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BIFURCATION DIAGRAM OF A CUBIC THREE-PARAMETER AUTONOMOUS SYSTEM

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Abstract. In this paper, we study the cubic three-parameter autonomous planar system

$$\dot{x}_1 = k_1 + k_2 x_1 - x_1^3 - x_2,$$
$$\dot{x}_2 = k_3 x_1 - x_2,$$

where $k_2, k_3 > 0$. Our goal is to obtain a bifurcation diagram; i.e., to divide the parameter space into regions within which the system has topologically equivalent phase portraits and to describe how these portraits are transformed at the bifurcation boundaries. Results may be applied to the macroeconomical model IS-LM with Kaldor's assumptions. In this model existence of a stable limit cycles has already been studied (Andronov-Hopf bifurcation). We present the whole bifurcation diagram and among others, we prove existence of more difficult bifurcations and existence of unstable cycles.

1. Introduction

In the present paper we shall consider the real dynamical autonomous system

$$\begin{aligned} \dot{x}_1 &= k_1 + k_2 x_1 - x_1^3 - x_2, \\ \dot{x}_2 &= k_3 x_1 - x_2, \end{aligned} \tag{1.1}$$

where $x_1, x_2 \in \mathbb{R}$ and $K = \{(k_1, k_2, k_3) \in \mathbb{R}^3 : k_2 > 0, k_3 > 0\}$ is a parameter space. Note that if $x_1(t)$, $x_2(t)$ are solutions of (1.1), $\tilde{x}_1(t) = -x_1(t)$, $\tilde{x}_2(t) = -x_2(t)$ are solutions of the system

$$\dot{x}_1 = -k_1 + k_2 x_1 - x_1^3 - x_2,$$
$$\dot{x}_2 = k_3 x_1 - x_2.$$

This implies that the bifurcation sets of (1.1) are symmetric with respect to the plane $k_1 = 0$, because the phase portraits of (1.1) with the parameters $(k_1, k_2, k_3) = (\tilde{k}_1, \tilde{k}_2, \tilde{k}_3)$ and $(k_1, k_2, k_3) = (-\tilde{k}_1, \tilde{k}_2, \tilde{k}_3)$ are symmetric about the origin. We

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denote

$$A = \begin{pmatrix} k_2 - 3x_1^2 & -1 \\ k_3 & -1 \end{pmatrix},$$
$$\operatorname{tr} A = k_2 - 3x_1^2 - 1,$$
$$\det A = 3x_1^2 - k_2 + k_3,$$
$$p_A(\lambda) = \det(A - \lambda I) = \lambda^2 - \lambda \operatorname{tr} A + \det A,$$

where A is Jacobi's matrix of the system (1.1), its trace $\operatorname{tr} A$, determinant $\det A$ and characteristic polynomial $p_A(\lambda)$ are functions of variable x_1.

All equilibrium points (ξ_1, ξ_2) of the system (1.1) have to be solutions of the equations

$$k_1 + k_2 x_1 - x_1^3 - x_2 = 0,$$
$$k_3 x_1 - x_2 = 0,$$

which gives that ξ_1 has to satisfy the equality

$$k_1 + (k_2 - k_3)\xi_1 - \xi_1^3 = 0 \tag{1.2}$$

and $\xi_2 = k_3 \xi_1$. System (1.1) has from one to three equilibrium points.

Lemma 1.1. *Let (ξ_1, ξ_2) be an equilibrium point of* (1.1)*. Then the set*

$$\{(x_1, x_2) \in \mathbb{R}^2 : k_3(x_1 - \xi_1)^2 + (x_2 - k_3\xi_1)^2 \le R\},$$

where

$$R = -k_3 \min_{x_1 \in \mathbb{R}} \{x_1^2(x_1^2 + 3\xi_1 x_1 - k_2 + 3\xi_1^2 - 1)\}$$

is globally attractive.

For the proof of the above lemma se [2, Theorems 5.1 and 5.2] .

Remark 1.2. A planar dynamical system

$$\begin{aligned} \dot{y} &= \alpha[i(y, r) - s(y, r)], \\ \dot{r} &= \beta[l(y, r) - m], \end{aligned} \tag{1.3}$$

where $\alpha, \beta > 0$, may represent a macroeconomical model IS-LM (see [2]). The variable $y = \ln Y$ is the logarithm of the product (GNP), r is the interest rate. Functions i and s are propensities to invest and save, l and the constant m - demand and supply of money. Using basic economic properties of the functions i, s and l (including Kaldor's assumptions), we can concretize the system (1.3) to the most simple one - a cubic system

$$\begin{aligned} \dot{y} &= \alpha(a_0 + a_1 y + br + a_2 y^2 + a_3 y^3), \\ \dot{r} &= \beta(c_0 + cy + dr), \end{aligned} \tag{1.4}$$

where $\alpha > 0$, $\beta > 0$, $b < 0$, $a_3 < 0$, $c > 0$, $d < 0$ and the quadratic equation $a_1 + 2a_2 x + 3a_3 x^2 = 0$ has two different real roots. System (1.4) can be replaced by the system (1.1) using some efficient transformation (see [8]).

The aim of this paper is to continue in the study of the dynamical system (1.4) from [2](the system (1.1) respectively) and to obtain deeper results concerning its stability, topological properties and types of bifurcations, especially existence and stability of limit cycles. From the economic point of view stable limit cycles correspond to business cycles. Economists are used to presume that economic equilibrium is globally stable always, i.e. they assume there exists some mechanism of adaptation in economy. This is true for a linear IS-LM model, with $a_2 = 0$, $a_3 = 0$. If the economy satisfies the Kaldor's assumptions, such mechanism need not exist. This was pointed out already in the original Kaldor's paper [5], but dealing with this problem all authors provided just numerical results or made some other specific assumptions to the model and to the best of my knowledge never found any unstable cycle. Although the system (1.4) is "only" cubic, we will show that even more than one cycle can appear and surely it need not be stable. Moreover, the described cycles are not evoked by external influences, but they are entirely determined by internal structure of the system, which is a problem passed by so called "invisible hand" that should lead the economy to the globally stable equilibrium.

2. Local bifurcations

Lemma 2.1. *Let (ξ_1, ξ_2) be an equilibrium point of (1.1) and let*

$$k_2 = k_3 + 3\xi_1^2, \quad k_3 \neq 1.$$

Then the equilibrium point (ξ_1, ξ_2) is a saddle-node for $\xi_1 \neq 0$. The origin is topologically equivalent to a node in the case $\xi_1 = 0$.

Proof. After transformation of the equilibrium point (ξ_1, ξ_2) to the origin by the change of variables $u_1 = x_1 - \xi_1$, $u_2 = x_2 - \xi_2$ we get the system

$$\begin{aligned} \dot{u}_1 &= k_3 u_1 - 3\xi_1 u_1^2 - u_1^3 - u_2, \\ \dot{u}_2 &= k_3 u_1 - u_2. \end{aligned}$$

For $k_3 \neq 1$, the following regular transformation

$$u_1 = y_1 + y_2, \quad u_2 = k_3 y_1 + y_2$$

(the matrix of the trasformation is given by the eigenvectors corresponding with one zero and one non-zero eigenvalues) and the time change $\tau = (k_3 - 1)t$ give the canonical form of system (1.1):

$$\begin{aligned} \dot{y}_1 &= F(y_1, y_2), \\ \dot{y}_2 &= y_2 - k_3 F(y_1, y_2), \end{aligned}$$

where

$$F(y_1, y_2) = \frac{3\xi_1}{(k_3 - 1)^2}(y_1 + y_2)^2 + \frac{1}{(k_3 - 1)^2}(y_1 + y_2)^3.$$

Let $y_2 = \varphi(y_1)$ be a solution of the equation

$$y_2 - k_3 F(y_1, y_2) = 0$$

in the neighbourhood of the origin. We approximate this solution corresponding with the central manifold of the system by a Taylor expansion

$$\varphi(y_1) = \sum_{i=0}^{\infty} a_i y_1^i$$

in the neighbourhood of the origin and get

$$\sum_{i=0}^{\infty} a_i y_1^i = \frac{3k_3\xi_1}{(k_3 - 1)^2}(y_1 + \sum_{i=0}^{\infty} a_i y_1^i)^2 + \frac{k_3}{(k_3 - 1)^2}(y_1 + \sum_{i=0}^{\infty} a_i y_1^i)^3.$$

We equate coefficients of equal powers of x on the left and the righthand side and find

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = \frac{3k_3\xi_1}{(k_3-1)^2} \neq 0.$$

The equilibrium point (ξ_1, ξ_2) of the system (1.1) is a saddle-node according to [1, Theorem 65 (par. 21)].

In the case that $\xi_1 = 0$, the system (1.1) has a unique equilibrium point $(0, 0)$. We analogically aproximate the central manifold by the Taylor expansion with zero coefficients up to the second order (including) and get

$$a_3 = \frac{k_3}{(k_3-1)^2} > 0.$$

Consequently, the origin is topologically equivalent to a node according to [1, Theorem 65 (par. 21)]. □

Theorem 2.2. *The subset M_T of the parameter space K,*

$$M_T = \{(k_1, k_2, k_3) \in K : k_1 = -2\xi_1^3, k_2 = k_3 + 3\xi_1^2, k_3 \neq 1, \xi_1 \in \mathbb{R} - \{0\}\},$$

is a bifurcation set of codimension 1 - double equilibrium (also called "saddle-node bifurcation"). The double equilibrium point $(\xi_1, k_3\xi_1)$ is a saddle-node.

Proof. Let (ξ_1, ξ_2) be an equilibrium point of the system (1.1). The bifurcation "double equilibrium" occurres in the case that the parameters k_1, k_2, k_3 satisfy the following condition

$$3\xi_1^2 - k_2 + k_3 = 0. \tag{2.1}$$

In this case two equilibrium points coincide to one. So called non-degeneracy condition is $\xi_1 \neq 0$, because the equilibrium point is triple for $\xi_1 = 0$. Conditions (1.2) and (5) together with the non-degeneracy condition define the subset of K, where the system (1.1) has exactly two equilibrium points: the double equilibrium point $(\xi_1, k_3\xi_1)$ and the single equilibrium point $(-2\xi_1, -2k_3\xi_1)$. In the case that $k_3 = 1$, the double equilibrium point has two zero eigenvalues and bifurcation of codimension 2 takes place (this case is studied in Theorem 2.7).

The set M_T consists of two components M_{Tl} and M_{Tr}. They correspond with the case $\xi_1 < 0$ (the double equilibrium point lies left of the single one) and $\xi_1 > 0$ (the double equilibrium point lies right of the single one). These sets are symmetrical according to the axis $k_1 = 0$.

The closure of the set M_T divides the parameter space K into two sets M_1, M_3

$$M_1 = \{(k_1, k_2, k_3) \in K : k_1 = -2\xi_1^3, k_2 < k_3 + 3\xi_1^2, \xi_1 \in \mathbb{R}\},$$
$$M_3 = \{(k_1, k_2, k_3) \in K : k_1 = -2\xi_1^3, k_2 > k_3 + 3\xi_1^2, \xi_1 \in \mathbb{R}\}.$$

The set M_1 contsists of all the parameters from K, for which the system (1.1) has a unique equilibrium point (non-saddle), the set M_3 consists of those, for which the system (1.1) has 3 equilibrium points (non-saddle, saddle, non-saddle). While crossing the boundary M_T from the set M_3 to M_1, two equilibrium points coincide and disappear then. According to Lemma 2.1, the double equilibrium point is a saddle-node. A qualitative local change of the phase portraits occurres, a local bifurcation of codimension 1 - "saddle-node". □

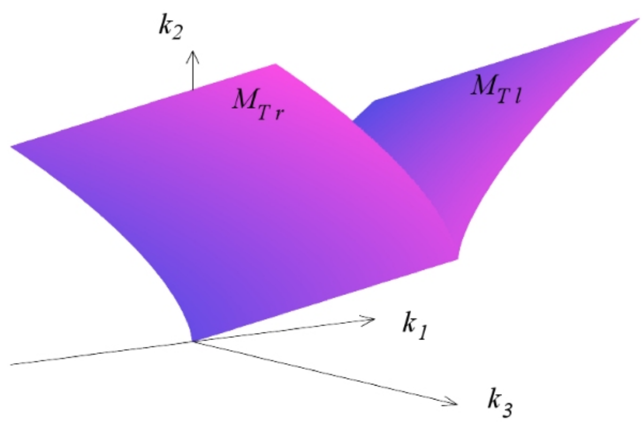

FIGURE 1. The set M_T.

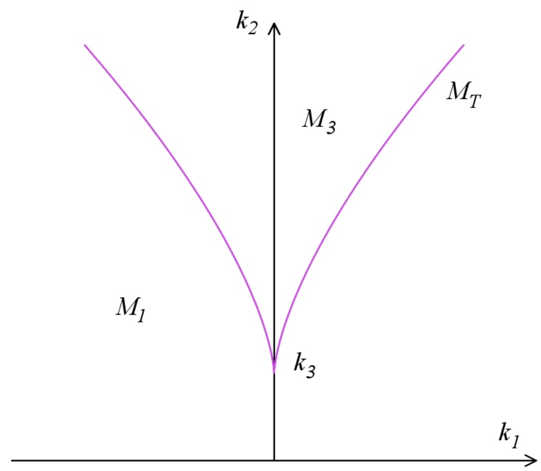

FIGURE 2. The section of M_T and the parameter space in k_3.

Theorem 2.3. *The subset M_H of the parameter space K,*

$$M_H = \{(k_1, k_2, k_3) \in K : k_1 = \xi_1(k_3 - 1 - 2\xi_1^2), k_2 = 1 + 3\xi_1^2, k_3 > 1, \xi_1 \in \mathbb{R}\},$$

is a bifurcation set corresponding with Andronov-Hopf bifurcation. The equilibrium point $(\xi_1, k_3\xi_1)$ is a multiple focus.

Proof. Let (ξ_1, ξ_2) be an equilibrium point of (1.1). The trace $\operatorname{tr} A = 0$ and the determinant $\det A > 0$ if and only if the Jacobi's matrix A has two purely imaginary eigenvalues. We get the following conditions

$$k_1 + (k_2 - k_3)\xi_1 - \xi_1^3 = 0,$$
$$k_2 - 3\xi_1^2 - 1 = 0,$$
$$3\xi_1^2 - k_2 + k_3 > 0.$$

These three conditions define the set M_H (see fig. 3).

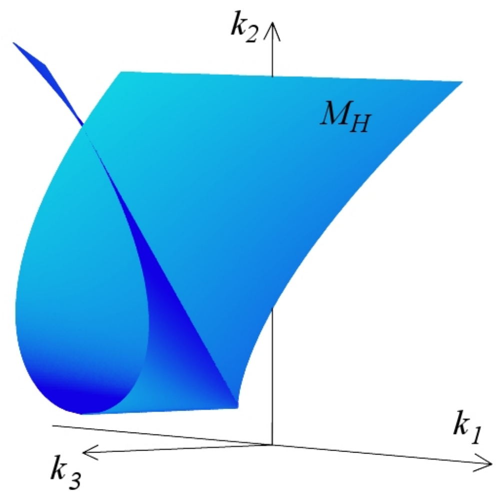

FIGURE 3. The set M_H.

The eigenvalues are purely imaginary on M_H,

$$\lambda_{1,2} = \pm i\omega, \quad \omega = \sqrt{\det A(\xi_1)} = \sqrt{k_3 - 1},$$

and the equilibrium point (ξ_1, ξ_2) is a multiple focus. While crossing the bound M_H, the equilibrium point may change its stability. We will compute the value of $\frac{d\,\mathrm{Re}\,\lambda_{1,2}}{dk_2}$ to describe the change of stability. Since

$$\frac{dp_A}{d\lambda} = 2\lambda - \operatorname{tr} A,$$

we have

$$\frac{dp_A}{d\lambda}\Big|_{M_H} = \pm i2\sqrt{k_3 - 1} \neq 0 \tag{2.2}$$

on the set M_H and we can apply the implicit function theorem and get

$$\frac{d\lambda}{dk_2}\Big|_{M_H} = -\frac{\frac{dp_A}{dk_2}}{\frac{dp_A}{d\lambda}}\Big|_{M_H}. \tag{2.3}$$

The coordinates of the equilibrium point depend on the parameters. Let us denote $\xi_1 = \varphi(k_1, k_2, k_3)$. Then we get

$$\frac{dp_A}{dk_2} = -(\lambda + 1)\big(1 - 6\varphi\frac{\partial\varphi}{\partial k_2}\big). \tag{2.4}$$

Since the equality (1.2) gives

$$\varphi + (k_2 - k_3)\frac{\partial \varphi}{\partial k_2} - 3\varphi^2 \frac{\partial \varphi}{\partial k_2} = 0,$$

we can express the partial derivative $\frac{\partial \varphi}{\partial k_2}$ on the set M_H as

$$\frac{\partial \varphi}{\partial k_2} = \frac{\xi_1}{k_3 - 1}.$$

Using this expression, equalities (2.2) and (2.4) in (2.3), we get

$$\frac{d\lambda}{dk_2}\Big|_{M_H} = \frac{\big(1 - 6\frac{\xi_1^2}{k_3-1}\big)(\pm i\sqrt{k_3-1}+1)}{\pm i\, 2\sqrt{k_3-1}}.$$

That yields

$$\frac{d\,\mathrm{Re}\,\lambda}{dk_2}\Big|_{M_H} = \frac{k_3 - 1 - 6\xi_1^2}{2(k_3-1)} = \frac{k_3 + 1 - 2k_2}{2(k_3-1)}. \tag{2.5}$$

Taking M_H as a parametric function of ξ_1, we have

$$\frac{dk_1}{d\xi_1} = k_3 - 1 - 6\xi_1^2.$$

The derivative $\frac{d\,\mathrm{Re}\,\lambda}{dk_2}$ is zero if and only if $\frac{dk_1}{d\xi_1} = 0$, that is in the case that the tangent to M_H is parallel to the axis k_2. In this situation, there is no crossing of M_H (just a contact) and there is also no change in stability of the focus. In the case that $k_3 + 1 > 2k_2$, a stable focus changes to an unstable focus, while crossing M_H in the direction of the axis k_2. In the opposite case, an unstable focus changes to a stable focus. (These results correspond to Theorem 2.5 on subcritical and supercritical bifurcation.)

While crossing the bifurcation bound M_H, the focus changes its stability and a limit cycle arises in its neighbourhood. There occurres a local qualitative change of the phase portraits called Andronov-Hopf bifurcation.

The set M_H is divided by the set M_T into three parts M_{Hr}, M_{Hl} and M_{Hu} (see fig. 4).

These sets correspond with Andronov-Hopf bifurcation of the right, left (in the case of three equilibrium points) and unique equilibrium point. □

Remark 2.4. Stability of the limit cycle depends on stability of the multiple focus and is determined by the sign of the first Lyapunov number of this multiple focus. The cycle is stable for $l_1 < 0$ and unstable for $l_1 > 0$. Parameters corresponding with zero values of the first Lyapunov number l_1 determine a subset of codimension 2 of M_H - degenerate Andronov-Hopf bifurcation.

Theorem 2.5. *The subset M_{DH} of the parameter space K*

$$M_{DH} = \{(k_1, k_2, k_3) \in K : k_1 = 4\xi_1^3, k_2 = 1 + 3\xi_1^2, k_3 = 1 + 6\xi_1^2, \xi_1 \in \mathbb{R} - \{0\}\}$$

is a bifurcation set of codimension 2 corresponding with degenerate Andronov-Hopf bifurcation.

Proof. Let (ξ_1, ξ_2) be an equilibrium point of the system (1.1). We transform the system (1.1) by a substitution $u = x - \xi_1$, $v = k_1 + k_2 x - x^3 - y$ to an equivalent

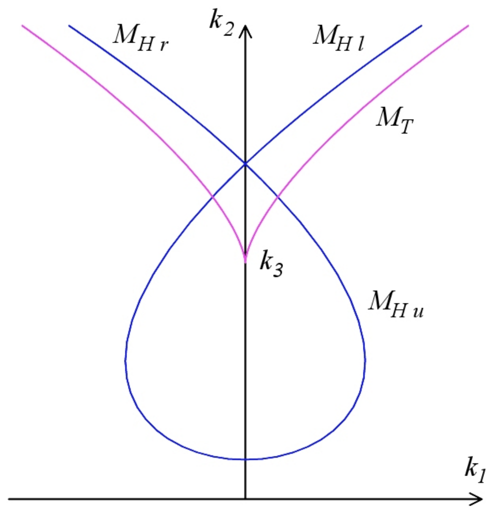

Figure 4. The section of M_T and M_H in $k_3 > 1$.

system of Lienard's type

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= p(u) + q(u)v \equiv p_0 + p_1 u + p_2 u^2 + p_3 u^3 + (q_0 + q_1 u + q_2 u^2)v, \end{aligned} \tag{2.6}$$

where

$$\begin{gathered} p_0 = k_1 + (k_2 - k_3)\xi_1 - \xi_1^3, \quad p_1 = k_2 - k_3 - 3\xi_1^2, \quad p_2 = -3\xi_1, \quad p_3 = -1, \\ q_0 = -1 + k_2 - 3\xi_1^2, \quad q_1 = -6\xi_1, \quad q_2 = -3. \end{gathered} \tag{2.7}$$

Since (1.2) holds for the equilibrium point (ξ_1, ξ_2), we have $p_0 = 0$, system (10) has an equilibrium point at the origin. The origin is a multiple focus if and only if $p_1 < 0$ and $q_0 = 0$. According to [8] or [3], we can express the first and the second Lyapunov numbers as

$$l_1 = p_2 q_1 - p_1 q_2, \quad l_2 = -p_3 q_2.$$

Consequently from (2.7)

$$l_1 = 3(k_2 - k_3 + 3\xi_1^2), \quad l_2 = -3.$$

Since $\operatorname{tr} A = 0$ on M_H, we get

$$l_1 = 3(1 - k_3 + 6\xi_1^2) = 3(2k_2 - k_3 - 1), \quad l_2 = -3 \neq 0.$$

The condition $l_1 = 0$ determines the subset M_{DH} on M_H (see fig. 5) that corresponds with the degenerate Andronov-Hopf bifurcation of codimension 2 (since $l_2 \neq 0$). The curve M_{DH} divides the surface M_H into parts M_{DH-} corresponding with the supercritical bifurcation ($l_1 < 0$, a stable limit cycle occurres) and M_{DH+} corresponding with the subcritical bifurcation ($l_1 > 0$, an unstable limit cycle occurres). $\square$

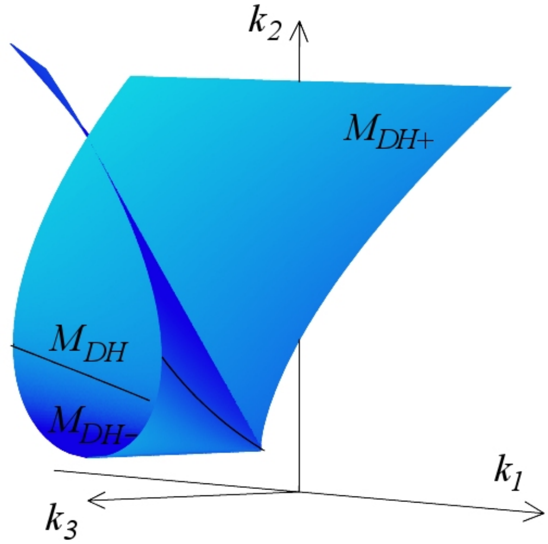

FIGURE 5. The set M_{DH}.

Remark 2.6. The set M_{DH-} is entirely contained in the set M_{Hu}, which imply that the stable limit cycle (caused by Andronov-Hopf bifurcation) may occur only in the case of the unique equilibrium point.

Theorem 2.7. *The subset M_{BT} of the parameter space K,*

$$M_{BT} = \{(k_1, k_2, k_3) \in K : k_1 = -2\xi_1^3, k_2 = 1 + 3\xi_1^2, k_3 = 1, \xi_1 \in \mathbb{R} - \{0\}\},$$

is a bifurcation set of codimension 2 corresponding with Bogdanov-Takens bifurcation.

Proof. Let (ξ_1, ξ_2) be an equilibrium point of the system (1.1). The bifurcation set of codimension 2 corresponding with Bogdanov-Takens bifurcation includes such parameters from K that both eigenvalues of Jacobi's matrix A are zero. The set M_{BT} is determined by two conditions $\det A = 0$ and $\operatorname{tr} A = 0$. The set M_{BT} lies in the intersection of the closure of M_H and the set M_T. In the case $\xi_1 = 0$, that is for $k_1 = 0, k_2 = k_3 = 1$, bifurcation of higher codimension occurres. Further analysis of this bifurcation is presented in Theorem 3.2. □

Theorem 2.8. *The subset M_C of the parameter space K,*

$$M_C = \{(k_1, k_2, k_3) \in K : k_1 = 0, k_2 = k_3, k_3 \neq 1\},$$

is a bifurcation set of codimension 2 - triple equilibrium point. The unique equilibrium point $(0,0)$ of (1.1) is topologically equivalent to a stable node for $k_3 < 1$, or an unstable node surrounded by a stable limit cycle for $k_3 > 1$.

Proof. The Jacobi's matrix on M_C is

$$A = \begin{pmatrix} k_2 & -1 \\ k_2 & -1 \end{pmatrix}$$

and its eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = k_2 - 1$. The origin is the unique equilibrium point of (1.1) and it is stable for $k_3 < 1$, unstable for $k_3 > 1$. The unstable unique equilibrium is surrounded by a stable limit cycle according to Lemma 1.1 on existence of a globally attractive set and the Poincaré's theorem. The origin is topologically equivalent to a node according to Lemma 2.1. □

3. Non-local bifurcations

In contradiction to local bifurcations, where the bifurcation sets could be expressed explicitly, bifurcation sets corresponding with non-local bifurcations can only be studied numerically or can be approximated with accuracy to a particular order in the neighbourhood of some important bifurcation points.

Non-local bifurcation of codimension 1 - multiple cycle. The curve M_{DH} is a boundary of a surface M_D corresponding with non-local bifurcation of codimension 1 - multiple cycle. While crossing the set M_D, two limit cycles (stable and unstable) merge into one semi-stable cycle that disappears then. Closures of sets M_D and M_H are tangent to each other in each point of the curve M_{DH}. The following schematic figure 6 shows the lay-out of the sets M_H, M_T and M_D only. They are figured by their intersections with the plane $k_3 =$ const. > 1. The numerical computations shows, that these sets lie closely to each other and there are technical problems with their rendering on the same scale.

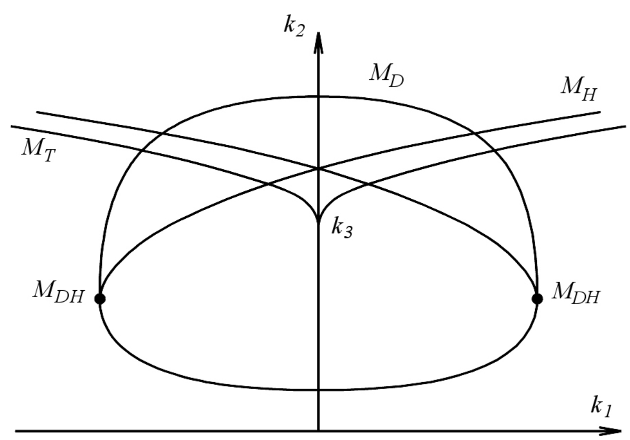

Figure 6. The section of M_{DH} in $k_3 > 1$.

Non-local bifurcation of codimension 1 - separatrix loop. The curve M_{BT} is a boundary of the surface M_L corresponding with non-local bifurcation of codimension 1 - separatrix loop. The surface M_L is tangent to M_T and M_H at each point of M_{BT}. The set M_L is contained in the half-space $k_3 > 1$ and consists of two components M_{Lr} and M_{Ll} corresponding with existence of the separatrix loop surrounding the right or the left equilibrium point respectively. While crossing the bound M_L, the unstable limit cycle (originated near M_H in consequence of the subcritical Andronov-Hopf bifurcation) merge into the separatrix loop and splits.

Let (ξ_1, ξ_2) be the right double equilibrium point of the system (1.1). Then the parameters of the system (1.1) lie in the set M_{BT} (Bogdanov-Takens bifurcation) and the coordinates of the double equilibrium point satisfy

$$\xi_1 = \sqrt{\frac{k_2 - 1}{3}}, \quad \xi_2 = k_3 \sqrt{\frac{k_2 - 1}{3}}$$

according to Theorem 2.7. Using the following substitution

$$x = x_1 - \sqrt{\frac{k_2 - 1}{3}}, \quad y = x_2 - k_3 \sqrt{\frac{k_2 - 1}{3}},$$

we transform the system (1.1) into a system

$$\begin{gathered}\dot{x} = k_1 + \sqrt{\frac{k_2 - 1}{3}} \Big(k_2 - k_3 - \frac{k_2 - 1}{3} \Big) + x - \sqrt{3(k_2 - 1)} x^2 - x^3 - y, \\ \dot{y} = k_3 x - y.\end{gathered} \tag{3.1}$$

The origin is a double equilibrium point of the system (12) with two zero eigenvalues for parameters from M_{BT}.

System (3.1) can be transformed by the linear transformation $x_1 = y$, $x_2 = k_3 x - y$ into the system

$$\begin{gathered}\dot{x}_1 = x_2, \\ \dot{x}_2 = h_{00} + h_{10} x_1 + \frac{1}{2} h_{20} x_1^2 + h_{11} x_1 x_2 + \frac{1}{2} h_{02} x_2^2 + R(x_1, x_2, k_1, k_2, k_3),\end{gathered} \tag{3.2}$$

where

$$\begin{gathered}h_{00} = k_3 \Big(k_1 + \sqrt{\frac{k_2 - 1}{3}} \big(k_2 - k_3 - \frac{k_2 - 1}{3} \big) \Big), \quad h_{10} = 1 - k_3, \\ h_{20} = -\frac{2}{k_3} \sqrt{3(k_2 - 1)}, \quad h_{11} = -\frac{2}{k_3} \sqrt{3(k_2 - 1)}, \\ h_{02} = -\frac{2}{k_3} \sqrt{3(k_2 - 1)}, \quad R(x_1, x_2, k_1, k_2, k_3) = -\frac{(x_1 + x_2)^3}{k_3^2}.\end{gathered}$$

This transformation keeps the equilibrium point at the origin as well as its zero eigenvalues. In the further analysis, we will study system (3.2) instead of the equivalent system (1.1).

Remark 3.1. For $(k_1, k_2, k_3) \in M_{BT}$, the following statements hold

$$h_{00} = 0, \quad h_{10} = 0, \quad h_{11} = h_{20} = h_{02} \neq 0.$$

Theorem 3.2. *The system* (3.2) *can be transformed by a smooth non-degenerate change of parameters to the Bogdanov-Takens normal canonical form*

$$\begin{gathered}\dot{x}_1 = x_2, \\ \dot{x}_2 = \beta_1 + \beta_2 x_1 + x_1^2 + x_1 x_2 + O(\|x\|^3),\end{gathered} \tag{3.3}$$

where

$$\begin{gathered}\beta_1 = \frac{h_{11}}{(-h_{10} + \frac{1}{4} h_{02} h_{00} + \frac{1}{2})^3} h_{00}, \\ \beta_2 = \frac{1}{(-h_{10} + \frac{1}{4} h_{02} h_{00} + \frac{1}{2})^2} (h_{10} - h_{00} h_{02}).\end{gathered} \tag{3.4}$$

In the neighbourhood of the Bogdanov-Takens curve M_{BT} corresponding with the right double equilibrium point, the set M_{Lr} can be expressed at the form

$$M_{Lr} = \big\{(k_1,k_2,k_3) \in \mathbb{R}^3 : \beta_2 < 0,\, \beta_1 = -\frac{6}{25}\beta_2^2 + o(\beta_2^2)\big\}. \tag{3.5}$$

The set M_{Ll} is symmetrical to M_{Lr} according to the plane $k_1 = 0$.

Proof. The change of time $dt = (1 - \frac{h_{02}}{2}x_1)d\tau$ and the substitution

$$u_1 = x_1, \quad u_2 = x_2 - \frac{h_{02}}{2}x_1x_2$$

eliminates the term with x_2^2. We get a system of the form

$$\begin{gathered}\dot u_1 = u_2,\\ \dot u_2 = \nu_1 + \nu_2 u_1 + C_1u_1^2 + C_2u_1u_2 + O(\|u\|^3),\end{gathered}$$

where

$$\nu_1 = h_{00}, \quad \nu_2 = h_{10} - h_{00}h_{02}, \quad C_1 = -h_{02}h_{10} + \frac{1}{4}h_{02}^2h_{00} + \frac{1}{2}h_{20}, \quad C_2 = h_{11}.$$

Note that $C_1 = \frac{1}{2}h_{20} \neq 0$ on M_{BT} according to Remark 3.1. Introducing a new time (denoted again with t)

$$t = \big|\frac{C_2}{C_1}\big|\tau$$

and new variables (denoted again with x_1 and x_2)

$$x_1 = \frac{C_2^2}{C_1}u_1, \quad x_2 = \operatorname{sgn}\Big(\frac{C_2}{C_1}\Big)\frac{C_2^3}{C_1^2}u_2,$$

we get the Bogdanov-Takens normal canonical form (3.3), where

$$\begin{gathered}\beta_1 = \frac{h_{11}^4}{(-h_{02}h_{10} + \frac{1}{4}h_{02}^2h_{00} + \frac{1}{2}h_{20})^3}h_{00},\\ \beta_2 = \frac{h_{11}^2}{(-h_{02}h_{10} + \frac{1}{4}h_{02}^2h_{00} + \frac{1}{2}h_{20})^2}(h_{10} - h_{00}h_{02}).\end{gathered}$$

With respect to the fact that $h_{20} = h_{11} = h_{02}$, we get the expressions (3.4).

The coefficient of the term with x_1x_2 corresponds to

$$s = \operatorname{sgn}\Big(\frac{C_2}{C_1}\Big)\big|_{M_{BT}} = \operatorname{sgn}\Big(\frac{h_{11}}{-h_{02}h_{10} + \frac{1}{4}h_{02}^2h_{00} + \frac{1}{2}h_{20}}\Big)\big|_{M_{BT}}.$$

According to Remark 3.1, we have $s = \operatorname{sgn} 2 = 1$. The Bogdanov-Takens bifurcation is non-degenerate, since

$$h_{11} = -2\sqrt{3(k_2 - 1)} = -6\xi_1 \neq 0$$

and $h_{20} \neq 0$ on M_{BT}. The change of parameters is invertible in the neighbourhood of the origin. It can be verified by a direct computation of the following determinants and finding

$$\begin{vmatrix} \frac{\partial\beta_1}{\partial k_1} & \frac{\partial\beta_1}{\partial k_2}\\ \frac{\partial\beta_2}{\partial k_1} & \frac{\partial\beta_2}{\partial k_2}\end{vmatrix} \neq 0, \quad \begin{vmatrix} \frac{\partial\beta_1}{\partial k_2} & \frac{\partial\beta_1}{\partial k_3}\\ \frac{\partial\beta_2}{\partial k_2} & \frac{\partial\beta_2}{\partial k_3}\end{vmatrix} \neq 0, \quad \begin{vmatrix} \frac{\partial\beta_1}{\partial k_3} & \frac{\partial\beta_1}{\partial k_1}\\ \frac{\partial\beta_2}{\partial k_3} & \frac{\partial\beta_2}{\partial k_1}\end{vmatrix} \neq 0.$$

This fact implies that the change of parameters cause no degeneration of the bifurcation manifold according to the parameter space. (In the bifurcation theory this regularity of the parameter transformation is called the transversality condition.)

The expression for the set M_L can be found in [6, Theorem 8.5, Appendix] or in [4]. The set M_{Ll} has to be symmetric to M_{Lr} about to the plane $k_1 = 0$. □

Non-local bifurcation of codimension 2 - two separatrix loops. The curve M_{LL}, which is an intersection of the sets M_{Lr} and M_{Ll} and lies in the plane $k_1 = 0$ (because of the symmetry of the parameter portrait) corresponds with the non-local bifurcation of codimension 2 - two separatrix loops. Two separatrix loops surround both the right and the left equilibrium points (see fig. 7).

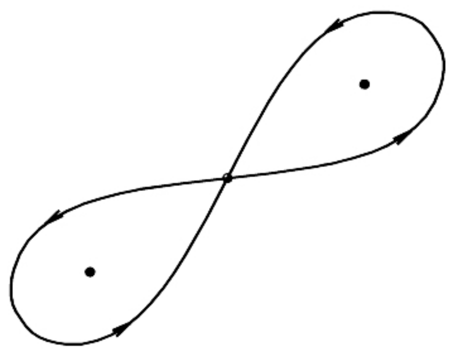

Figure 7. Structurally unstable two separatrix loops.

Non-local bifurcation of codimension 1 - "big separatrix loop". According to [7], the curve M_{LL} is a boundary of a bifurcation set M_{BL} corresponding with non-local bifurcation of codimension 1 - ,,big separatrix loop". While crossing the set M_{BL}, separatrix loop surrounding both equilibrium points appears and consequently gives to arise to an unstable limit cycle containing the saddle and both remaining equilibrium points in its interior (see fig. 8).

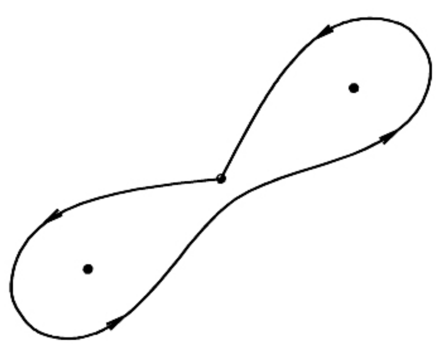

Figure 8. A structurally unstable big separatrix loop.

Figure 9 presents the lay-out of the sets M_T, M_H, M_L and M_{BL}, showing the section of the parameter space K by the plane $k_3 = const. > 1$, near 1.

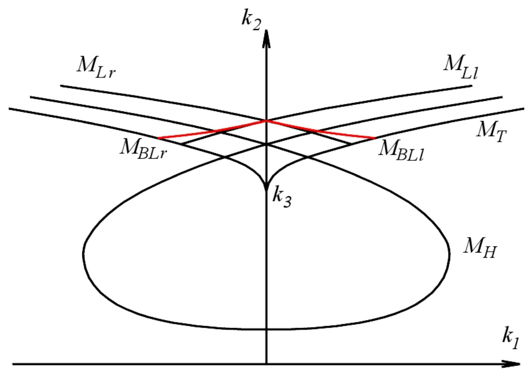

Figure 9. The section of M_L and M_{BL} in $k_3 > 1$.

4. Global bifurcation diagram

The bifurcation sets described above divide the parameter space K into parts, where the phase portraits of system (1.1) are topologically equivalent and structurally stable. The bifurcation sets contain those parameters, for which the phase portraits are structurally unstable.

Figure 10 shows a section of the global bifurcation diagram by the plane k_3 =const. for $k_3 \in (0, 1]$, and figure 11 this section for $k_3 > 1$. Figure 12 shows the structurally stable phase portraits corresponding to the marked regions for $k_1 < 0$. The half-space $k_1 > 0$ is symmetrical to the opposite one and the phase portraits are symmetrical according to the origin.

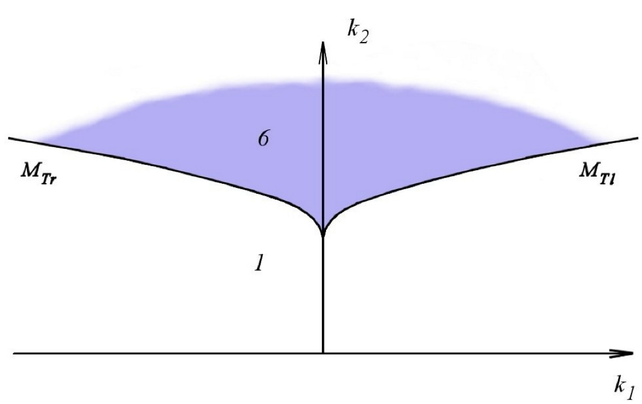

Figure 10. The section of the bifurcation diagram in $k_3 \in (0, 1]$.

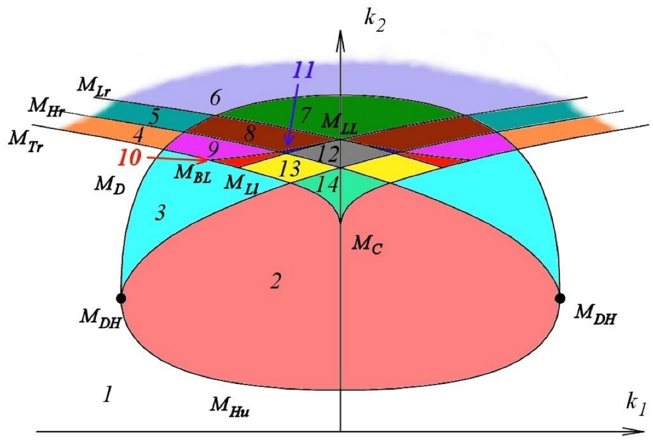

FIGURE 11. The section of the bifurcation diagram in $k_3 > 1$.

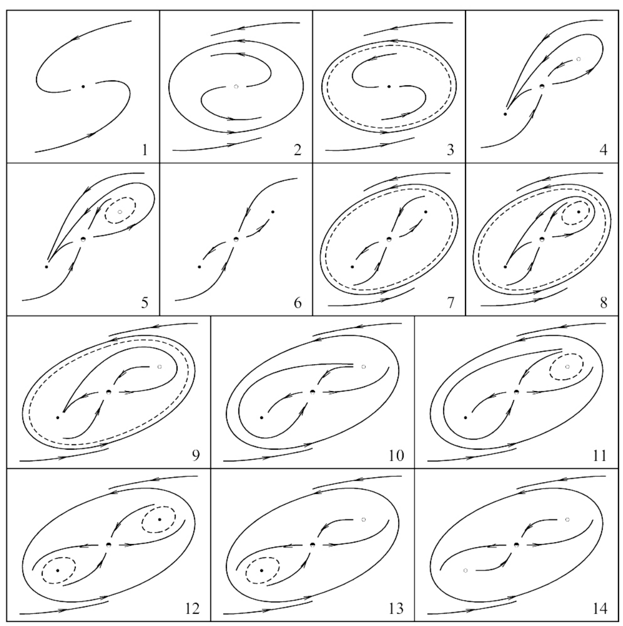

FIGURE 12. Phase portraits corresponding to the marked regions.

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