}-1)\ \exp \frac{b}{\beta}(e^{\beta t}-1)] \operatorname{diag}[ae^{\alpha t} be^{\beta t}]$. Hence

$$w(t,v) = \begin{pmatrix} w_1(t,v_1) \\ w_2(t,v_2) \end{pmatrix} = \begin{pmatrix} w_1(t, \frac{-ae^{\alpha t}}{h} \int_\theta^{\theta+s} e^{-\alpha t} u_1(-\tau)d\tau \\ w_2(t, \frac{-be^{\alpha t}}{h} \int_\theta^{\theta+s} e^{-\beta t} u_2(-\tau)d\tau \end{pmatrix}$$

It can be shown easily that the commutant of $A(t)$ and H satisfy $[A(t), H(t))] = \phi$, $\phi = (0,0) \in \mathbb{R}^+$ and

$$\det(H^{ast}(t)A^{ast}(t)A(t)H(t)) = a^2b^2e^{2(\alpha+\beta)t} > 0, \quad \alpha+\beta > 0\,.$$

Therefore, $v_i(t)$ are estimated as

$$\begin{aligned} v_1(t) &\le \frac{a^2}{h}\int_0^t \int_{\theta+s}^{\theta} u_1(-\tau)d\tau ds, \\ v_2(t) &\le \frac{b^2}{h}\int_0^t \int_{\theta+s}^{\theta} u_2(-\tau)d\tau ds, \\ v_{*1}(t) &\le \frac{a^2}{h}\int_0^t \int_{\theta+s}^{\theta} \max_{\tau\in[0,\theta+s]} u_1(-\tau)d\tau ds = \frac{a^2}{h} t^* |u_1(-\tau)|_{R_0} \\ v_{*2}(t) &\le \frac{b^2}{h}\int_0^t \int_{\theta+s}^{\theta} \max_{\tau\in[0,\theta+s]} u_2(-\tau)d\tau ds = \frac{b^2}{h} t^* |u_1(-\tau)|_{R_0} \end{aligned}$$

Here t^* is the threshold value of t taken across the interval $[0, \theta+s]$. Therefore, applying theorem 4.6 to (llS) yields. where

$$\begin{aligned} u^*(t) = {}& \int_{t_0}^t D(\tau)A^*(\tau)w(tau, \int_{t_0}^t G(t,s,A_1(s)H(s)u^*(s)ds)d\tau \\ &+ \int_{t_0}^t d\tau D(\tau)A^*(\tau) \begin{pmatrix} w_1(t, \frac{a^2t^*}{h}|u_1(-\tau)|R_0) \\ w_2(t, \frac{b^2t^*}{h}|u_2(-\tau)|_{R_0} \end{pmatrix} \end{aligned}$$

Therefore, the right-hand side provides the upper bound for $u(t)$.

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