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INTERVAL OSCILLATION CRITERIA FOR SECOND ORDER FORCED NONLINEAR MATRIX DIFFERENTIAL EQUATIONS

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Abstract. New oscillation criteria are established for the nonlinear matrix differential equations with a forced term

$$[r(t)Y'(t)]' + p(t)Y'(t) + Q(t)G(Y'(t))F(Y(t)) = e(t)I_n.$$

Our results extend and improve the recent results of Li and Agarwal for scalar cases. Furthermore, one example that dwell upon the importance of our results is included.

1. Introduction

In this paper, we consider the oscillatory behavior of solutions of the forced second order nonlinear matrix differential equation

$$[r(t)Y'(t)]' + p(t)Y'(t) + Q(t)G(Y'(t))F(Y(t)) = e(t)I_n, \tag{1.1}$$

where $t \geq t_0$, $r(t) \in C^1([t_0,\infty),(0,\infty))$, $p(t) \in C([t_0,\infty),(-\infty,\infty))$, $Q(t), G(Y'(t))$ are positive semi-definite matrices, $Q(t)$ is continuous, $F \in C^1(R^{n^2}, R^{n^2})$, and the inverse of the matrix $F(Y(t))$ exists for all $Y(t) \neq 0$ and is denoted by $[F(Y(t))]^{-1}$. Moreover, $[F(Y(t))]^{-1}$ is positive definite and satisfies [18]

$$([F(Y(t))]^{-1})^T (Y'(t))^T = Y'(t)[F(Y(t))]^{-1} \tag{1.2}$$

for every solution $Y(t)$ of (1.1), where A^T is the transpose of A.

We call a matrix function $Y(t) \in C^2([t_0,\infty), R^{n^2})$ a prepared nontrivial solution of (1.1) if $\det Y(t) \neq 0$ for at least one $t \in [t_0,\infty)$, $r(t)Y'(t) \in C^1([t_0,\infty), R^{n^2})$ and $Y(t)$ satisfies (1.2).

A prepared solution $Y(t)$ of (1.1) is called oscillatory if $\det Y(t)$ has arbitrary large zeros. (1.1) is called oscillatory if every nontrivial prepared solution of (1.1) is oscillatory. Otherwise it is called non-oscillatory.

For $n = 1$, $p(t) = 0$ and $G = 1$, (1.1) has been studied by many authors, for example, Jaros, Kusano and Yoshida [1] and their references. On the one hand, many authors assume that $Q(t)$ is nonnegative; see Skidmore and Leighton [3] and Tenfel [4]. In this case, one can usually establish oscillation criteria for more general nonlinear differential equation by employing a technique introduced by Kartsators [2] where it is additionally assumed that $e(t)$ is the second derivative of an oscillatory

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function $h(t)$. On the other hand, most oscillation results involve the integral of $Q(t)$ and hence require the information of r and Q on the entire half-line $[t_0, \infty)$, see Li and Yan [5] and their references.

For $n > 1$, Erbe, Kong and Ruan [6], Meng, Wang and Zheng [7] and Etgen and Pawlowski [8] obtain some generalized Kamenev type oscillation criterion for the linear matrix differential equation

$$(R(t)Y'(t))' + Q(t)Y(t) = 0. \tag{1.3}$$

In 1999, Kong [9] employed the technique from Philos [10] for the second-order linear differential equations, and presented several interval oscillation criteria for (1.3) with $n = 1$ (see Theorems 2.1 and 2.2 and their corollaries 2.1-2.4 in [9]) involving the Kamenev's type condition. These results have been generalized by Li [11], Li and Agarwal [13, 15] and Li and Cheng [16].

Recently, Zhuang [17] and Yang [18] extended the results of [9] to the matrix differential equation (1.3) and to the nonlinear matrix differential equation

$$[r(t)Y'(t)]' + p(t)Y'(t) + Q(t)G(Y'(t))F(Y(t)) = 0, t \geq t_0,$$

respectively. However, the above results cannot be applied to the non-homogeneous nonlinear matrix differential equation (1.1).

Motivated by the ideas from Li and Agarwal [13, 15], in this paper we obtain, by using a matrix Riccati type transformation, some results of [13, 15] are generalized to the nonlinear matrix differential equation (1.1).

For convenience of the reader, we introduce the following notation. Let M be the linear space of $n \times n$ real matrices, $M_0 \subset M$ be the subspace of symmetric matrices. For any real symmetric matrices $A, B, C \in M_0$, we write $A \geq B$ to mean that $A - B \geq 0$, that is, $A - B$ is positive semi-definite and $A > B$ to mean that $A - B > 0$, that is $A - B$ is positive definite. We will use some properties of this ordering, viz., $A \geq B$ implies that $CAC \geq CBC$ and $A \geq B$ and $B \geq 0$ implies $A \geq 0$. Moreover, $A \geq B$ implies $\int_a^b A ds \geq \int_a^b B ds$

2. Main Results

In the sequel we say that a function $H = H(t)$ belongs to a function class $D(a, b) = \{H \in C^1[a, b] : H(t) \not\equiv 0, H(a) = H(b) = 0\}$, denoted by $H \in D(a, b)$.

Lemma 2.1. *If $Y(t)$ is a nontrivial prepared solution of* (1.1) *and* $\det Y(t) > 0$ *for $t \geq t_0$, then, for any $\rho(t) \in C^1([t_0, \infty), (0, \infty))$, the matrix*

$$W(t) = \rho(t)r(t)Y'(t)[F(Y(t))]^{-1} \tag{2.1}$$

satisfies the equation

$$\begin{aligned} W'(t) = \Big[\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)}\Big]W(t) - \rho(t)Q(t)G(Y'(t)) \\ - \frac{W(t)F'(Y(t))W(t)}{\rho(t)r(t)} + \rho(t)e(t)[F(Y(t))]^{-1}. \end{aligned} \tag{2.2}$$

Proof. From (1.1), we obtain

$$\begin{aligned}W'(t) &= \frac{\rho'(t)}{\rho(t)}W(t) + \rho(t)(r(t)Y'(t))'[F(Y(t))]^{-1} + \rho(t)r(t)Y'(t)[[F(Y(t))]^{-1}]' \\ &= \frac{\rho'(t)}{\rho(t)}W(t) + \rho(t)[e(t)I_n - p(t)Y'(t) - Q(t)G(Y'(t))F(Y(t))][F(Y(t))]^{-1} \\ &\quad - \rho(t)r(t)Y'(t)[F(Y(t))]^{-1}F'(Y(t))Y'(t)[F(Y(t))]^{-1} \\ &= [\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)}]W(t) - \rho(t)Q(t)G(Y'(t)) \\ &\quad - \frac{W(t)F'(Y(t))W(t)}{\rho(t)r(t)} + \rho(t)e(t)[F(Y(t))]^{-1}.\end{aligned}$$

The proof is complete. □

Theorem 2.2. *Suppose that for any $T \geq t_0$, there exist $T \leq a < b$ such that $e(t) < 0$, $t \in [a, b]$. If there exist $H \in D(a, b)$ and a function $\rho(t) \in C^1([t_0, \infty), (0, \infty))$ such that*

$$\begin{aligned}&\int_a^b H^2(t)\rho(t)Q(t)G(Y'(t))dt \\ &\geq \frac{1}{4}\int_a^b \rho(t)r(t)[2H'(t) + (\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)})H(t)]^2[F'(Y(t))]^{-1}dt,\end{aligned}$$

then (1.1) *is oscillatory.*

Proof. Suppose the contrary. Then without loss of generality we assume that there is a nontrivial prepared solution $Y(t)$ of (1.1), which is nonsingular on $[t_0, \infty)$, and $W(t) = \rho(t)r(t)Y'(t)[F(Y(t))]^{-1}$ exists on $[t_0, \infty)$.

Since $Y(t)$ is prepared, by (1.2), $W(t) \in M_0$ and by Lemma 2.1, $W(t)$ satisfies the equation

$$\begin{aligned}W'(t) = [\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)}]W(t) - \rho(t)Q(t)G(Y'(t)) \\ - \frac{W(t)F'(Y(t))W(t)}{\rho(t)r(t)} + \rho(t)e(t)[F(Y(t))]^{-1}.\end{aligned} \tag{2.3}$$

That is,

$$\begin{aligned}\rho(t)Q(t)G(Y'(t)) = -W'(t) + [\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)}]W(t) \\ - \frac{W(t)F'(Y(t))W(t)}{\rho(t)r(t)} + \rho(t)e(t)[F(Y(t))]^{-1}.\end{aligned} \tag{2.4}$$

By assumption, we can choose $b > a \geq T_0$ such that $e(t) < 0$ on the interval $I = [a, b]$. From (2.4) we see that $W(t)$ satisfies

$$\rho(t)Q(t)G(Y'(t)) < -W'(t) + [\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)}]W(t) - \frac{W(t)F'(Y(t))W(t)}{\rho(t)r(t)}. \tag{2.5}$$

Let $H \in D(a,b)$ be given as in hypothesis. Multiplying H^2 through (2.5) and integrating over $I = [a,b]$, we have

$$\begin{aligned} &\int_a^b H^2(t)\rho(t)Q(t)G(Y'(t))dt \\ &< -\int_a^b H^2(t)W'(t)dt + \int_a^b H^2(t)\Big[\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)}\Big]W(t)dt \\ &\quad - \int_a^b H^2(t)\frac{W(t)F'(Y(t))W(t)}{\rho(t)r(t)}dt. \end{aligned} \tag{2.6}$$

Integrating (2.6) by parts and using that $H(a) = H(b) = 0$, we have

$$\begin{aligned} &\int_a^b H^2(t)\rho(t)Q(t)G(Y'(t))dt \\ &< -\int_a^b \Big\{\frac{H^2(t)W(t)F'(Y(t))W(t)}{\rho(t)r(t)} \\ &\quad - 2H(t)H'(t)W(t) - H^2(t)\Big[\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)}\Big]W(t)\Big\}dt \\ &= -\int_a^b \Big\{\frac{H^2(t)W(t)F'(Y(t))W(t)}{\rho(t)r(t)} \\ &\quad - \Big[2H'(t) + \Big(\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)}\Big)H(t)\Big]H(t)W(t)\Big\}dt \\ &= -\int_a^b \Big\{\frac{H(t)W(t)}{\sqrt{\rho(t)r(t)}} \\ &\quad - \frac{\sqrt{\rho(t)r(t)}\big[2H'(t) + (\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)})H(t)\big][F'(Y(t))]^{-1}}{2}\Big\}F'(Y(t)) \\ &\quad \times \Big\{\frac{H(t)W(t)}{\sqrt{\rho(t)r(t)}} - \frac{\sqrt{\rho(t)r(t)}[2H'(t) + (\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)})H(t)][F'(Y(t))]^{-1}}{2}\Big\}dt \\ &\quad + \frac{1}{4}\int_a^b \rho(t)r(t)\Big[2H'(t) + \Big(\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)}\Big)H(t)\Big]^2[F'(Y(t))]^{-1}dt \\ &\le \frac{1}{4}\int_a^b \rho(t)r(t)\Big[2H'(t) + \Big(\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)}\Big)H(t)\Big]^2[F'(Y(t))]^{-1}dt, \end{aligned}$$

which contradicts the condition (2.3). Hence every solution of (1.1) is oscillatory. The proof is complete. □

From Theorem 2.2, it is easy to see that the following important corollary is true.

Corollary 2.3. *Under the assumptions in Theorem 2.2, assume that $F'(Y) \ge A > 0$ and $G(Y) \ge B > 0$, where $A, B \in M_0$ are constant positive definite matrices such that*

$$\int_a^b H^2(t)\rho(t)Q(t)Bdt \ge \frac{1}{4}\int_a^b \rho(t)r(t)\Big[2H'(t) + \Big(\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)}\Big)H(t)\Big]^2 A^{-1}dt\,. \tag{2.7}$$

Then every solution of (1.1) *is oscillatory.*

We remark that if $n = 1$, then Corollary 2.3 reduces to the main result of Li and Agarwal [15].

Example. Consider the linear $n \times n$ matrix differential equation

$$(\sqrt{t}Y'(t))' - 2Y'(t) + \frac{5}{4\sqrt{t}}Y(t) = \frac{1}{\sqrt{t}}(\sin\sqrt{t} - \cos\sqrt{t})I_n, \tag{2.8}$$

where $r(t) = \sqrt{t}$, $p(t) = -2$, $Q(t) = \frac{5}{4\sqrt{t}}$, $G(Y') = I_n$, $F(Y) = Y(t)$, and $F'(Y) = I_n$.

Clearly, the zeros of the forcing term $\frac{1}{\sqrt{t}}(\sin\sqrt{t} - \cos\sqrt{t})I_n$ are $[k\pi + \frac{\pi}{4}]^2$. Let

$$H(t) = \sin(\sqrt{t} - \frac{\pi}{4}).$$

For any $T > 1$, choose k sufficient large so that $((2k+1)\pi + \frac{\pi}{4}) > T$ and set

$$a = [(2k+1)\pi + \frac{\pi}{4}]^2, \quad b = [2(k+1)\pi + \frac{\pi}{4}]^2,$$

then $e(t) \le 0$ for $t \in [a, b]$. Pick up $\rho(t) \equiv 1$. It is easy to verify that

$$\begin{aligned}
\int_a^b H^2(t)Q(t)B\,dt &= \int_a^b \sin^2(\sqrt{t} - \frac{\pi}{4})\frac{5}{4\sqrt{t}}I_n\,dt \\
&= \int_{(2k+1)\pi+\frac{\pi}{4}}^{2(k+1)\pi+\frac{\pi}{4}} \sin^2(s - \frac{\pi}{4})\frac{5}{4s}2sI_n\,ds \\
&= \int_{(2k+1)\pi+\frac{\pi}{4}}^{2(k+1)\pi+\frac{\pi}{4}} \frac{5}{2}\sin^2(s - \frac{\pi}{4})I_n\,ds = \frac{5\pi}{4}I_n,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{4}\int_a^b \rho(t)r(t)\Big[2H'(t) + \big(\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)}\big)H(t)\Big]^2 A^{-1}\,dt \\
&= \frac{1}{4}\int_a^b \sqrt{t}\Big[2\frac{\cos(\sqrt{t} - \frac{\pi}{4})}{2\sqrt{t}} + \frac{2}{\sqrt{t}}\sin(\sqrt{t} - \frac{\pi}{4})\Big]^2 I_n\,dt \\
&= \frac{1}{4}\int_{(2k+1)\pi+\frac{\pi}{4}}^{2(k+1)\pi+\frac{\pi}{4}} s\Big[\frac{\cos(s - \frac{\pi}{4})}{s} + \frac{2}{s}\sin(s - \frac{\pi}{4})\Big]^2 2sI_n\,ds \\
&= \frac{1}{2}\int_{(2k+1)\pi+\frac{\pi}{4}}^{2(k+1)\pi+\frac{\pi}{4}} \Big[\cos^2(s - \frac{\pi}{4}) + 2\sin(2s - \frac{\pi}{2}) + 4\sin^2(s - \frac{\pi}{4})\Big]I_n\,ds \\
&= \frac{5\pi}{4}I_n,
\end{aligned}$$

which implies that (2.7) holds. It follows from Corollary 2.3 that every solution of (2.8) is oscillatory. Obverse that $Y(t) = \sin\sqrt{t}I_n$ is such a solution.

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