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MAXIMUM PRINCIPLE AND EXISTENCE OF POSITIVE SOLUTIONS FOR NONLINEAR SYSTEMS ON $\mathbb{R}^N$

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Abstract. In this paper, we study the following non-linear system on $\mathbb{R}^N$

$$\begin{gathered}-\Delta_p u = a(x)|u|^{p-2}u + b(x)|u|^{\alpha}|v|^{\beta}v + f \quad x \in \mathbb{R}^N \\ -\Delta_q v = c(x)|u|^{\alpha}|v|^{\beta}u + d(x)|v|^{q-2}v + g \quad x \in \mathbb{R}^N \\ \lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = 0, \quad u, v > 0 \quad \text{in } \mathbb{R}^N\end{gathered}$$

where $\Delta_p u = \operatorname{div}|\nabla u|^{p-2}\nabla u)$ with $p > 1$ and $p \neq 2$ is the "p-Laplacian", $\alpha, \beta > 0$, $p, q > 1$, and f, g are given functions. We obtain necessary and sufficient conditions for having a maximum principle; then we use an approximation method to prove the existence of positive solution for this system.

1. Introduction

The operator $-\Delta_p$ occurs in problems arising in pure mathematics, such as the theory of quasiregular and quasiconformal mappings and in a variety of applications, such as non-Newtonian fluids, reaction-diffusion problems, flow through porous media, nonlinear elasticity, glaciology, petroleum extraction, astronomy, etc (see [14, 16]).

We are concerned with existence of positive solutions and with the following form of the maximum principle: If $f, g \geq 0$ then $u, v \geq 0$ for any solution (u, v) of (1.3).

The maximum principle for linear elliptic systems with constant coefficients and the same differential operator in all the equations, have been studied in [7, 9]. Systems defined on unbounded domains and involving Schrödinger operators have been considered in [1, 2, 22]. In [18, 19], the authors presented necessary and sufficient conditions for having the maximum principle and for existence of positive solutions for linear systems involving Laplace operator with variable coefficients. These results have been extended in [15], to the nonlinear system

$$\begin{gathered}-\Delta_p u_i = \sum_{j=1}^{n} a_{ij}|u_j|^{p-2}u_j + f_i \quad u_i \text{ in } \Omega \\ u_i = 0 \quad \text{on } \partial\Omega\end{gathered} \tag{1.1}$$

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In [6], it has been proved the validity of the maximum principle and the existence of positive solutions for the following system defined on bounded domain Ω of $\mathbb{R}^N$, and with cooperative constant coefficients a, b, c, d:

$$\begin{gathered} -\Delta_p u = a|u|^{p-2}u + b|u|^{\alpha}|v|^{\beta}v + f, \quad \text{in } \Omega \\ -\Delta_q v = c|u|^{\alpha}|v|^{\beta}u + d|v|^{p-2}v + g, \quad \text{in } \Omega \\ u = v = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.2}$$

Here, we study system

$$\begin{gathered} -\Delta_p u = a(x)|u|^{p-2}u + b(x)|u|^{\alpha}|v|^{\beta}v + f \quad x \in \mathbb{R}^N \\ -\Delta_q v = c(x)|u|^{\alpha}|v|^{\beta}u + d(x)|v|^{q-2}v + g \quad x \in \mathbb{R}^N \\ \lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = 0, \quad u, v > 0 \quad \text{in } \mathbb{R}^N \end{gathered} \tag{1.3}$$

where $\Delta_p u = \operatorname{div} |\nabla u|^{p-2}\nabla u)$ with $p > 1$ and $p \neq 2$ is the "p-Laplacian", $\alpha, \beta > 0$, $p, q > 1$, and f, g are given functions.

System (1.3) is a generalization for (1.2) to the whole space $\mathbb{R}^N$ and the coefficients $a(x), b(x), c(x), d(x)$ are variables. We obtain necessary and sufficient conditions on the coefficients for having a maximum principle for system (1.3). Then using the method of sup and super solutions, we prove the existence of positive solutions under some conditions on the functions f and g.

This article is organized as follows: In section 2, we give some assumptions on the coefficients $a(x), b(x), c(x), d(x)$ and on the functions f, g to insure the existence of a solution of (1.3) in a suitable Sobolev space. We also introduce some technical results and some notation, which are established in [3, 4, 16]. Section 3 is devoted to the maximum principle of system (1.3), while section 4 is devoted to the existence of positive solutions.

2. Technical results

In this section, we introduce some technical results concerning the eigenvalue problem (see [16])

$$\begin{gathered} -\Delta_p u = \lambda g(x)\Psi_p(u) \qquad \text{in } \mathbb{R}^N \\ u(x) \to 0 \text{ as } |x| \to \infty, \quad u > 0 \text{ in } \mathbb{R}^N \end{gathered} \tag{2.1}$$

where $\Psi_p(u) = |u|^{p-2}u$ and $g(x)$ satisfies

$$g(x) \in L^{N/p}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N), \quad g(x) \geq 0 \text{ almost everywhere in } \mathbb{R}^N \tag{2.2}$$

For $1 < p < N$, let $p^* = \frac{pN}{N-p}$ be the critical Sobolev exponent of p. Let us introduce the Sobolev space $D^{1,p}(\mathbb{R}^N)$ defined as the completion of $C_0^{\infty}(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{D^{1,p}} = \Big(\int_{\mathbb{R}^N} |\nabla u|^p\Big)^{1/p}$$

It can be shown that

$$D^{1,p} = \big\{u \in L^{\frac{Np}{N-p}}(\mathbb{R}^N) : \nabla u \in \big(L^p(\mathbb{R}^N)\big)^N\big\}$$

and that there exists $k > 0$ such that for all $u \in D^{1,p}$,

$$\|u\|_{L^{Np/(N-p)}} \leq k\|u\|_{D^{1,p}} \tag{2.3}$$

Clearly $D^{1,p}(\mathbb{R}^N)$ is a reflexive Banach space, which is embedded continuously in $L^{Np/(N-p)}(\mathbb{R}^N)$ (see [11]).

Lemma 2.1. (i) *If $\{u_n\}$ is a sequence in $D^{1,p}$, with $u_n \to u$ weakly, then there is a subsequence, denoted again by $\{u_n\}$, such that $B(u_n) \to B(u)$*
(ii) *If $B'(u) = 0$, then $B(u) = 0$, where $B(u) = \int_{\mathbb{R}^N} g(u)|u|^p dx$.*

Theorem 2.2. *Let g satisfy* (2.2)*. Then* (2.1) *admits a positive first eigenvalue $\lambda_g(p)$. Moreover, it is characterized by*

$$\lambda_g(p) \int_{\mathbb{R}^N} g(u)|u|^p \le \|u\|^p_{D^{1,p}} \tag{2.4}$$

Theorem 2.3. *Let g satisfy* (2.2)*. Then*

(i) *the eigenfunction associated to $\lambda_g(p)$ is of constant sign; i.e., $\lambda_g(p)$ is a principal eigenvalue.*
(ii) *$\lambda_g(p)$ is the only eigenvalue of* (2.1) *which admits positive eigenfunction.*

3. Maximum Principle

We assume that $1 < p,\, q < N$ and that the coefficients $a(x), b(x), c(x)$, and $d(x)$ are smooth positive functions such that

$$a(x), d(x) \in L^{p/N}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \tag{3.1}$$

and

$$\begin{gathered} b(x) < (a(x))^{\alpha+1/p}(d(x))^{\beta+1/q} \\ c(x) < (a(x))^{\alpha+1/p}(d(x))^{\beta+1/q}, \end{gathered} \tag{3.2}$$

where

$$\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1, \quad \alpha+\beta+2 < N, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad \frac{1}{q} + \frac{1}{q'} = 1 \tag{3.3}$$

Theorem 3.1. *Assume that* (3.1) *and* (3.2) *hold. For $f \in L^{\frac{Np}{N(p-1)+p}}(\mathbb{R}^N)$, $g \in L^{\frac{Nq}{N(q-1)+q}}(\mathbb{R}^N)$, system* (1.3) *satisfies the maximum principle if the following conditions are satisfied:*

$$\lambda_a(p) > 1, \quad \lambda_d(q) > 1 \tag{3.4}$$

$$\big(\lambda_a(p) - 1\big)^{(\alpha+1)/p}\big(\lambda_d(q) - 1\big)^{(\beta+1)/q} - 1 > 0 \tag{3.5}$$

Conversely, if the maximum principle holds, then (3.4) *holds and*

$$[(\lambda_a(p) - 1)]^{(\alpha+1)/p}[\lambda_d(q) - 1]^{(\beta+1)/q} > \Theta\Big(\frac{b(x)}{a(x)}\Big)^{(\alpha+1)/p}\Big(\frac{c(x)}{d(x)}\Big)^{(\beta+1)/q}, \tag{3.6}$$

where

$$\Theta = \frac{\inf_x \big(\phi^p/\psi^q\big)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}}{\sup_x \big(\phi^p/\psi^q\big)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}}$$

and ϕ (respectively ψ) is the positive eigenfunction associated to $\lambda_a(p)$ (respectively $\lambda_d(q)$).

Proof. The necessary condition: Assume that $\lambda_a(p) \le 1$, then the function $f := a(x)(1-\lambda_a(p))\phi^{p-1}$ and $g := 0$ are nonnegative; nevertheless $(-\phi, 0)$ satisfies (1.3), which contradicts the maximum principle.

Similarly, if $\lambda_d(q) \le 1$, then the functions $g := d(x)(1-\lambda_d(q))\psi^{q-1}$ and $f := 0$ are nonnegative; nevertheless $(0, -\psi)$ satisfies (1.3), which contradicts the maximum principle.

Now suppose that $\lambda_a(p) > 1$, $\lambda_d(q) > 1$ and (3.6) does not hold; i.e.,

$$[(\lambda_a(p)-1)]^{(\alpha+1)/p}[\lambda_d(q)-1]^{(\beta+1)/q} \le \Theta\Big(\frac{b(x)}{a(x)}\Big)^{(\alpha+1)/p}\Big(\frac{c(x)}{d(x)}\Big)^{(\beta+1)/q},$$

Then, there exists $\xi > 0$ such that

$$\Big(a(x)\frac{(\lambda_a(p)-1)}{b(x)}\Big)^{(\alpha+1)/p}\Big(\frac{\phi^p}{\psi^q}\Big)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}} \le \xi \le \frac{\big(\frac{c(x)}{d(x)}\big)^{(\beta+1)/q}}{(\lambda_d(q)-1)^{(\beta+1)/q}}\Big(\frac{\phi^p}{\psi^q}\Big)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}$$

Let $\xi = \big(\frac{D^q}{C^p}\big)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}$ with $C, D > 0$. Then

$$\begin{aligned}&\Big(a(x)\frac{(\lambda_a(p)-1)}{b(x)}\Big)^{(\alpha+1)/p}\Big(\frac{(C\phi)^p}{(D\psi)^q}\Big)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}\\ &\le 1 \le \frac{\big(\frac{c(x)}{d(x)}\big)^{(\beta+1)/q}}{(\lambda_d(q)-1)^{(\beta+1)/q}}\Big(\frac{(C\phi)^p}{(D\psi)^q}\Big)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}\end{aligned}$$

So

$$a(x)(\lambda_a(p)-1)((C\phi)^p)^{(\beta+1)/q} \le b(x)(D\psi)^{\beta+1} \tag{3.7}$$

$$d(x)(\lambda_d(q)-1)((D\psi)^q)^{(\alpha+1)/p} \le c(x)(C\phi)^{\alpha+1} \tag{3.8}$$

From the two expression above, we have

$$\begin{gathered}a(x)(\lambda_a(p)-1)((C\phi))^{p-1} \le b(x)(D\psi)^{\beta+1}(C\phi)^{\alpha},\\ d(x)(\lambda_d(q)-1)((D\psi)^{q-1}) \le c(x)(C\phi)^{\alpha+1}(D\psi)^{\beta}.\end{gathered}$$

Hence

$$\begin{gathered}f = -a(x)(\lambda_a(p)-1)((C\phi))^{p-1} + b(x)(D\psi)^{\beta+1} + (C\phi)^{\alpha} \ge 0,\\ g = -d(x)(\lambda_d(q)-1)((D\psi))^{q-1} + c(x)(D\psi)^{\beta} + (C\phi)^{\alpha+1} \ge 0\end{gathered}$$

Since f and g are nonnegative functions, and $(-C\phi, -D\psi)$ is a solution of (1.3) and the maximum principle does not hold. □

Now, we show that the condition is sufficient. Assume that (3.4) and (3.5) hold. If (u, v) is a solution of (1.3), then for $f, g \ge 0$, we obtain by multiplying the first equation of (1.3) by $\overline{u} := \max(0, -u)$ and integrating over $\mathbb{R}^N$:

$$\begin{aligned}-\int_{\mathbb{R}^N}|\nabla\overline{u}|^p = -\int_{\mathbb{R}^N} a(x)|\overline{u}|^p &+ \int_{\mathbb{R}^N} b(x)|\overline{u}|^{\alpha+1}|v^+|^{\beta}v^+\\ &- \int_{\mathbb{R}^N} b(x)|\overline{u}|^{\alpha+1}|v^-|^{\beta}v^- + \int_{\mathbb{R}^N} fu^-.\end{aligned}$$

Then

$$\int_{\mathbb{R}^N}|\nabla\overline{u}|^p \le \int_{\mathbb{R}^N} a(x)|\overline{u}|^p + \int_{\mathbb{R}^N} b(x)|\overline{u}|^{\alpha+1}|v^-|^{\beta+1}.$$

From (2.4), we get

$$(\lambda_a(p)-1)\int_{\mathbb{R}^N} a(x)|\overline{u}|^p \le \int_{\mathbb{R}^N} b(x)|\overline{u}|^{\alpha+1}|v^-|^{\beta+1}$$

Applying Holder inequality and using (3.2), we find

$$(\lambda_a(p)-1)\int_{\mathbb{R}^N} a(x)|\overline{u}|^p \le \Big(\int_{\mathbb{R}^N} a(x)|\overline{u}|^p\Big)^{(\alpha+1)/p}\Big(\int_{\mathbb{R}^N} d(x)|v^-|^q\Big)^{(\beta+1)/q}$$

Hence

$$\begin{aligned}&\Big[(\lambda_a(p)-1)\Big(\int_{\mathbb{R}^N} a(x)|\overline{u}|^p\Big)^{(\beta+1)/q}\\ &-\Big(\int_{\mathbb{R}^N} d(x)|v^-|^q\Big)^{(\beta+1)/q}\Big]\Big(\int_{\mathbb{R}^N} a(x)|\overline{u}|^p\Big)^{(\alpha+1)/p}\le 0\end{aligned}$$

If $\big(\int_{\mathbb{R}^N} a(x)|\overline{u}|^p\big)^{(\alpha+1)/p}=0$, then $\overline{u}=0$. If not, we have

$$(\lambda_a(p)-1)^{(\alpha+1)/p}\Big(\int_{\mathbb{R}^N} a(x)|\overline{u}|^p\Big)^{\frac{\alpha+1}{p}\,\frac{\beta+1}{q}} \le \Big(\int_{\mathbb{R}^N} d(x)|v^-|^q\Big)^{\frac{\beta+1}{q}\,\frac{\alpha+1}{p}} \tag{3.9}$$

Similarly

$$(\lambda_d(q)-1)^{(\beta+1)/q}\Big(\int_{\mathbb{R}^N} d(x)|\overline{v}|^q\Big)^{\frac{\alpha+1}{p}\,\frac{\beta+1}{q}} \le \Big(\int_{\mathbb{R}^N} a(x)|u^-|^p\Big)^{\frac{\beta+1}{q}\,\frac{\alpha+1}{p}} \tag{3.10}$$

From (3.9) and (3.10), we obtain

$$\Big((\lambda_a(p)-1)^{(\alpha+1)/p}(\lambda_d(q)-1)^{(\beta+1)/q}-1\Big)\Big(\int_{\mathbb{R}^N} d(x)|\overline{v}|^q\int_{\mathbb{R}^N} a(x)|u^-|^p\Big)^{\frac{\beta+1}{q}\,\frac{\alpha+1}{p}}\le 0$$

From (3.5), we have $\overline{u}=\overline{v}=0$ and hence $u\ge 0$, $v\ge 0$ i.e. the maximum principle holds.

Corollary 3.2. *If $p=q$, then the maximum principle holds for system* (1.3) *if and only if conditions* (3.4) *and* (3.5) *are satisfied.*

4. Existence of positive solutions

By an approximation method used in [5], we prove now that the system (1.3) has a positive solution in the space $D^{1,p}\times D^{1,q}$. For $\epsilon\in(0,1)$, we introduce the system

$$\begin{gathered}-\Delta_p u_\epsilon = a(x)\frac{(|u_\epsilon|^{p-2}u_\epsilon)}{(1+|\epsilon^{1/p}u_\epsilon|^{p-1})}+b(x)\frac{|v_\epsilon|^\beta v_\epsilon}{(1+|\epsilon^{1/q}v_\epsilon|^{\beta+1})}\frac{|u_\epsilon|^\alpha}{(1+|\epsilon^{1/p}u_\epsilon|^{\alpha})}+f \text{ in } \mathbb{R}^N\\ -\Delta_q v_\epsilon = d(x)\frac{(|v_\epsilon|^{q-2}v_\epsilon)}{(1+|\epsilon^{1/q}v_\epsilon|^{q-1})}+c(x)\frac{|v_\epsilon|^\beta}{(1+|\epsilon^{1/q}v_\epsilon|^{\beta})}\frac{|u_\epsilon|^\alpha u_\epsilon}{(1+|\epsilon^{1/p}u_\epsilon|^{\alpha+1})}+g \text{ in } \mathbb{R}^N\\ \lim_{|x|\to\infty} u_\epsilon=\lim_{|x|\to\infty} v_\epsilon=0,\quad u_\epsilon,\, v_\epsilon>0\quad\text{in } \mathbb{R}^N\end{gathered} \tag{4.1}$$

Letting $(\xi,\eta)=(u_\epsilon,v_\epsilon)$ then system above can be written as

$$\begin{gathered}-\Delta_p\xi = h(\xi,\eta)+f\quad\text{in } \mathbb{R}^N\\ -\Delta_q\eta = k(\xi,\eta)+g\quad\text{in } \mathbb{R}^N\\ \xi,\eta\to 0 \text{ as } |x|\to\infty\quad \xi,\eta>0 \text{ in } \mathbb{R}^N\end{gathered}$$

where

$$h(\xi,\eta) = a(x)\frac{|\xi|^{p-2}\xi}{(1+|\epsilon^{1/p}\xi|^{p-1})} + b(x)\frac{|\eta|^{\beta}\eta}{(1+|\epsilon^{1/q}\eta|^{\beta+1})}\frac{|\xi|^{\alpha}}{(1+|\epsilon^{1/p}\xi|^{\alpha})},$$
$$k(\xi,\eta) = c(x)\frac{|\eta|^{\beta}}{(1+|\epsilon^{1/q}\eta|^{\beta})}\frac{|\xi|^{\alpha}\xi}{(1+|\epsilon^{1/p}\xi|^{\alpha+1})} + d(x)\frac{|\eta|^{q-2}\eta}{(1+|\epsilon^{1/q}\eta|^{q-1})}.$$

From (3.1) and (3.2) $h(\xi,\eta)$, $k(\xi,\eta)$ are bounded functions; i.e., there exists $M > 0$ such that $|h(\xi,\eta)| \le M$, and $|k(\xi,\eta)| \le M$ for all ξ,η. Then, as in [6], we can prove the following lemma.

Lemma 4.1. *If $(\xi_k,\eta_k) \to (\xi,\eta)$, weakly in $L^{Np/(N-p)}(\mathbb{R}^N) \times L^{Nq/(N-q)}(\mathbb{R}^N)$, then*

$$\Big\|a(x)\Big(\frac{|\xi_k(x)|^{p-2}\xi_k(x)}{(1+|\epsilon^{1/p}\xi_k(x)|^{p-1})} - \frac{|\xi(x)|^{p-2}\xi(x)}{(1+|\epsilon^{1/p}\xi(x)|^{p-1})}\Big)\Big\|_{L^{Np/(N(p-1)+p)}(\mathbb{R}^N)} \to 0 \quad (4.2)$$

$$\Big\|b(x)\Big(\frac{|\eta_k|^{\beta}\eta_k}{(1+|\epsilon^{1/q}\eta_k|^{\beta+1})}\frac{|\xi_k|^{\alpha}}{(1+|\epsilon^{1/p}\xi_k|^{\alpha})} - \frac{|\eta|^{\beta}\eta}{(1+|\epsilon^{1/p}\eta|^{\beta+1})}\frac{|\xi|^{\alpha}}{(1+|\epsilon^{1/p}\xi|^{\alpha})}\Big)\Big\| \quad (4.3)$$

approaches 0 *under the norm of $L^{Np/(N(p-1)+p)}(\mathbb{R}^N)$, a.e. in $\mathbb{R}^N$ as k approaches infinity in $L^{Nq/(N(q-1)+q)}(\mathbb{R}^N)$.*

$$\Big\|d(x)\Big(\frac{|\eta_k(x)|^{q-2}\eta_k(x)}{(1+|\epsilon^{1/q}\eta_k(x)|^{q-1})} - \frac{|\eta(x)|^{q-2}\eta(x)}{(1+|\epsilon^{1/q}\eta(x)|^{q-1})}\Big)\Big\| \to 0 \quad (4.4)$$

$$\Big\|c(x)\Big(\frac{|\xi_k|^{\alpha}\xi_k}{(1+|\epsilon^{1/p}\xi_k|^{\alpha+1})}\frac{|\eta_k|^{\beta}}{(1+|\epsilon^{1/q}\eta_k|^{\beta})} - \frac{|\xi|^{\alpha}\xi}{(1+|\epsilon^{1/p}\xi|^{\alpha+1})}\frac{|\eta|^{\beta}}{(1+|\epsilon^{1/q}\eta|^{\beta})}\Big)\Big\| \to 0 \quad (4.5)$$

a.e. in $\mathbb{R}^N$ as $k \to \infty$ in $L^{Nq/(N(q-1)+q)}(\mathbb{R}^N)$

Lemma 4.2. *System* (4.1) *has a solution $U_\epsilon =: (u_\epsilon, v_\epsilon)$ in $D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$.*

Proof. We complete the proof in four steps:
Step 1. Construction of sub-super solutions of (4.1): Let $\xi^0 \in D^{1,p}$ (respectively $\eta^0 \in D^{1,q}$ be a solution of

$$-\Delta_p \xi^0 = M + f \quad (\text{resp. } -\Delta_q \eta^0 = M + g) \quad (4.6)$$

and let $\xi_0 \in D^{1,p}$ (respectively $\eta_0 \in D^{1,q}$ be a solution of

$$-\Delta_p \xi_0 = -M + f \quad (\text{resp. } -\Delta_q \eta_0 = -M + g) \quad (4.7)$$

Then (η^0, ξ^0) is a super solution of (4.1) and η_0, ξ_0 is a sub solution since

$$-\Delta_p\xi^0 - h(\xi^0,\eta) - f \ge -\Delta_p\xi^0 - M - f = 0 \quad \forall \eta \in [\eta_0,\eta^0]$$
$$-\Delta_p\xi_0 - h(\xi_0,\eta) - f \le -\Delta_p\xi_0 - M - f = 0 \quad \forall \eta \in [\eta_0,\eta^0]$$
$$-\Delta_p\eta^0 - h(\xi,\eta^0) - g \ge -\Delta_q\eta^0 - (M+g) = 0 \quad \forall \xi \in [\xi_0,\xi^0]$$
$$-\Delta_p\eta_0 - h(\xi,\eta_0) - g \le -\Delta_q\eta_0 - (M+g) = 0 \quad \forall \xi \in [\xi_0,\xi^0]$$

Let us assume that $K = [\xi_0,\xi^0] \times [\eta_0 \times \eta^0]$.

Step 2. Definition of the operator T: We define the operator $T:(\xi,\eta)\to(w,z)$, where (w,z) is the solution of the system

$$\begin{gathered}
-\Delta_p w = h(\xi,\eta)+f \quad \text{in } \mathbb{R}^N\\
-\Delta_q z = k(\xi,\eta)+g \quad \text{in } \mathbb{R}^N\\
w=z\to 0 \quad \text{as } |x|\to\infty
\end{gathered} \tag{4.8}$$

Sept 3. Construction of an invariant set under T. We have to prove that $T(k)\subset K$: From (4.6) and (4.8), we get

$$-\Delta_p w-\Delta_p\xi^0 \le h(\xi,\eta)-M \tag{4.9}$$

Multiplying this equation by $(w-\xi^0)^+$ and integrating over $\mathbb{R}^N$, we obtain

$$\int_{\mathbb{R}^N}[\Psi_p(\nabla w)-\Psi_p(\nabla\xi^0)][\nabla(w-\xi^0)^+]\le\int_{\mathbb{R}^N}(h(\xi,\eta)-M)(w-\xi^0)^+\le 0.$$

By monotonicity of p-Laplacian, we have $(w-\xi^0)^+=0$ and hence $w\le\xi^0$. Similarly $w\ge\xi_0$, so that step is complete.

Step 4. T is completely continuous: We prove that T maps weakly convergent sequence to strongly convergence ones. From (4.8), we get

$$\begin{aligned}
&-\Delta_p w_k-\Delta_p w\\
&=a(x)\Big[\frac{|\xi_k^{p-2}|\xi_k}{(1+|\epsilon^{1/p}\xi_k|^{p-1})}-\frac{|\xi|^{p-2}\xi}{(1+|\xi^{1/p}\xi|^{p-1})}\Big]\\
&\quad+b(x)\Big[\frac{|\eta_k|^\beta\eta_k}{(1+|\epsilon^{1/q}|\eta_k^{\beta+1})}\frac{|\xi_k|^\alpha}{(1+|\epsilon^{1/p}\xi_k|^\alpha)}-\frac{|\eta|^\beta\eta}{(1+|\epsilon^{1/q}\eta|^{\beta+1})}\frac{|\xi_i|^\alpha}{(1+|\epsilon^{1/p}\xi_i|^\alpha)}\Big]
\end{aligned}$$

Multiplying by (w_k-w) and integrating over $\mathbb{R}^N$, we obtain

$$\begin{aligned}
&\int[\Psi_p(\nabla w_k)-\Psi_p(\nabla w)][\nabla(w_k-w)]\\
&=\int a(x)\Big[\frac{|\xi_k^{p-2}|}{(1+|\epsilon^{1/p}\xi_k|^{p-1})}-\frac{|\xi|^{p-2}\xi}{(1+|\xi^{1/p}\xi|^{p-1})}\Big](w_k-w)\\
&\quad+\int b(x)\Big[\frac{|\eta_k|^\beta\eta_k}{(1+|\epsilon^{1/q}|\eta_k^{\beta+1})}\frac{|\xi_k|^\alpha}{(1+|\epsilon^{1/p}\xi_k|^\alpha)}\\
&\quad-\frac{|\eta|^\beta\eta}{(1+|\epsilon^{1/q}\eta|^{\beta+1})}\frac{|\xi_i|^\alpha}{(1+|\epsilon^{1/p}\xi_i|^\alpha)}\Big](w_k-w)
\end{aligned}\tag{4.10}$$

Using Hölder's inequality, we obtain

$$\begin{aligned}
&\int[\Psi_p(\nabla w_k)-\Psi_p(\nabla w)][\nabla(w_k-w)]\\
&\le\Big\|a(x)\Big[\frac{|\xi_k^{p-2}|}{(1+|\epsilon^{1/p}\xi_k|^{p-1})}-\frac{|\xi|^{p-2}\xi}{(1+|\xi^{1/p}\xi|^{p-1})}\Big]\Big\|_{L^{Np/(N(p-1)+p}(\mathbb{R}^N)}\\
&\quad\times\|(w_k-w)\|_{L^{Np/(N-p)}(\mathbb{R}^N)}\\
&\quad+\Big\|b(x)\Big[\frac{|\eta_k|^\beta\eta_k}{(1+|\epsilon^{1/q}|\eta_k^{\beta+1})}\frac{|\xi_k|^\alpha}{(1+|\epsilon^{1/p}\xi_k|^\alpha)}\\
&\quad-\frac{|\eta|^\beta\eta}{(1+|\epsilon^{1/q}\eta|^{\beta+1})}\frac{|\xi_i|^\alpha}{(1+|\epsilon^{1/p}\xi_i|^\alpha)}\Big]\Big\|_{L^{Np/(N(p-1)+p}(\mathbb{R}^N)}\|(w_k-w)\|_{L^{Np/(N-p)}(\mathbb{R}^N)}
\end{aligned}$$

It is well known [23], that

$$|\xi-\xi'|^p \le c\{[|\xi|^{p-2}\xi-|\xi'|^{p-2}\xi'](\xi-\xi')\}^{\alpha/2}\{|\xi|^p+|\xi'|^p\}^{1-(\alpha/2)} \quad \forall \xi,\xi' \in \mathbb{R}^N \tag{4.11}$$

where $\alpha = p$ if $1 \le p \le 2$ and $\alpha = 2$ if $p > 2$. From (4.11) and the continuous imbedding of $D^{1,p}(\mathbb{R}^N)$ in $L^{Np/(N-p)}(\mathbb{R}^N)$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla(w_k - w)|^p \\
&\le k\Big(\Big\|a(x)\Big[\frac{|\xi_k^{p-2}|}{(1+|\epsilon^{1/p}\xi_k|^{p-1})} - \frac{|\xi|^{p-2}\xi}{(1+|\xi^{1/p}\xi|^{p-1})}\Big]\Big\|_{L^{Np/(N(p-1)+p}(\mathbb{R}^N)} \\
&\quad + \Big\|b(x)\Big[\frac{|\eta_k|^\beta\eta_k}{(1+|\epsilon^{1/q}|\eta_k^{\beta+1})}\frac{|\xi_k|^\alpha}{(1+|\epsilon^{1/p}\xi_k|^\alpha)} \\
&\quad - \frac{|\eta|^\beta\eta}{(1+|\epsilon^{1/q}\eta|^{\beta+1})}\frac{|\xi_i|^\alpha}{(1+|\epsilon^{1/p}\xi_i|^\alpha)}\Big]\Big\|_{L^{Np/(N(p-1)+p}(\mathbb{R}^N)}\Big)\|(w_k-w)\|
\end{aligned}$$

Applying lemma 4.1, we obtain $\|w_k - w\|^{p-1}_{D^{1,p}} \to 0$ as $k \to +\infty$ in $D^{1,p}(\mathbb{R}^N)$ which implies

$$(w_k) \to (w) \quad \text{strongly as } k \to +\infty \text{ in } D^{1,p}(\mathbb{R}^N).$$

Similarly we can prove that

$$(z_k) \to (z) \quad \text{as } k \to +\infty \text{ in } D^{1,p},$$

So T is completely continuous operator.

Since k is a convex, bounded, closed in: $D^{1,p} \times D^{1,q}$, we can apply Schauder's Fixed Point Theorem and obtain the existence of a fixed point for T, which gives the existence of solution $U_\epsilon =: (u_\epsilon, v_\epsilon)$ of S_ϵ . □

Now, we can prove the existence of solution for system (1.3).

Theorem 4.3. *If (2.2)–(2.4) are satisfied, then for any* $f \in L^{\frac{Np}{N(p-1)+p}}(\mathbb{R}^N)$, $g \in L^{\frac{Nq}{N(q-1)+q}}(\mathbb{R}^N)$, *system (1.3) has a nonnegative solution* $U = (u,v)$ *in the space:* $D^{1,p} \times D^{1,q}$.

Proof. This proof is done in three steps.
Step 1. First we prove that $U_\epsilon =: (\epsilon^{1/p}u_\epsilon, \epsilon^{1/q}v_\epsilon)$ is bounded in $D^{1,p} \times D^{1,q}$. Multiplying the first equation of (4.1) by (ϵu_ϵ) and integrating over $\mathbb{R}^N$, we obtain

$$\begin{aligned}
&\int |\nabla \epsilon^{1/p}u_\epsilon|^p \\
&\le \int a(x)|\epsilon^{1/p}u_\epsilon| + \int b(x)|\epsilon^{1/p}u_\epsilon| + \epsilon^{1/p'}\int |\epsilon^{1/p}u_\epsilon||f| \\
&\le \|a(x)\|_{N/p}\|\epsilon^{1/p}u_\epsilon\|_{Np/(N-p)} + \|b(x)\|_{Np/(N(p-1)+p)}\|\epsilon^{1/p}u_\epsilon\|_{Np/(N-p)} \\
&\quad + \epsilon^{1/p'}\|\epsilon^{1/p}u_\epsilon\|_{Np/(N-p)}\|f\|_{Np(N(p-1)+p)} \\
&\le M\|\epsilon^{1/p}u_\epsilon\|_{Np/(N-p)} \\
&\le kM\|\epsilon^{1/p}u_\epsilon\|_{D^{1,p}}
\end{aligned}$$

so $\|\epsilon^{1/p}u_\epsilon\|^{p-1}_{D^{1,p}} \le kM$ which implies $U_\epsilon =: (\epsilon^{1/p}u_\epsilon)$ is bounded in $D^{1,p}$. Similarly for $(\epsilon^{1/q}v_\epsilon)$.

Step 2. $U_\epsilon =: (\epsilon^{1/p}u_\epsilon, \epsilon^{1/q}v_\epsilon)$ converges to (0,0) strongly in $D^{1/p}\times D^{1/q}$. From step 1, $U_\epsilon =: (\epsilon^{1/p}u_\epsilon, \epsilon^{1/q}v_\epsilon)$ converges weakly to (u^*, v^*) in $D^{1,p}\times D^{1,q}$ and strongly in $L^{\frac{Np}{N-p}}(\mathbb{R}^N)\times L^{\frac{Nq}{N-q}}(\mathbb{R}^N)$. Multiplying the first equation of (4.1), by $(\epsilon^{1/p'})$, we get

$$-\Delta_p(\epsilon^{1/p}u_\epsilon) = a(x)\frac{|\epsilon^{1/p}u_\epsilon|^{p-2}(\epsilon^{1/p}u_\epsilon)}{(1+|\epsilon^{1/p}u_\epsilon|^{p-1})}+b(x)\frac{(\epsilon^{1/q}v_\epsilon)^\beta\epsilon^{1/q}v_\epsilon}{1+|\epsilon^{1/q}v_\epsilon|^{\beta+1}}\frac{|(\epsilon^{1/p}u_\epsilon)|^\alpha}{1+|(\epsilon^{1/p}u_\epsilon)|^\alpha}+f\epsilon^{1/p'}$$

Again using Lemma 4.1, we have

$$a(x)\frac{|\epsilon^{1/p}u_\epsilon|^{p-2}(\epsilon^{1/p}u_\epsilon)}{(1+|\epsilon^{1/p}u_\epsilon|^{p-1})}\to a(x)\frac{|u^*|^{p-2}(u^*)}{(1+|u^*|^{p-1})}\quad\text{strongly in } L^{\frac{Np}{N(p-1)+p}}(\mathbb{R}^N)$$

and similarly

$$b(x)\frac{(\epsilon^{1/q}v_\epsilon)^\beta\epsilon^{1/q}v_\epsilon}{1+|\epsilon^{1/q}v_\epsilon|^{\beta+1}}\frac{|(\epsilon^{1/p}u_\epsilon)|^\alpha}{1+|(\epsilon^{1/p}u_\epsilon)|^\alpha}\to b(x)\frac{|v^*|^\beta v^*}{(1+|v^*|^{\beta+1})}\frac{|u^*|^\alpha}{(1+|u^*|^\alpha)}$$

strongly in $L^{\frac{Np}{N(p-1)+p}}(\mathbb{R}^N)$. Using a classical result in [21], we have

$$-\Delta_p(\epsilon^{1/p}u_\epsilon)\to -\Delta_p(u_*)\quad\text{strongly in } L^{\frac{Np}{N(p-1)+p}}(\mathbb{R}^N).$$

So

$$-\Delta_p(u^*) = a(x)\frac{|u^*|^{p-2}(u^*)}{(1+|u^*|^{p-1})}+b(x)\frac{|v^*|^\beta v^*}{(1+|v^*|^{\beta+1})}\frac{|u^*|^\alpha}{(1+|u^*|^\alpha)}\tag{4.12}$$

Multiplying this equality by $(u^*)^-$ and integrating over $\mathbb{R}^N$, then applying (2.4) we get

$$(\lambda_a(p)-1)\int a(x)|u^{*-}|^p\le\int|\nabla u^{*-}|^p\le\int d(x)|v^{*-}|^{\beta+1}|u^{*-}|^{\alpha+1}$$

Using Hölder inequality and (2.3), as in the proof of Theorem 3.1, we deduce

$$(\lambda_a(p)-1)^{(\alpha+1)/p}\Big(\int a(x)|u^{*-}|^p\Big)^{\frac{\beta+1}{q}\frac{\alpha+1}{p}}\le\Big(\int d(x)|v^{*-}|^q\Big)^{\frac{\beta+1}{q}\frac{\alpha+1}{p}}\tag{4.13}$$

Similarly, from the second equation of (4.1), we have

$$(\lambda_d(q)-1)^{(\beta+1)/q}\Big(\int d(x)|v^{*-}|^q\Big)^{\frac{\beta+1}{q}\frac{\alpha+1}{p}}\le\Big(\int a(x)|u^{*-}|^p\Big)^{\frac{\beta+1}{q}\frac{\alpha+1}{p}}\tag{4.14}$$

Multiplying (4.13) by (4.14), we obtain

$$\Big((\lambda_d(q)-1)^{(\beta+1)/q}(\lambda_a(p)-1)^{(\alpha+1)/p}-1\Big)\Big(\int d(x)|v^{*-}|^q\int a(x)|u^{*-}|^p\Big)^{\frac{\beta+1}{q}\frac{\alpha+1}{p}}\le 0$$

From conditions (3.4) and (3.5), we have $u^*_- = v^*_- = 0$, which implies that $u^*, v^*\ge 0$. We show that $(u^* = v^* = 0)$. Multiplying (4.12) by u^* , we get

$$\Big((\lambda_d(q)-1)^{(\beta+1)/q}(\lambda_a(p)-1)^{(\alpha+1)/p}-1\Big)\Big(\int d(x)|v^*|^q\int a(x)|u^*|^p\Big)^{\frac{\beta+1}{q}\frac{\alpha+1}{p}}\le 0$$

which implies that $u^* = v^* = 0$, and step 2 is complete.
Step 3. (u_ϵ, v_ϵ) is bounded in $D^{1,p}\times D^{1,q}$: Assume that $\|u_\epsilon\|_{D^{1,p}}\to\infty$ or $\|v_\epsilon\|_{D^{1,q}}\to\infty$ Set $t_\epsilon = \max(\|u_\epsilon\|_{D^{1,p}}, \|v_\epsilon\|_{D^{1,q}})$ and $z_\epsilon = u_\epsilon t_\epsilon^{-1/p}$, $w_\epsilon = v_\epsilon t_\epsilon^{-1/q}$.

Dividing the first equation of (4.1) by $(t_\epsilon^{1/p'})$ and the second by $(t_\epsilon^{1/q'})$, we have

$$-\Delta_p(z_\epsilon) = a(x)\frac{|z_\epsilon|^{p-2}z_\epsilon}{(1+|\epsilon^{1/p}u_\epsilon|^{p-1})} + b(x)\frac{|w_\epsilon\|^\beta w_\epsilon}{(1+|\epsilon^{1/q}v_\epsilon|)^{\beta+1}}\frac{|z_\epsilon|^\alpha}{(1+|\epsilon^{1/p}u_\epsilon|)^\alpha} + ft_\epsilon^{-1/p'}$$

$$-\Delta_q(w_\epsilon) = d(x)\frac{|w_\epsilon|^{q-2}w_\epsilon}{(1+|\epsilon^{1/q}v_\epsilon|^{q-1})} + c(x)\frac{|z_\epsilon\|^\alpha z_\epsilon}{(1+|\epsilon^{1/p}u_\epsilon|)^{\beta+1}}\frac{|w_\epsilon|^\beta}{(1+|\epsilon^{1/q}v_\epsilon|)^\alpha} + gt_\epsilon^{-1/q'}$$

As above, we can prove that $(z_\epsilon, w_\epsilon) \to (z, w)$ strongly in $D^{1,p} \times D^{1,q}$; and taking the limit, we obtain

$$-\Delta_p z = a(x)\Psi_p(z) + b(x)|w|^\beta |z|^\alpha w$$
$$-\Delta_q w = d(x)\Psi_q(w) + c(x)|w|^\beta |z|^\alpha z$$

and we deduce $w = z = 0$. Since there exists a sequence $(\epsilon_n)_{n\in N}$ such that either $\|z_{\epsilon_n}\| = 1$ or $\|w_{\epsilon_n}\| = 1$ we obtain a contradiction.

Hence (u_ϵ, v_ϵ) is bounded in $D^{1,p} \times D^{1,q}$, we can extract a subsequence denoted (u_ϵ, v_ϵ) which converges to (u_0, v_0) weakly in $D^{1,p} \times D^{1,q}$ as $\epsilon \to 0$. By using similar procedure as above, we can prove that that (u_ϵ, v_ϵ) converges strongly to (u_0, v_0) in $D^{1,p} \times D^{1,q}$.

Indeed, since $(\epsilon^{1/p}u_\epsilon(x), \epsilon^{1/q}v_\epsilon(x)) \to (0,0)$ a.e. on $\mathbb{R}^N$, then, as in [6], we have

$$a(x)\frac{|u_\epsilon(x)|^{p-2}u_\epsilon(x)}{(1+|\epsilon^{1/p}u_\epsilon(x)|^{p-1})} \to a(x)|u_0(x)|^{p-2}u_0(x) \quad \text{a.e. in } \mathbb{R}^N \text{ as } \epsilon \to 0,$$

$$a(x)\frac{|u_\epsilon|^{p-2}u_\epsilon}{(1+|\epsilon^{1/p}u_\epsilon|^{p-1})} \le M|u_\epsilon|^{p-1} \le Mh^{p-1} \in L^{p^*}(\mathbb{R}^N),$$

$$b(x)\frac{|v_\epsilon|^\beta v_\epsilon}{(1+|\epsilon^{1/q}v_\epsilon|)^{\beta+1}}\frac{|u_\epsilon|^\alpha}{(1+|\epsilon^{1/p}u_\epsilon|)^{\alpha_i}} \to b(x)|v_0|^\beta v_0|u_0|^\alpha \quad \text{a.e. in } \mathbb{R}^N \text{ as } \epsilon \to 0,$$

$$b(x)\frac{|v_\epsilon|^\beta v_\epsilon}{(1+|\epsilon^{1/q}v_\epsilon|)^{\beta+1}}\frac{|u_\epsilon|^\alpha}{(1+|\epsilon^{1/p}u_\epsilon|)^\alpha} \le M_2h^\alpha l^{\beta+1} \in L^{p^*}(\mathbb{R}^N)$$

Hence from the Dominated Convergence Theorem and Lemma 4.1, we obtain

$$\Big[\int_{\mathbb{R}^N} a(x)\Big(\frac{|u_\epsilon|^{p-2}u_\epsilon}{(1+|\epsilon^{1/p}u_\epsilon|^{p-1})}\Big) - (|u_0|^{p-2}u_0)^{p^*}\Big]^{1/p^*} \to 0,$$

$$\Big[\int_{\mathbb{R}^N} d(x)\Big(\frac{|v_\epsilon|^{q-2}u_\epsilon}{(1+|\epsilon^{1/q}v_\epsilon|^{p-1})}\Big) - (|v_0|^{p-2}v_0)^{q^*}\Big]^{1/q^*} \to 0,$$

$$\Big(\int_{\mathbb{R}^N}\Big(b(x)\frac{|v_\epsilon|^\beta v_\epsilon}{(1+|\epsilon^{1/q}v_\epsilon|)^{\beta+1}}\frac{|u_\epsilon|^\alpha}{(1+|\epsilon^{1/p}u_\epsilon|Big)^\alpha} - |v_0|^\beta v_0|u_0|^\alpha\Big)^{p^*}\Big)^{1/p^*} \to 0,$$

$$\Big(\int_{\mathbb{R}^N}\Big(c(x)\frac{|u_\epsilon|^\alpha u_\epsilon}{(1+|\epsilon^{1/p}u_\epsilon|)^{\alpha+1}}\frac{|v_\epsilon|^\beta}{(1+|\epsilon^{1/q}v_\epsilon|)^\beta} - |u_0|^\alpha u_0|v_0|^\beta\Big)^{q^*}\Big)^{1/q^*} \to 0,$$

as $\epsilon \to 0$. Therefore, passing to the limit, $(u_\epsilon, v_\epsilon) \to (u_0, v_0)$ we obtain from (4.1),

$$-\Delta_p u_0 = a(x)|u_0|^{p-2}u_0 + b(x)|v_0|^\beta|u_0|^\alpha v_0 + f$$
$$-\Delta_q v_0 = d(x)|v_0|^{q-2}v_0 + c(x)|u_0|^\alpha|v_0|^\beta u_0 + g$$

Hence (u_0, v_0) satisfies (1.3). □

We remark that if $\alpha = \beta = 0$ and $p = q = 2$, we obtain the results presented in [18, 19].

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