Electronic Journal of Differential Equations, Vol. 2005(2005), No. 87, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

STEKLOV PROBLEM WITH AN INDEFINITE WEIGHT FOR THE p-LAPLACIAN

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ABSTRACT. Let $\Omega \subset \mathbb{R}^N$, with $N \geq 2$, be a Lipschitz domain and let $1 < p < \infty$. We consider the eigenvalue problem $\Delta_p u = 0$ in Ω and $|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda m|u|^{p-2}u$ on $\partial\Omega$, where λ is the eigenvalue and $u \in W^{1,p}(\Omega)$ is an associated eigenfunction. The weight m is assumed to lie in an appropriate Lebesgue space and may change sign. We sketch how a sequence of eigenvalues may be obtained using infinite dimensional Ljusternik-Schnirelman theory and we investigate some of the nodal properties of eigenfunctions associated to the first and second eigenvalues. Amongst other results we find that if $m^+ \not\equiv 0$ and $\int_{\partial\Omega} m\, d\sigma < 0$ then the first positive eigenvalue is the only eigenvalue associated to an eigenfunction of definite sign and any eigenfunction associated to the second positive eigenvalue has exactly two nodal domains.

1. INTRODUCTION

Let $N \geq 2$ and let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz continuous boundary. Let $1 < p < \infty$ and let $\frac{N-1}{p-1} < q < \infty$ if $p < N$ and $q \geq 1$ if $p \geq N$. Let $m \in L^q\partial\Omega)$ be a weight function which may change sign and denote $m^{\pm} = \max\{0, \pm m\}$. The Steklov eigenvalue problem is defined by

$$\begin{gathered} \Delta_p u = 0 \quad \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda m|u|^{p-2}u \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

where $\lambda \in \mathbb{R}$ is the eigenvalue and $u \in W^{1,p}(\Omega)$ is an associated eigenfunction. We are interested in weak solutions of (1.1), i.e. functions satisfying

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi\, dx = \lambda \int_{\partial\Omega} m|u|^{p-2}u\varphi\, d\sigma \quad \forall \varphi \in W^{1,p}(\Omega) \tag{1.2}$$

where $d\sigma$ is the $N-1$ dimensional Hausdorff measure.

This eigenvalue problem was first introduced in [11] by M. W. Steklov when $p = 2$ and $m \equiv 1$ in which case it appears in a model of an elastic membrane whose mass is concentrated on the boundary. Various properties of the spectrum have been considered in the literature in the case when $p = 2$ and m is nonnegative (see [3] for a review of some of these results). The case when Ω is a Riemannian

2000 *Mathematics Subject Classification.* 35J70, 35P30.
Key words and phrases. Nonlinear eigenvalue problem; Steklov problem; p-Laplacian; nonlinear boundary condition; indefinite weight.

Submitted August 10, 2004. Published August 14, 2005.

manifold with boundary and $p = 2$ and $m \equiv 1$ has been considered in [7] (see also references therein).

Eigenvalue problems involving the p-Laplacian have been the topic of many studies. The classical eigenvalue problem

$$\begin{gathered} -\Delta_p u = \lambda m|u|^{p-2}u \quad \text{in } \Omega \\ u = 0 \text{ or } \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.3}$$

has attracted considerable attention. Assuming m satisfies appropriate integrability conditions and $m^+ \not\equiv 0$, it has been shown using Ljusternik-Schnirelman theory that there exists an unbounded sequence of positive eigenvalues. Various properties are known for the first eigenvalues in the spectrum. Let us recall a few of them. In the case of the Dirichlet boundary condition it is proved that there exists a first eigenvalue which is simple and isolated and that it is the only positive eigenvalue associated to an eigenfunction of constant sign (see [5] and references therein). Furthermore, there is a second positive eigenvalue and when $m \equiv 1$ it is shown in [6] that any associated eigenfunction has exactly two nodal domains. In the case of the Neumann boundary condition it is shown in [8] that the existence of a nonzero simple eigenvalue associated to an eigenfunction of definite sign depends on the sign of $\int_\Omega m\,dx$. A problem in which the eigenvalue appears in the boundary condition has also been considered in the literature. Indeed consider the problem

$$\begin{gathered} \Delta_p u = |u|^{p-2}u \quad \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda m|u|^{p-2}u \quad \text{on } \partial\Omega \end{gathered} \tag{1.4}$$

Assuming $m^+ \not\equiv 0$ the authors of [4] show that there is an unbounded sequence of positive eigenvalues. Furthermore they show that, as in the Dirichlet case above, there exists a first positive eigenvalue which is simple and isolated and that it is the only positive eigenvalue associated to an eigenfunction of constant sign.

Many of the properties of the above mentioned eigenvalue problems carry over to the Steklov problem. However due to the presence of an indefinite weight, care is required in some of the proofs. Also, the weight function plays a role in determining, via the sign of $\int_{\partial\Omega} m\,d\sigma$, some qualitative properties of the beginning of the spectrum such as the existence of a nonzero eigenvalue associated to an eigenfunction of constant sign. In this sense the Steklov problem bears stronger resemblance to the classical p-Laplacian eigenvalue problem with a Neumann condition than to (1.4). Some properties and methods of proof which we present below for the Steklov problem may also be of interest when adapted to the eigenvalue problems (1.3) and (1.4).

A sequence of Steklov eigenvalues can be obtained as follows. Let

$$\Sigma^\pm = \Big\{u \in W^{1,p}(\Omega);\ \frac{1}{p}\int_{\partial\Omega} m|u|^p\,d\sigma = \pm 1\Big\}$$

For any integer $n \ge 1$ let

$$\mathcal{C}_n^\pm = \{C \subset \Sigma^\pm;\ C \text{ is symmetric, compact and } \gamma(C) \ge n\}$$

where γ is the Krasnoselski genus, and let

$$\lambda_n^\pm = \pm \inf_{C \in \mathcal{C}_n^\pm} \sup_{u \in C} \frac{1}{p}\int_\Omega |\nabla u|^p\,dx \tag{1.5}$$

Theorem 1.1. *Let $1 < p < \infty$ and let $m \in L^q\partial\Omega)$ where q is as above.*

(1) *If $m^+ \not\equiv 0$ then λ_n^+ given by (1.5) is a nondecreasing and unbounded sequence of positive Steklov eigenvalues.*
(2) *If $m^- \not\equiv 0$ then λ_n^- given by (1.5) is a nonincreasing and unbounded sequence of negative Steklov eigenvalues.*

Moreover if $\lambda_n^\pm = \lambda_{n+j}^\pm$ for some integer $j \geq 1$ then

$$\gamma(\{u \in \Sigma^\pm;\ u \text{ is an eigenfunction associated to } \lambda_n^\pm\}) \geq j + 1$$

This theorem is proved by applying a general result from infinite dimensional Ljusternik-Schnirelman theory. However it must first be shown that the corresponding Palais-Smale condition is satisfied. This fact is perhaps not obvious since the functional $\frac{1}{p}\int_\Omega |\nabla u|^p\,dx$ is not necessarily coercive on $\Sigma^\pm$, as will be shown by an example below.

Next we consider qualitative properties of the begining of the spectrum.

Theorem 1.2. *Let $1 < p < \infty$ and let $m \in L^q\partial\Omega)$ where q is as above.*

(1) *Assume that $m^+ \not\equiv 0$ and that $\int_{\partial\Omega} m\,d\sigma < 0$. Then $\lambda_1^- = 0$ and $\lambda_1^+ > 0$ is the first positive Steklov eigenvalue. Moreover λ_1^+ is simple and isolated and it is the only nonzero Steklov eigenvalue associated to an eigenfunction of definite sign. Also, λ_2^+ is the second positive Steklov eigenvalue and λ_2^- is the first negative Steklov eigenvalue.*
(2) *Assume that $m^- \not\equiv 0$ and that $\int_{\partial\Omega} m\,d\sigma > 0$. Then $\lambda_1^+ = 0$ and $\lambda_1^- < 0$ is the first negative Steklov eigenvalue. Moreover λ_1^- is simple and isolated and it is the only nonzero Steklov eigenvalue associated to an eigenfunction of definite sign. Also, λ_2^+ is the first positive Steklov eigenvalue and λ_2^- is the second negative Steklov eigenvalue.*
(3) *Assume that $\int_{\partial\Omega} m\,d\sigma = 0$. Then $\lambda_1^+ = \lambda_1^- = 0$ and there does not exist a nonzero eigenvalue associated to an eigenfunction of definite sign. Also, λ_2^+ is the first positive Steklov eigenvalue and λ_2^- is the first negative Steklov eigenvalue.*

In each of the above cases any eigenfunction associated to λ_2^+ or λ_2^- has exactly two nodal domains. Moreover, if in case (1) there holds $m^+ \equiv 0$ then there does not exist a positive eigenvalue, however the assertions concerning the negative eigenvalues remain true. A similar statement holds when $m^- \equiv 0$ in case (2).

Note that $\lambda = 0$ is always a Steklov eigenvalue and that the associated eigenfunctions are just the constant functions.

In [13], L. Véron discovered an interesting formula for the first nonzero eigenvalue in the usual p-Laplacian spectrum on a Riemannian manifold without boundary. There holds a similar formula for the Steklov problem and it will be used to deduce, as in [13], some of the assertions in the above theorem. Let us now state the formula. If $\omega \subset \Omega$ is an open subset, let $W_*^{1,p}(\omega)$ denote the subset of $W^{1,p}(\Omega)$ consisting of functions which are zero almost everywhere in $\Omega \setminus \bar{\omega}$. Let

$$\mu^\pm(\omega) = \inf\{\frac{1}{p}\int_\Omega |\nabla u|^p\,dx;\ u \in W_*^{1,p}(\omega),\ \frac{1}{p}\int_{\partial\Omega} m|u|^p\,d\sigma = \pm 1\} \tag{1.6}$$

if this quantity is well defined and $\mu^\pm(\omega) = +\infty$ if not. Lastly let $\mathcal{A}$ denote the set of pairs $(\omega, \tilde{\omega})$ where ω and $\tilde{\omega}$ are disjoint nonempty opens subsets of Ω.

Theorem 1.3. *Let $1 < p < \infty$ and let $m \in L^q\partial\Omega)$ where q is as above. If $m^\pm \not\equiv 0$ then we have the characterisation*

$$\lambda_2^\pm = \pm \min_{(\omega,\tilde{\omega})\in\mathcal{A}} \max\left\{\mu^\pm(\omega), \mu^\pm(\tilde{\omega})\right\} \tag{1.7}$$

Moreover the minimum is achieved if and only if ω and $\tilde{\omega}$ are the nodal domains of some eigenfunction associated to $\lambda_2^\pm$.

Let us remark that this formula can be adapted to the eigenvalue problems (1.3) and (1.4).

2. Existence of eigenvalues

Theorem 1.1 is proved by applying theorem 5.3, page 209 in [9]. Using the assumption $m^\pm \not\equiv 0$, a standard argument shows that $\lambda_n^\pm$ is well defined. It remains only to show that the relevant Palais-Smale condition is satisfied. Let us first remark that it is not immediately obvious that the Palais-Smale condition holds since $\frac{1}{p}\int_\Omega |\nabla u|^p\,dx$ is not always coercive on Σ^+. Indeed, assume $p = 2$ and $\int_{\partial\Omega} m\,d\sigma = 0$ and consider a sequence of functions $u_k \in \Sigma^+$ such that $\int_{\partial\Omega} m u_k\,d\sigma = 0$. Then the unbounded sequence $u_k + k$ lies in Σ^+ and $\frac{1}{p}\int_\Omega |\nabla(u_k + k)|^p\,dx$ is bounded. Hence coercivity does not hold. In fact it will follow from the proof of theorem 1.2 in the next section that $\frac{1}{p}\int_\Omega |\nabla u|^p\,dx$ is coercive on $\Sigma^\pm$ if and only if $\int_{\partial\Omega} m\,d\sigma \neq 0$. Lastly, let us note that the assumption on q ensures that the trace mapping $W^{1,p}(\Omega) \to L^{pq/(q-1)}\partial\Omega)$ is compact.

Lemma 2.1. *Let $W = \left\{w \in W^{1,p}(\Omega);\ \int_\Omega w\,dx = 0\right\}$. Let $\phi \in W^{1,p}(\Omega)'$ be such that $\langle\phi,\alpha\rangle = 0$ for any constant function α. Then there exists a unique $w \in W$ such that $-\Delta_p w = \phi$. Moreover w depends continuously on ϕ.*

Proof. Consider the minimization problem

$$\inf_{w\in W} \frac{1}{p}\int_\Omega |\nabla w|^p\,dx - \langle\phi, w\rangle$$

By Poincaré-Wirtinger's inequality any minimizing sequence is bounded so that the infimum is achieved by some function w satisfying

$$\int_\Omega |\nabla w|^{p-2}\nabla w \nabla\varphi\,dx = \langle\phi,\varphi\rangle + \eta\int_\Omega \varphi \quad \forall\varphi \in W^{1,p}(\Omega) \tag{2.1}$$

where η is a Lagrange multiplier. Since $\langle\phi,\alpha\rangle = 0$ for any constant function α, it follows that $\eta = 0$. Thus $-\Delta_p w = \phi$. Uniqueness and continuity follow from standard estimates. □

For $u \in W^{1,p}(\Omega)$ define $J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx$ and $B(u) = \frac{1}{p}\int_{\partial\Omega} m|u|^p\,d\sigma$. Also define $Du \in W^{1,p}(\Omega)'$ by $Du = J'(u) - J(u)B'(u)$. We are now ready to prove the Palais-Smale condition.

Lemma 2.2 (Palais-Smale condition)**.** *Let $u_k \in \Sigma^\pm$ be a sequence such that $J(u_k) \to \lambda \neq 0$ in $\mathbb{R}$ and $Du_k \to 0$ in $W^{1,p}(\Omega)'$ as $k \to \infty$. Then u_k contains a convergent subsequence.*

Proof. Let us first show that the sequence u_k is bounded. Fix $\varphi \in W^{1,p}(\Omega)$ such that $\|\varphi\|_{W^{1,p}(\Omega)} = 1$ and $\int_{\partial\Omega} m\varphi \, d\sigma \neq 0$. Since $\langle Du_k, \varphi\rangle \to 0$ as $k \to \infty$ we have that

$$\Big| \int_{\partial\Omega} m|u_k|^{p-2}u_k\varphi \, d\sigma \Big| = \Big| \frac{1}{J(u_k)} \Big(\int_\Omega |\nabla u_k|^{p-2}\nabla u_k \nabla\varphi \, dx - \langle Du_k, \varphi\rangle \Big) \Big| \leq C$$

where $C > 0$ is a positive constant. Now let $\alpha_k = \frac{1}{|\Omega|}\int_\Omega u_k \, dx$ and let $\tilde{u}_k = u_k - \alpha_k$. Since $J(\tilde{u}_k) = J(u_k)$ is bounded it follows from the Poincaré-Wirtinger inequality that the sequence $\tilde{u}_k$ is bounded in $W^{1,p}(\Omega)$. We may write

$$\int_{\partial\Omega} m|u_k|^{p-2}u_k\varphi \, d\sigma = |\alpha_k|^{p-2}\alpha_k \int_{\partial\Omega} m\Big|\frac{\tilde{u}_k}{\alpha_k} + 1\Big|^{p-2}\Big(\frac{\tilde{u}_k}{\alpha_k} + 1\Big)\varphi \, d\sigma \tag{2.2}$$

If the sequence $|\alpha_k|$ is not bounded we may assume $|\alpha_k| \to \infty$ so that the integral on the right hand side of (2.2) goes to $\int_{\partial\Omega} m\varphi \, d\sigma \neq 0$ and we have a contradiction. Since α_k is bounded it follows that the sequence u_k is bounded. Using the compactness property of the trace mapping it follows that u_k contains a subsequence, again noted u_k, such that $B'(u_k) \to B'(u)$ for some $u \in W^{1,p}(\Omega)$. Since $-\Delta_p u_k = Du_k + J(u_k)B'(u_k) \to \lambda B'(u)$ in $W^{1,p}(\Omega)'$ it follows from lemma 2.1 that $u_k - \alpha_k \to (-\Delta_p)^{-1}(\lambda B'(u))$ in $W^{1,p}(\Omega)$. Since α_k is bounded we have that u_k converges in the $W^{1,p}(\Omega)$ sense. □

3. Qualitative properties of the first and second eigenvalues

In this section, we prove theorems 1.2 and 1.3. Let us note that all Steklov eigenfunctions are of class $C^{1,\alpha}(\Omega)$ since they are p-harmonic. Moreover, following the procedure outlined in [10] one may show that $u \in L^\infty(\Omega)$. The proof of this fact is carried out in detail in [12].

The following lemma derives from Picone's identity (see [1]) and is a standard tool in this context. A little extra care is required in the proof in the Steklov setting.

Lemma 3.1. *Let $1 < p < \infty$ and let $m \in L^q(\partial\Omega)$ where q is as above. Let u and v be two nonnegative Steklov eigenfunctions associated to some eigenvalues λ and $\tilde{\lambda}$, respectively. Then*

$$0 \leq (\lambda - \tilde{\lambda}) \int_{\partial\Omega} mu^p \, d\sigma \tag{3.1}$$

and equality holds if and only if v is a multiple of u.

Proof. We first show that the trace of v satisfies $v > 0$ on $\partial\Omega$. Let $\varepsilon > 0$. By the maximum principle of Vazquez $v > 0$ in Ω so that $\frac{v}{v+\varepsilon}$ converges in $L^p(\Omega)$ to 1_Ω as $\varepsilon \to 0$. On the other hand $\nabla\frac{v}{v+\varepsilon}$ converges to 0 a.e. as $\varepsilon \to 0$. Taking $\varphi = \frac{1}{(v+\varepsilon)^{p-1}}$ as testing function in equation (1.2) satisfied by v we have

$$(p-1)\int_\Omega \frac{|\nabla v|^p}{(v+\varepsilon)^p} \, dx = \lambda \int_{\partial\Omega} m\Big(\frac{v}{v+\varepsilon}\Big)^{p-1} d\sigma$$

so that

$$\Big|\nabla\frac{v}{v+\varepsilon}\Big|^p = \Big(\frac{\varepsilon}{v+\varepsilon}\Big)^p \frac{|\nabla v|^p}{(v+\varepsilon)^p} \leq \frac{|\nabla v|^p}{v^p} \in L^1(\Omega).$$

By the dominated convergence theorem we have that $\frac{v}{v+\varepsilon} \to 1_\Omega$ in $W^{1,p}(\Omega)$. By continuity of the trace mapping we have that $\frac{v}{v+\varepsilon} \to 1_{\partial\Omega}$ in $L^1\partial\Omega)$ as $\varepsilon \to 0$ and it follows that $v > 0$ on $\partial\Omega$.

Now let $\varepsilon > 0$. By Picone's identity we have

$$\begin{aligned} 0 &\le \int_\Omega |\nabla u|^p\,dx - \int_\Omega |\nabla v|^{p-2}\nabla v \nabla\big(\frac{u^p}{(v+\varepsilon)^{p-1}}\big)\,dx \\ &= \lambda \int_{\partial\Omega} m u^p\,d\sigma - \tilde{\lambda}\int_{\partial\Omega} m\big(\frac{v}{v+\varepsilon}\big)^{p-1} u^p\,d\sigma \end{aligned}$$

and equality holds if and only if v is a multiple of u. Going to the limit $\varepsilon \to 0$ and using the fact that $v > 0$ on $\partial\Omega$ we get the desired inequality. $\square$

Proof of theorem 1.2. We begin by proving the assertions relating to $\lambda_1^\pm$. It follows immediatly from (1.5) that we have

$$\lambda_1^\pm = \pm\inf\{\int_\Omega |\nabla u|^p\,dx;\ u \in W^{1,p}(\Omega),\ \int_{\partial\Omega} m|u|^p\,d\sigma = \pm 1\} \tag{3.2}$$

Assume first that $\int_{\partial\Omega} m\,d\sigma < 0$ and $m^+ \not\equiv 0$. The minimum λ_1^- is achieved by a constant function so that $\lambda_1^- = 0$. Now let u_k be a minimizing sequence for λ_1^+. Let $\alpha_k = \frac{1}{|\Omega|}\int_\Omega u_k\,dx$. By Poincaré-Wirtinger's inequality the sequence $\tilde{u}_k = u_k - \alpha_k$ is bounded and, moreover, we may write

$$1 = |\alpha_k|^p \int_{\partial\Omega} m\big|\frac{\tilde{u}_k}{\alpha_k} + 1\big|^p\,d\sigma$$

If the sequence $|\alpha_k|$ is not bounded we may assume that $|\alpha_k| \to \infty$ so that the integral goes to $\int_{\partial\Omega} m\,d\sigma < 0$ and we have a contradiction. Thus α_k is bounded and we have that u_k is bounded. It follows that the infimum λ_1^+ is achieved. Since the minimiser cannot be constant we have that $\lambda_1^+ > 0$ and it is clear that λ_1^+ is the first positive eigenvalue. Now let u be an eigenfunction associated to λ_1^+ so that $|u|$ is a minimiser for (3.2) and is thus an eigenfunction associated to λ_1^+. It follows from the maximum principle of Vazquez that $|u| > 0$ in Ω and we conclude that u has constant sign. Taking $\lambda = \lambda_1^+$ in (3.1) we see that no eigenvalue $\tilde{\lambda} > \lambda_1^+$ can be associated to a positive eigenfunction. Taking $\lambda = 0$ in (3.1) and $u \equiv 1$ an associated eigenfunction, we see that no eigenvalue $\tilde{\lambda} < 0$ can be associated to a positive eigenfunction. Thus λ_1^+ is the only nonzero eigenvalue associated to an eigenfunction of definite sign. Taking $\lambda = \tilde{\lambda} = \lambda_1^+$ in (3.1) we see that any eigenfunction v associated to λ_1^+ must be a multiple of u, so that λ_1^+ is simple.

The case where $\int_{\partial\Omega} m\,d\sigma > 0$ and $m^- \not\equiv 0$ follows from the previous case by taking $-m$ as weight function.

Now assume that $\int_{\partial\Omega} m\,d\sigma = 0$ and suppose by contradiction that $\lambda_1^+ > 0$. If v is an eigenfunction associated to λ_1^+ then $|v|$ is a minimiser in (3.2) and it is thus also an eigenfunction associated to λ_1^+. However, taking $\tilde{\lambda} = \lambda_1^+$, $\lambda = 0$ and $u \equiv 1$ in (3.1) we see that v is constant, since it must be a multiple of u. But this implies that $\lambda_1^+ = 0$ which is a contradiction. In the same way it can be shown that $\lambda_1^- = 0$.

Now we prove the assertions concerning $\lambda_2^\pm$. Since λ_1^+ is simple we have

$$\gamma\big(\{u \in \Sigma^+;\ u \text{ is an eigenfunction associated to } \lambda_1^+\}\big) = 1$$

Thus by theorem 1.1 there holds $\lambda_1^+ < \lambda_2^+$. Likewise we have $\lambda_2^- < \lambda_1^-$. Let $\lambda \neq \lambda_1^\pm$ be an eigenvalue. Following a similar reasoning to [2] we now show that either $\lambda \le \lambda_2^-$ or $\lambda_2^+ \le \lambda$. First assume $\lambda > 0$ and let u be an eigenfunction associated to λ. Since u must change sign we may assume that u is normalized in such a way that

$\frac{1}{p}\int_{\partial\Omega} m|u^+|^p\,d\sigma = \frac{1}{p}\int_{\partial\Omega} m|u^-|^p\,d\sigma = 1$ and $\frac{1}{p}\int_\Omega |\nabla u^+|^p\,dx = \frac{1}{p}\int_\Omega |\nabla u^-|^p\,dx = \lambda$. The set $C = \{\alpha u^+ + \beta u^-;\ \alpha,\beta\in\mathbb{R}$ such that $|\alpha|^p + |\beta|^p = 1\}$ is in $\mathcal{C}_2^+$. Thus

$$\lambda_2^+ \le \max_{|\alpha|^p+|\beta|^p=1} \frac{1}{p}\int_\Omega |\nabla(\alpha u^+ + \beta u^-)|^p\,dx = \lambda$$

Similarly if $\lambda < 0$ we may show that $\lambda \le \lambda_2^-$. It remains only to show that any eigenfunction associated to λ_2^+ or λ_2^- has exactly two nodal domains. To prove this we use formula (1.7) which will be proved below. Let u be an eigenfunction associated to $\lambda_2^\pm$. Assume that u has at least three nodal domains ω_1, ω_2 and ω_3 with, say, $u > 0$ in ω_1 and ω_2 and $u < 0$ in ω_3. In a similar situation the authors of [6] show that $\partial\omega_1\cap\Omega \not\subseteq \partial\omega_2\cap\Omega$ and $\partial\omega_2\cap\Omega \not\subseteq \partial\omega_1\cap\Omega$. Their proof relies on the maximum principle and carries over to the case considered here. It follows that there exist disjoint open sets $\tilde\omega_1, \tilde\omega_2 \subset \Omega$ such that $\omega_1 \subsetneq \tilde\omega_1$ and $\omega_2 \subsetneq \tilde\omega_2$. Now it follows from standard arguments that $\mu^\pm(\tilde\omega_1) < \mu^\pm(\omega_1)$ and $\mu^\pm(\tilde\omega_2) < \mu^\pm(\omega_2)$ thus contradicting (1.7). $\square$

Proof of theorem 1.3. We only prove the formula for λ_2^+ since the proof for λ_2^- is similar. Denote

$$\mu = \inf_{(\omega,\tilde\omega)\in\mathcal{A}} \max\{\mu^+(\omega), \mu^-(\tilde\omega)\} \tag{3.3}$$

Let $(\omega,\tilde\omega)\in\mathcal{A}$ and let ψ and $\tilde\psi$ be minimizers for $\mu^+(\omega)$ and $\mu^+(\tilde\omega)$ respectively, normalized so that $\frac{1}{p}\int_{\partial\Omega} m\psi^p\,d\sigma = \frac{1}{p}\int_{\partial\Omega} m\tilde\psi^p\,d\sigma = 1$. The set $C = \{\alpha\psi + \beta\tilde\psi;\ \alpha,\beta\in\mathbb{R}$ such that $|\alpha|^p + |\beta|^p = 1\}$ is in $\mathcal{C}_2^+$. Consequently

$$\begin{aligned}
\lambda_2^+ &\le \max_{|\alpha|^p+|\beta|^p=1} \tfrac{1}{p}\textstyle\int_\Omega |\nabla(\alpha\psi + \beta\tilde\psi)|^p\,dx \\
&= \max_{|\alpha|^p+|\beta|^p=1} |\alpha|^p\mu^+(\omega) + |\beta|^p\mu^+(\tilde\omega) \\
&\le \max\{\mu^+(\omega), \mu^+(\tilde\omega)\}
\end{aligned}$$

Hence $\lambda_2^+ \le \mu$. Now let u be an eigenfunction associated to λ_2^+ normalized so that $\frac{1}{p}\int_{\partial\Omega} m|u^+|^p\,d\sigma = \frac{1}{p}\int_{\partial\Omega} m|u^-|^p\,d\sigma = 1$. Denote $\Omega^\pm = \{x\in\Omega;\ \pm u(x) > 0\}$. We have that

$$\mu^+(\Omega^\pm) \le \frac{1}{p}\int_\Omega |\nabla u^\pm|^p\,dx = \lambda_2^+$$

Hence $\mu \le \lambda_2^+$. In fact using Picone's identity it can be shown that $\mu^+(\Omega^\pm) = \lambda_2^+$ so that the infimum (3.3) is achieved by the nodal domains of an eigenfunction associated to λ_2^+. It remains to show that this is the only case in which (3.3) is achieved. Let $(\omega,\tilde\omega)\in\mathcal{A}$ be such that $\max\{\mu^+(\omega),\mu^+(\tilde\omega)\} = \lambda_2^+$. Let ψ and $\tilde\psi$ be minimizers for $\mu^+(\omega)$ and $\mu^+(\tilde\omega)$ respectively, normalized as usual, and let C be as above. Then the infimum λ_2^+ given by (1.5) is in fact a minimum which is achieved by C. It is then a straightforward consequence of Ljusternik-Schnirelman theory that some element of C must be an eigenfunction associated to λ_2^+. Since any function in C has ω and $\tilde\omega$ as nodal domains the proof is complete. $\square$

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