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ASYMPTOTIC PROFILE OF A RADIALLY SYMMETRIC SOLUTION WITH TRANSITION LAYERS FOR AN UNBALANCED BISTABLE EQUATION

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Abstract. In this article, we consider the semilinear elliptic problem

$$-\varepsilon^2\Delta u = h(|x|)^2(u-a(|x|))(1-u^2)$$

in $B_1(0)$ with the Neumann boundary condition. The function a is a C^1 function satisfying $|a(x)|<1$ for $x\in[0,1]$ and $a'(0)=0$. In particular we consider the case $a(r)=0$ on some interval $I\subset[0,1]$. The function h is a positive C^1 function satisfying $h'(0)=0$. We investigate an asymptotic profile of the global minimizer corresponding to the energy functional as $\varepsilon\to 0$. We use the variational procedure used in [4] with a few modifications prompted by the presence of the function h.

1. Introduction and Statement of Main Results

In this article, we consider the boundary value problem

$$\begin{gathered}-\varepsilon^2\Delta u = h(|x|)^2(u-a(|x|))(1-u^2)\quad\text{in } B_1(0)\\ \frac{\partial u}{\partial\nu}=0\quad\text{on } \partial B_1(0)\end{gathered}\tag{1.1}$$

where ε is a small positive parameter, $B_1(0)$ is a unit ball in $\mathbb{R}^N$ centered at the origin, and the function a is a C^1 function on $[0,1]$ satisfying $-1<a(|x|)<1$ and $a'(0)=0$. The function h is a positive C^1 function on $[0,1]$ satisfying $h'(0)=0$. We set $r=|x|$.

Problem (1.1) appears in various models such as population genetics, chemical reactor theory and phase transition phenomena. See [1] and the references therein. If the function h satisfies $h(r)\equiv 1$ and the function a satisfies $a(r)\not\equiv 0$, then this problem (1.1) has been studied in [1], [4] and [7]. In this case, it is shown that there exist radially symmetric solutions with transition layers near the set $\{x\in B_1(0)|a(|x|)=0\}$. If the set $\{r\in\mathbb{R}|a(r)=0\}$ contains an interval I, then the problem to decide the configuration of transition layer on I is more delicate.

When $N=1$, if the function h satisfies $h(r)\not\equiv 1$ and the function a satisfies $a(r)\equiv 0$, then problem (1.1) has been studied in [8] and [9]. In this case, it is

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shown that there exist stable solutions with transition layers near prescribed local minimum points of h.

In this paper, we consider the case where the function a satisfies $a(r) \not\equiv 0$ with $a(r) = 0$ on some interval $I \subset (0, 1)$. We show the minimum point of the function $r^{N-1}h(r)$ on I has very important role to decide the configuration of transition layer on I in this case.

We note that in [4], Dancer and Shusen Yan considered a problem similar to ours. They assume that $N \geq 2$, $h \equiv 1$ and the nonlinear term is $u(u - a|x|)(1 - u)$ satisfying $a(r) = 1/2$ on $I = [l_1, l_2]$ and $a(r) < 1/2$ for $l_1 - r > 0$ small and $a(r) > 1/2$ for $r - l_2 > 0$ small, then a global minimizer of the corresponding functional has a transition layer near the l_1, that is, the minimum point of r^{N-1} on I (see [4, Theorem 1.3]). In this sense, we can say that our results are natural extension of the results in [4]. We are going to follow throughout the variational procedure used in [4] with a few modifications prompted by the presence of the function h.

Here we state the energy functional, corresponding to (1.1),

$$J_\varepsilon(u) = \int_{B_1(0)} \frac{\varepsilon^2}{2} |\nabla u|^2 - F(|x|, u) dx,$$

where $F(|x|, u) = \int_{-1}^{u} f(|x|, s) ds$ and $f(|x|, u) = h(|x|)^2(u - a(|x|))(1 - u^2)$. It is easy to see that the following minimization problem has a minimizer

$$\inf\{J_\varepsilon(u) | u \in H^1(B_1(0))\}. \tag{1.2}$$

Let $A_- = \{x \in B_1(0) | a(|x|) < 0\}$ and $A_+ = \{x \in B_1(0) | a(|x|) > 0\}$.

In this paper, we will analyze the profile of the minimizer of (1.2), and prove the following results.

Theorem 1.1. *Let u_ε be a global minimizer of* (1.2). *Then u_ε is radially symmetric and*

$$u_\varepsilon \to \begin{cases} 1, & \text{uniformly on each compact subset of } A_-, \\ -1, & \text{uniformly on each compact subset of } A_+, \end{cases}$$

as $\varepsilon \to 0$. In particular u_ε converges uniformly near the boundary of $B_1(0)$, that is, if $a(r) < 0$ on $[r_0, 1]$ for some $r_0 > 0$, $u_\varepsilon \to 1$ uniformly on $\overline{B_1(0)} \backslash B_{r_0}(0)$ and if $a(r) > 0$ on $[r_0, 1]$ for some $r_0 > 0$, $u_\varepsilon \to -1$ uniformly on $\overline{B_1(0)} \backslash B_{r_0}(0)$. Moreover, for any $0 < r_1 \leq r_2 < 1$ with $a(r_i) = 0$, $i = 1, 2$, $a(r) \neq 0$ for $r_1 - r > 0$ small and for $r - r_2 > 0$ small, $a(r) = 0$ if $r \in [r_1, r_2]$, we have:

- (i) *If $a(r) < 0$ for $r_1 - r > 0$ small and $a(r) > 0$ for $r - r_2 > 0$, then for any small $\eta > 0$ and for any small $\theta > 0$, there exists a positive number ε_0 which has the following properties:*
  - (a) *For all $\varepsilon \in (0, \varepsilon_0]$, there exist $t_{\varepsilon,1} < t_{\varepsilon,2}$ such that*

$$\begin{gathered} u_\varepsilon(r) > 1 - \eta \quad \text{for } r \in [r_1 - \theta, t_{\varepsilon,1}), \\ u_\varepsilon(t_{\varepsilon,1}) = 1 - \eta, \\ u_\varepsilon(t_{\varepsilon,2}) = -1 + \eta, \\ u_\varepsilon(r) < -1 + \eta, \quad \text{for } r \in (t_{\varepsilon,2}, r_2 + \theta]. \end{gathered}$$

  - (b) *The function $u_\varepsilon(r)$ is decreasing on the interval $(t_{\varepsilon,1}, t_{\varepsilon,2})$*
  - (c) *The inequality $0 < R_1 \leq \frac{t_{\varepsilon,2} - t_{\varepsilon,1}}{\varepsilon} \leq R_2$ holds, where R_1 and R_2 are two constants independent of $\varepsilon > 0$.*

(d) *If* $t_{\varepsilon_j,1}$, $t_{\varepsilon_j,2} \to \bar{t}$ *for some positive sequence* $\{\varepsilon_j\}$ *converging to zero as* $j \to \infty$, *then* $\bar{t}$ *satisfies* $h(\bar{t})\bar{t}^{N-1} = \min_{s\in[r_1,r_2]} h(s)s^{N-1}$.

(ii) *If* $a(r) > 0$ *for* $r_1 - r > 0$ *small and* $a(r) < 0$ *for* $r - r_2 > 0$, *then for each small* $\eta > 0$ *and for each small* $\theta > 0$, *there exists a positive number* ε_0 *which has the following properties: For each* $\varepsilon \in (0, \varepsilon_0]$, *there exist* $t_{\varepsilon,1} < t_{\varepsilon,2}$ *such that*

(a)
$$\begin{gathered} u_\varepsilon(r) < -1 + \eta \quad \text{for } r \in [r_1 - \theta, t_{\varepsilon,1}), \\ u_\varepsilon(t_{\varepsilon,1}) = -1 + \eta, \\ u_\varepsilon(t_{\varepsilon,2}) = 1 - \eta, \\ u_\varepsilon(r) > 1 - \eta, \quad \text{for } r \in (t_{\varepsilon,2}, r_2 + \theta]. \end{gathered}$$

(b) *The function* $u_\varepsilon(r)$ *is increasing in* $(t_{\varepsilon,1}, t_{\varepsilon,2})$.

(c) *The inequality* $0 < R_1 \le \frac{t_{\varepsilon,2} - t_{\varepsilon,1}}{\varepsilon} \le R_2$ *holds, where* R_1 *and* R_2 *are two constants independent of* $\varepsilon > 0$.

(d) *If* $t_{\varepsilon_j,1}$, $t_{\varepsilon_j,2} \to \bar{t}$ *for some positive sequence* $\{\varepsilon_j\}$ *converging to zero as* $j \to \infty$, *then* $\bar{t}$ *satisfies* $h(\bar{t})\bar{t}^{N-1} = \min_{s\in[r_1,r_2]} h(s)s^{N-1}$.

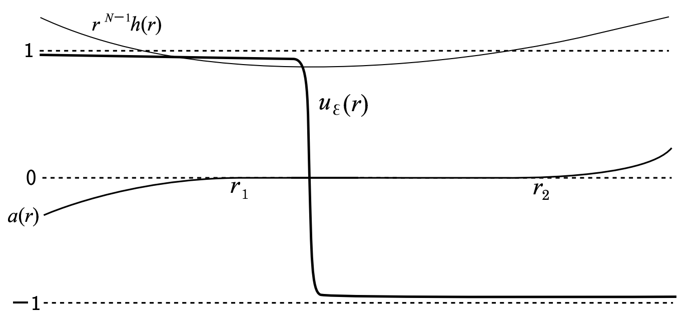

FIGURE 1. Profile of the global minimizer u_ε

Remarks.

- Note that results from (a) to (c) both in cases (i) and (ii) are not related to the presence of the function h. The effect of presence of function h appears in the result (d) in (i) and (ii).
- If $\min_{s\in[r_1,r_2]} s^{N-1}h(s)$ is attained at a unique point $\bar{t}$, we can show $t_{\varepsilon,1}$, $t_{\varepsilon,2} \to \bar{t}$ as $\varepsilon \to 0$ without taking subsequences.
- If the function $r^{N-1}h(r)$ is constant on $[r_1, r_2]$, it is a very difficult problem to know the location of the point $\bar{t} \in [r_1, r_2]$.

This paper is organized as follows: In section 2, we present some preliminary results. In section 3, we prove the main theorem.

2. Preliminary Results

Let D is a bounded domain in $\mathbb{R}^N$. Let $\overline{f}(x,t)$ be a function defined on $\overline{D} \times \mathbb{R}$ which is bounded on $\overline{D} \times [-1, 1]$. Suppose $\overline{f}$ is continuous on $t \in \mathbb{R}$ for each $x \in \overline{D}$

and is measurable in D for each $t \in \mathbb{R}$. We also assume

$$\begin{aligned} \overline{f}(x,t) > 0 \quad \text{for } x \in \overline{D},\ t < -1; \\ \overline{f}(x,t) < 0 \quad \text{for } x \in \overline{D},\ t > 1. \end{aligned} \tag{2.1}$$

Consider the minimization problem

$$\inf\left\{ \overline{J}_\varepsilon(u,D) := \int_D \frac{\varepsilon^2}{2}|\nabla u|^2 - \overline{F}(x,u)dx : u - \eta \in H_0^1(D) \right\}, \tag{2.2}$$

where $\eta \in H^1(D)$ with $-1 \le \eta \le 1$ on D and

$$\overline{F}(x,t) = \int_{-1}^{t} \overline{f}(x,s)ds.$$

We can prove next two lemmas by methods similar to [4]. For the readers convenience, we prove these lemmas in this section.

Lemma 2.1. *Suppose that $\overline{f}(x,t)$ satisfies* (2.1)*. Let u_ε be a minimizer of* (2.2)*. Then $-1 \le u_\varepsilon \le 1$ on D.*

Proof. We prove $-1 \le u_\varepsilon$ on D. Let $M = \{x : u_\varepsilon(x) < -1\}$. Define $\tilde{u}_\varepsilon$ by

$$\tilde{u}_\varepsilon(x) = \begin{cases} u_\varepsilon(x) & \text{if } x \in D\backslash M \\ -1 & \text{if } x \in M. \end{cases}$$

Since $u_\varepsilon(x) = \eta \ge -1$ on ∂D, we see that M is compactly contained in D. Thus $\tilde{u} - \eta \in H_0^1(D)$. If the measure $m(M)$ of M is positive, we have $\overline{J}_\varepsilon(\tilde{u}_\varepsilon, D) < \overline{J}_\varepsilon(u_\varepsilon, D)$. Because u_ε is a minimizer, we see $m(M) = 0$, where $m(A)$ denotes the Lebesgue measure of the set A. Thus $u_\varepsilon \ge -1$. Similarly we can prove that $u_\varepsilon \le 1$. □

Lemma 2.2. *Suppose that $\overline{f}_1(x,t)$ and $\overline{f}_2(x,t)$ both satisfy* (2.1) *and the same regularity assumption on $\overline{f}$. Assume that $\eta_i \in H^1(D)$ satisfy $-1 \le \eta_i \le 1$ on D for $i = 1,2$. Let $u_{\varepsilon,i}$ be a corresponding minimizer of* (2.2)*, where $\overline{f} = \overline{f}_i$ and $\eta = \eta_i$, $i = 1,2$. Suppose that $\overline{f}_1(x,t) \ge \overline{f}_2(x,t)$ for all $(x,t) \in \overline{D} \times [-1,1]$ and $1 \ge \eta_1 \ge \eta_2 \ge -1$. Then $u_{\varepsilon,1} \ge u_{\varepsilon,2}$.*

Proof. Let $M = \{x \in D : u_{\varepsilon,2} > u_{\varepsilon,1}\}$. Define $\varphi_\varepsilon = (u_{\varepsilon,2} - u_{\varepsilon,1})^+$. Since $\eta_1 \ge \eta_2$, we have $\varphi_\varepsilon \in H_0^1(D)$. Set $\overline{F}_i(x,u) = \int_{-1}^{u} \overline{f}_i(x,s)ds$. Since $u_{\varepsilon,i}$ is a minimizer of

$$J_{\varepsilon,i}(u) := \int_D \frac{\varepsilon^2}{2}|\nabla u|^2 - \overline{F}_i(x,u)dx$$

and $\varphi_\varepsilon = 0$ for $x \in D\backslash M$, we have

$$\begin{aligned} 0 &\le J_{\varepsilon,1}(u_{\varepsilon,1} + \varphi_\varepsilon) - J_{\varepsilon,1}(u_{\varepsilon,1}) \\ &= \int_M \frac{\varepsilon^2}{2}(|\nabla(u_{\varepsilon,1} + \varphi_\varepsilon)|^2 - |\nabla u_{\varepsilon,1}|^2)dx - \int_M \int_{u_{\varepsilon,1}}^{u_{\varepsilon,1}+\varphi_\varepsilon} \overline{f}_1(x,s)ds \\ &\le \int_M \frac{\varepsilon^2}{2}(|\nabla(u_{\varepsilon,1} + \varphi_\varepsilon)|^2 - |\nabla u_{\varepsilon,1}|^2)dx - \int_M \int_{u_{\varepsilon,1}}^{u_{\varepsilon,1}+\varphi_\varepsilon} \overline{f}_2(x,s)ds \\ &= J_{\varepsilon,2}(u_{\varepsilon,2}) - J_{\varepsilon,2}(u_{\varepsilon,2} - \varphi_\varepsilon) \le 0. \end{aligned}$$

This implies that $u_{\varepsilon,1}+\varphi_\varepsilon$ is also a minimizer of $J_{\varepsilon,1}(u)$. Let $L>0$ be large enough such that $\overline{f}_1(x,t)+Lt$ is strictly increasing for $x\in\overline{D}$, $t\in[-1,1]$. From

$$-\varepsilon^2\Delta(u_{\varepsilon,1}+\varphi_\varepsilon)=\overline{f}_1(u_{\varepsilon,1}+\varphi_\varepsilon),$$

we obtain

$$-\varepsilon^2\Delta\varphi_\varepsilon=\overline{f}_1(u_{\varepsilon,1}+\varphi_\varepsilon)-\overline{f}_1(u_{\varepsilon,1}).$$

Thus

$$-\varepsilon^2\Delta\varphi_\varepsilon+L\varphi_\varepsilon=\overline{f}_1(u_{\varepsilon,1}+\varphi_\varepsilon)+L(u_{\varepsilon,1}+\varphi_\varepsilon)-(\overline{f}_1(u_{\varepsilon,1})+Lu_{\varepsilon,1})>0$$

in D. Fix $z_0\in M$. Let $x_0\in\partial M$ such that $|x_0-z_0|=\operatorname{dist}(z_0,\partial M)$. Using the Strong maximum principle and Hopf's lemma in $B_{\operatorname{dist}(z_0,\partial M)}(z_0)$, we obtain that $\frac{\partial\varphi_\varepsilon}{\partial\nu}(x_0)<0$, where $\nu=(x_0-z_0)/|x_0-z_0|$. But $\varphi_\varepsilon(x)=0$ for $x\notin M$. Thus, $\frac{\partial\varphi_\varepsilon}{\partial\nu}(x_0)=0$. This is a contradiction. Thus we obtain $M=\emptyset$. □

3. Proof of Main Theorem

To prove Theorem 1.1, the following proposition is used as the first step.

Propositon 3.1. *Let u_ε be a global minimizer of the problem* (1.2). *Then u_ε satisfies*

$$u_\varepsilon\to\begin{cases}1 & \text{uniformly on each compact subset of } A_-\\ -1 & \text{uniformly on each compact subset of } A_+\end{cases}$$

as $\varepsilon\to0$.

Proof. Let $x_0\in A_-$. Choose $\delta>0$ small so that $B_\delta(x_0)\subset\subset A$. Take $b\in(\max_{z\in\overline{B_\delta(x_0)}}a(z),1/2)$. Define $f_{x_0,\delta,b}(t)=(\min_{z\in B_\delta(x_0)}h(|z|)^2)(t-b)(1-t^2)$. Then for $x\in\overline{B_\delta(x_0)}$, $t\in[-1,1]$, we have $f(|x|,t)\ge f_{x_0,\delta,b}(t)$. Let $u_{\varepsilon,x_0,\delta,b}$ be the minimizer of

$$\inf\Big\{\int_{B_\delta(x_0)}\frac{\varepsilon^2}{2}|\nabla u|^2-F_{x_0,\delta,b}(u)dx : u+1\in H_0^1(B_\delta(x_0))\Big\},$$

where $F_{x_0,\delta,b}(t)=\int_{-1}^t f_{x_0,\delta,b}(s)ds$. It follows from Lemmas 2.1 and 2.2 that

$$u_{\varepsilon,x_0,\delta,b}(x)\le u_\varepsilon(x)\le1,\quad \text{for } x\in B_\delta(x_0).$$

Since $\int_{-1}^1 f_{x_0,\delta,b}(s)ds>0$, it follows from [2, 3] that $u_{\varepsilon,x_0,\delta,b}(x)\to1$ as $\varepsilon\to0$ uniformly in $B_{\delta/2}(x_0)$, thus $u_\varepsilon(x)\to1$ as $\varepsilon\to0$ uniformly in $B_{\delta/2}(x_0)$. □

To prove the rest of Theorem 1.1, we need the following proposition and lemma.

Propositon 3.2. *Let u be a local minimizer of the problem*

$$\inf\Big\{\int_{B_1(0)}\frac12|\nabla u|^2-G(|x|,u)dx : u\in H^1(B_1(0))\Big\}.$$

Here $G(r,t)=\int_{-1}^t g(r,s)ds$, $g(r,t)$ is C^1 in $t\in\mathbb{R}$ for each $r\ge0$, $g(r,t)$ and $g_t(r,t)$ are measurable on $[0,+\infty)$ for each $t\in\mathbb{R}$, $g(r,t)<0$ if $t<-1$ or $t>1$ and $|g(r,t)|+|g_t(r,t)|$ is bounded on $[0,k]\times[-2,2]$ for any $k>0$. Then u is radial, i.e., $u(x)=u(|x|)$.

The proof of the above proposition can be found in [4, Proposition 2.6].

Lemma 3.3. *Let $0 < \eta < 1$ be any fixed constant and w satisfies*

$$-w_{zz} = w(1-w^2) \quad \text{on } \mathbb{R},$$
$$w(0) = -1+\eta \quad (\text{resp. } w(0) = 1-\eta),$$
$$w(z) \le -1+\eta \quad (\text{resp. } w(z) \ge 1-\eta) \quad \text{for } z \le 0,$$
$$w \text{ is bounded on } \mathbb{R}.$$

Then w is a unique solution of

$$-w_{zz} = w(1-w^2) \quad \text{on } \mathbb{R},$$
$$w(0) = -1+\eta \quad (\text{resp. } w(0) = 1-\eta),$$
$$w'(z) > 0 \quad (\text{resp. } w'(z) < 0) \quad z \in \mathbb{R},$$
$$w(z) \to \pm 1 \quad (\text{resp. } w(z) \to \mp 1) \quad \text{as } z \to \pm\infty.$$

The proof of the above lemma can be found in [6]. Now we prove the rest of Theorem 1.1.

Proof of Theorem 1.1. For the sake of simplicity, we prove for the case where $a(r) < 0$ on $[0, r_1)$, $a(r) = 0$ on $[r_1, r_2]$ and $a(r) > 0$ on $(r_2, 1]$ for some $0 < r_1 < r_2 < 1$ (see Figure 1 in Section 1).

Part 1. First we show that u_ε converges uniformly near the boundary of $B_1(0)$, that is, $u_\varepsilon \to -1$ uniformly on $\overline{B_1(0)} \backslash B_{r_2+\tau}(0)$ for any small $\tau > 0$. We note that we have $u_\varepsilon \to -1$ uniformly on $\overline{B_{1-\tau}(0)} \backslash B_{r_2+\tau}(0)$ as $\varepsilon \to 0$. Now we claim that $u_\varepsilon(r) \le u_\varepsilon(1-\tau) =: T_\varepsilon$ for $r \in [1-\tau, 1]$. We define the function $\tilde{u}_\varepsilon$ by

$$\tilde{u}_\varepsilon(r) = \begin{cases} u_\varepsilon(r) & \text{if } r \in [0, 1-\tau] \\ u_\varepsilon(r) & \text{if } u_\varepsilon(r) < T_\varepsilon \text{ and } r \in [1-\tau, 1], \\ T_\varepsilon & \text{if } u_\varepsilon(r) \ge T_\varepsilon \text{ and } r \in [1-\tau, 1]. \end{cases}$$

We note that $\tilde{u}_\varepsilon \in H^1(B_1(0))$ and $-F(r, T_\varepsilon) \le -F(r, t)$ for $\varepsilon > 0$ and $|r-1|$ small and $t \ge T_\varepsilon$. Hence we obtain $J_\varepsilon(\tilde{u}_\varepsilon) < J_\varepsilon(u_\varepsilon)$ and we have a contradiction if we assume that the measure of the set $\{r \in [0,1] | u_\varepsilon(r) > T_\varepsilon \text{ and } r \in [1-\tau, 1]\}$ is positive. Hence $-1 < u_\varepsilon(r) \le T_\varepsilon$ and $u_\varepsilon \to -1$ uniformly on $\overline{B_1(0)} \backslash B_{r_2+\tau}(0)$.

Part 2. We remark that, by Proposition 3.1, u_ε is radially symmetric and we note that for any $t_2 > t_1$, u_ε is a minimizer of the following problem

$$\inf\{J_\varepsilon(u, B_{t_2}(0) \backslash \overline{B_{t_1}(0)}) : u - u_\varepsilon \in H_0^1(B_{t_2}(0) \backslash \overline{B_{t_1}(0)})\},$$

where

$$J_\varepsilon(u, M) = \int_M \frac{\varepsilon^2}{2} |\nabla u|^2 - F(|x|, u) dx$$

for any open set M. Let $m_{\varepsilon, t_1, t_2}$ be the minimum value of this minimization problem.

In this part we show that u_ε has exactly one layer near the interval $[r_1, r_2]$.

Step 2.1. First we estimate the energy of transition layer. Let $\eta > 0$ and $\theta > 0$ be small numbers. Since $u_\varepsilon \to 1$ uniformly on $[0, r_1 - \theta]$ and $u_\varepsilon \to -1$ uniformly on $[r_2 + \theta, 1 - \theta]$, we can find $\overline{r}_\varepsilon \in (r_1 - \theta, r_2 + \theta)$ such that $u_\varepsilon(r) \ge 1 - \eta$ if $r \in [0, \overline{r}_\varepsilon]$, $u_\varepsilon(r) < 1 - \eta$ for $r - \overline{r}_\varepsilon > 0$ small. Let $\tilde{r}_\varepsilon > \overline{r}_\varepsilon$ be such that $u_\varepsilon(r) \le \eta$ if $r \in [\tilde{r}_\varepsilon, 1-\theta]$, $u_\varepsilon(r) > \eta$ for $\tilde{r}_\varepsilon - r > 0$ small. We may assume that $\overline{r}_\varepsilon \to \overline{r} \in [r_1, r_2]$ and $\tilde{r}_\varepsilon \to \tilde{r} \in [r_1, r_2]$

We employ the so-called blow-up argument. Let $v_\varepsilon(t) = u_\varepsilon(\varepsilon t + \overline{r}_\varepsilon)$. Then

$$-v_\varepsilon'' - \varepsilon \frac{N-1}{\varepsilon t + \overline{r}_\varepsilon} v_\varepsilon' = f(\varepsilon t + \overline{r}_\varepsilon, v_\varepsilon),$$

$-1 \le v_\varepsilon \le 1$ and $v_\varepsilon(0) = 1 - \eta$. Since $\overline{r}_\varepsilon \to \overline{r} \in [r_1, r_2]$, it is easy to see that $v_\varepsilon \to v$ in $C^1_{\rm loc}(\mathbb{R})$ and

$$-v'' = h(\overline{r})^2 (v - v^3), \quad t \in \mathbb{R}.$$

and $v(t) \ge 1 - \eta$ for $t \le 0$. If we set $v(t) = V(h(\overline{r})t)$, the function $V(t)$ satisfies

$$\begin{gathered} -V'' = V - V^3 \quad \text{on } \mathbb{R}, \\ V(0) = 1 - \eta, \\ V'(t) \ge 1 - \eta \quad t \le 0. \end{gathered} \tag{3.1}$$

Hence by Lemma 3.3, the function V is a unique solution for

$$\begin{gathered} -V'' = V - V^3 \quad \text{on } \mathbb{R}, \\ V(0) = 1 - \eta, \\ V'(t) < 0 \quad t \le 0. \\ V(t) \to \pm 1 \quad \text{as } t \to \mp\infty. \end{gathered} \tag{3.2}$$

Thus, we can find an $R > 0$ large, such that $v(R) = \eta$. Since $v_\varepsilon \to v$ in $C^1_{\rm loc}(\mathbb{R})$, we can find an $R_\varepsilon \in (R-1, R+1)$, such that $v_\varepsilon'(r) < 0$ if $r \in [0, R_\varepsilon]$ and $v_\varepsilon(R_\varepsilon) = -1+\eta$. Hence $u_\varepsilon'(r) < 0$ if $r \in [\overline{r}_\varepsilon, \overline{r}_\varepsilon + \varepsilon R_\varepsilon]$ and $u_\varepsilon(\overline{r}_\varepsilon + \varepsilon R_\varepsilon) = -1 + \eta$. Then we have

$$\begin{aligned} &J_\varepsilon(u_\varepsilon, B_{\overline{r}_\varepsilon + \varepsilon R_\varepsilon}(0) \backslash \overline{B_{\overline{r}_\varepsilon}(0)}) \\ &= \omega_{N-1}(\overline{r}_\varepsilon^{N-1} + o_\varepsilon(1)) \int_{\overline{r}_\varepsilon}^{\overline{r}_\varepsilon + \varepsilon R_\varepsilon} \Big(\frac{\varepsilon^2}{2} |u_\varepsilon'|^2 - F(t, u_\varepsilon) \Big) dt \\ &= \omega_{N-1}(\overline{r}_\varepsilon^{N-1} + o_\varepsilon(1)) \varepsilon \int_0^{R_\varepsilon} \Big(\frac{1}{2} |v_\varepsilon'|^2 - F(\varepsilon t + \overline{r}_\varepsilon, v_\varepsilon) \Big) dt \\ &= \omega_{N-1}(\overline{r}_\varepsilon^{N-1} + o_\varepsilon(1))(\beta_{h(\overline{r})} + O(\eta) + o_\varepsilon(1))\varepsilon, \end{aligned} \tag{3.3}$$

where ω_{N-1} is the area of the unit sphere in $\mathbb{R}^N$, $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$, $\beta_{h(s)}$ is the positive value defined by

$$\begin{aligned} \beta_{h(s)} &= \int_{-\infty}^{+\infty} \Big(\frac{1}{2} |w_{h(s)}'(t)|^2 + h(s)^2 \frac{(w_{h(s)}^2 - 1)^2}{4} \Big) dt \\ &= h(s) \int_{-\infty}^{+\infty} \frac{1}{2} |V'(t)|^2 + \frac{(V(t)^2 - 1)^2}{4} dt \\ &= h(s) \beta_1 \end{aligned}$$

and $w_{h(s)}(t) = V(h(s)t)$ for $s \in [0,1]$. We note that although the function V depends on η, the value

$$\beta_1 = \int_{-\infty}^{+\infty} \frac{1}{2} |V'(t)|^2 + \frac{(V(t)^2 - 1)^2}{4} dt$$

is independent of η.

Step 2.2. We claim u_ε has exactly one layer near the interval $[r_1, r_2]$. To show u_ε has exactly one layer near the interval $[r_1, r_2]$, it sufficient to prove the following claim

Claim. $\tilde{r}_\varepsilon = \overline{r}_\varepsilon + \varepsilon R_\varepsilon$.

Suppose that the claim is not true. Then we can find a $t_\varepsilon > \overline{r}_\varepsilon + R_\varepsilon\varepsilon$ such that $u_\varepsilon(r) < -1+\eta$ if $r \in (\overline{r}_\varepsilon + R_\varepsilon\varepsilon, t_\varepsilon)$, $u_\varepsilon(t_\varepsilon) = -1+\eta$. Thus we can use the blow-up argument again at t_ε to deduce that there is a $\tilde{t}_\varepsilon = t_\varepsilon + \varepsilon\tilde{R}_\varepsilon$ with $u_\varepsilon'(r) > 0$ if $r \in (t_\varepsilon, \tilde{t}_\varepsilon)$, $u_\varepsilon(\tilde{t}_\varepsilon) = 1-\eta$. We may assume that $t_\varepsilon, \tilde{t}_\varepsilon \to \overline{t}$ as $\varepsilon \to 0$ for some $\overline{t} \in [r_2, r_3]$. Moreover

$$J_\varepsilon(u_\varepsilon, B_{\tilde{t}_\varepsilon}(0)\backslash\overline{B_{t_\varepsilon}(0)}) = \omega_{N-1}(t_\varepsilon^{N-1} + o_\varepsilon(1))(\beta_{h(\overline{t})} + O(\eta))\varepsilon + o_\varepsilon(1) \tag{3.4}$$

Now we claim $\tilde{t}_\varepsilon \geq r_1$. Suppose $\tilde{t}_\varepsilon < r_1$. Let $F_a(t) = \int_{-1}^t (v-a)(1-v^2)dv$. Then for any $t > 0$ small and $s \in [-1+t, 1-t]$,

$$\begin{aligned}
&F_a(1-t) - F_a(s) \\
&= F_0(1-t) - F_0(s) + F_a(1-t) - F_0(1-t) - F_a(s) + F_0(s) \\
&= \big[\frac{(v^2-1)^2}{4}\big]_s^{1-t} - a\int_s^{1-t}(1-v^2)dv
\end{aligned} \tag{3.5}$$

Thus it follows from (3.5) that if $a < 0$, then

$$F_a(1-t) - F_a(s) > 0 \tag{3.6}$$

for $s \in [-1+t, 1-t]$. Define

$$\overline{u}_\varepsilon(r) := \begin{cases} 1-\eta & r \in [\overline{r}_\varepsilon, \overline{r}_\varepsilon + R_\varepsilon\varepsilon] \cup [t_\varepsilon, \tilde{t}_\varepsilon], \\ -u_\varepsilon(r) & r \in [\overline{r}_\varepsilon + R_\varepsilon\varepsilon, t_\varepsilon]. \end{cases}$$

By the assumption that $\tilde{t}_\varepsilon < r_1$ and using (3.6), we see $F(r, u_\varepsilon) < F(r, \overline{u}_\varepsilon)$ if $r \in [\overline{r}_\varepsilon, \tilde{t}_\varepsilon]$. Hence, we obtain

$$J_\varepsilon(\overline{u}_\varepsilon, B_{\tilde{t}_\varepsilon}(0)\backslash\overline{B_{\overline{r}_\varepsilon}(0)}) < J_\varepsilon(u_\varepsilon, B_{\tilde{t}_\varepsilon}(0)\backslash\overline{B_{\overline{r}_\varepsilon}(0)}).$$

Thus we obtain a contradiction. Therefore we have that $\tilde{t}_\varepsilon \geq r_1$.

Since $a(r) \geq 0$ for $r \in [r_1, 1]$, we see $F(r,t) \leq F(r,-1) = 0$ if $r \in [r_1, 1]$. Since $u_\varepsilon(r) \in (-1, -1+\eta)$ for $r \in [\overline{r}_\varepsilon + R_\varepsilon\varepsilon, t_\varepsilon]$, we have

$$\begin{aligned}
m_{\varepsilon,\overline{r}_\varepsilon,\tilde{r}_\varepsilon} &= J_\varepsilon(\overline{u}_\varepsilon, B_{\overline{r}_\varepsilon+\varepsilon R_\varepsilon}(0)\backslash\overline{B_{\overline{r}_\varepsilon}(0)}) + J_\varepsilon(\overline{u}_\varepsilon, B_{\tilde{t}_\varepsilon}(0)\backslash\overline{B_{t_\varepsilon}(0)}) \\
&\quad + J_\varepsilon(\overline{u}_\varepsilon, B_{t_\varepsilon}(0)\backslash\overline{B_{\overline{r}_\varepsilon+\varepsilon R_\varepsilon}(0)}) + J_\varepsilon(\overline{u}_\varepsilon, B_{\tilde{r}_\varepsilon}(0)\backslash\overline{B_{\tilde{t}_\varepsilon}(0)}) \\
&\geq \omega_{N-1}(\overline{r}_\varepsilon^{N-1}\beta_{h(\overline{r})}\varepsilon + t_\varepsilon^{N-1}\beta_{h(\overline{t})}\varepsilon) + O(\eta\varepsilon) + o(\varepsilon) \\
&\quad + \inf\Big\{-\int_{B_{t_\varepsilon}(0)\backslash B_{\overline{r}_\varepsilon+\varepsilon R_\varepsilon}(0)} F(r,w) : -1 \leq w \leq 1+\eta\Big\} \\
&\quad + \inf\Big\{-\int_{B_{\tilde{r}_\varepsilon}(0)\backslash B_{\tilde{t}_\varepsilon}(0)} F(r,w) : -1 \leq w \leq 1\Big\} \\
&\geq \omega_{N-1}(\overline{r}_\varepsilon^{N-1}\beta_{h(\overline{r})}\varepsilon + t_\varepsilon^{N-1}\beta_{h(\overline{t})}\varepsilon) + O(\eta\varepsilon) + o(\varepsilon)
\end{aligned} \tag{3.7}$$

Now we give an upper bound for $m_{\varepsilon,\overline{r}_\varepsilon,\tilde{r}_\varepsilon}$. Let $R > 0$ be such that $V(h(\overline{r})R) = \eta$, where V is a unique solution to (3.2). Define $\overline{u}_\varepsilon$ by

$$\overline{u}_\varepsilon(r) := \begin{cases} V(h(\overline{r})\frac{r-\overline{r}_\varepsilon}{\varepsilon}) & r \in [\overline{r}_\varepsilon, \overline{r}_\varepsilon + \varepsilon R] \\ -1+\eta - \frac{\eta}{\varepsilon}(r - \overline{r}_\varepsilon - \varepsilon R) & r \in [\overline{r}_\varepsilon + \varepsilon R, \overline{r}_\varepsilon + \varepsilon R + \varepsilon] \\ -1 & r \in [\overline{r}_\varepsilon + \varepsilon R + \varepsilon, \tilde{r}_\varepsilon - \varepsilon] \\ -1 + \frac{\eta}{\varepsilon}(r - \tilde{r}_\varepsilon + \varepsilon) & r \in [\tilde{r}_\varepsilon - \varepsilon, \tilde{r}_\varepsilon] \end{cases} \tag{3.8}$$

Now we note that $|F(r,t)| = O(\eta)$ for $r \in [\overline{r}_\varepsilon, \tilde{r}_\varepsilon]$ and $-1 \le t \le -1+\eta$. Then we have

$$\begin{aligned} m_{\varepsilon,\overline{r}_\varepsilon,\tilde{r}_\varepsilon} &\le J_\varepsilon(\overline{u}_\varepsilon, B_{\tilde{r}_\varepsilon}(0)\backslash\overline{B_{\overline{r}_\varepsilon}(0)}) \\ &\le J_\varepsilon(\overline{u}_\varepsilon, B_{\overline{r}_\varepsilon+R\varepsilon}(0)\backslash\overline{B_{\overline{r}_\varepsilon}(0)}) + J_\varepsilon(\overline{u}_\varepsilon, B_{\tilde{r}_\varepsilon}(0)\backslash\overline{B_{\tilde{r}_\varepsilon-\varepsilon}(0)}) \\ &\quad + J_\varepsilon(\overline{u}_\varepsilon, B_{\tilde{r}_\varepsilon-\varepsilon}(0)\backslash\overline{B_{\overline{r}_\varepsilon+\varepsilon R}(0)}) \\ &\le \omega_{N-1}\overline{r}_\varepsilon^{N-1}(\beta_{h(\overline{r})}+O(\eta))\varepsilon + o(\varepsilon) + O(\varepsilon\eta) + o(\varepsilon) \\ &= \omega_{N-1}\overline{r}_\varepsilon^{N-1}\beta_{h(\overline{r})} + O(\eta\varepsilon) + o(\varepsilon) \end{aligned} \tag{3.9}$$

By (3.7) and (3.9), we have

$$\omega_{N-1}(\overline{r}_\varepsilon^{N-1}\beta_{h(\overline{r})} + t_\varepsilon^{N-1}\beta_{h(\overline{t})})\varepsilon \le \omega_{N-1}\overline{r}_\varepsilon^{N-1}\beta_{h(\overline{r})}\varepsilon + O(\varepsilon\eta) + o(\varepsilon)$$

This is a contradiction. So we can conclude $\tilde{r}_\varepsilon = \overline{r}_\varepsilon + \varepsilon R_\varepsilon$.

Part 3. It remains to prove that if $\overline{r}_{\varepsilon_j} \to \overline{r}$ for some positive sequence $\{\varepsilon_j\}$ converging to zero as $j \to \infty$ then $\overline{r}$ satisfies

$$\overline{r}^{N-1}h(\overline{r}) = \min_{s\in[r_1,r_2]} s^{N-1}h(s).$$

Step 3.1. First we note that from Part 1, the function u_ε satisfies $-1 \le u_\varepsilon \le -1+\eta$ for $r \in [\overline{r}_\varepsilon + \varepsilon R_\varepsilon, 1]$ in this case.
Step 3.2. Set $H(s) = s^{N-1}h(s)$. Assume that the result is not true. Then there exists a subsequence of $\{\overline{r}_\varepsilon\}$ (denoted by $\overline{r}_\varepsilon$) such that $\overline{r}_\varepsilon \to r' \in [r_1, r_2]$ and $H(r') > \min_{s\in[r_1,r_2]} H(s)$. Then we can find a point $\overline{t} \in (r_1, r_2)$ such that $H(r') > H(\overline{t})$.

Now we give a lower estimate for $J_\varepsilon(u_\varepsilon)$. We have

$$J_\varepsilon(u_\varepsilon) = J_\varepsilon(u_\varepsilon, B_{\overline{r}_\varepsilon}(0)) + J_\varepsilon(u_\varepsilon, B_{\overline{r}_\varepsilon+\varepsilon R_\varepsilon}(0)\backslash\overline{B_{\overline{r}_\varepsilon}(0)}) + J_\varepsilon(u_\varepsilon, B_1(0)\backslash\overline{B_{\overline{r}_\varepsilon+R_\varepsilon\varepsilon}(0)}). \tag{3.10}$$

First we note that $1-\eta \le u_\varepsilon(r) \le 1$ for $r \le \overline{r}_\varepsilon$ and for sufficiently small $\eta > 0$, $-F(r,u) \ge -F(r,1)$ $(u \in [1-\eta, 1])$. We also remark that since $a(r) < 0$ for $r < r_1$ and $a(r) = 0$ for $r_1 \le r \le r_2$ and $a(r) > 0$ for $r > r_2$, we have $-F(r,1) < 0$ for $r < r_1$ and $-F(r,1) = 0$ for $r_1 \le r \le r_2$ and $-F(r,1) > 0$ for $r > r_2$. Hence we have $-\int_{r_1}^{\overline{r}_\varepsilon} r^{N-1}F(r,1)dr \ge 0$ and we obtain the estimate

$$\begin{aligned} J_\varepsilon(u_\varepsilon, B_{\overline{r}_\varepsilon}(0)) &\ge -\int_0^{\overline{r}_\varepsilon} r^{N-1}F(r,u_\varepsilon)dr \\ &\ge -\int_0^{\overline{r}_\varepsilon} r^{N-1}F(r,1)dr \\ &= -\int_0^{r_1} r^{N-1}F(r,1)dr - \int_{r_1}^{\overline{r}_\varepsilon} r^{N-1}F(r,1)dr \\ &\ge -\int_0^{r_1} r^{N-1}F(r,1)dr =: A. \end{aligned} \tag{3.11}$$

Using methods similar to those in the proof of (3.3), we obtain

$$J_\varepsilon(u_\varepsilon, B_{\overline{r}_\varepsilon+R_\varepsilon\varepsilon}(0)\backslash\overline{B_{\overline{r}_\varepsilon}(0)}) \ge \omega_{N-1}H(r')\beta_1\varepsilon + O(\eta\varepsilon) + o(\varepsilon). \tag{3.12}$$

Since $-1 \le u_\varepsilon(r) \le -1+\eta$ for $r \ge \overline{r}_\varepsilon + \varepsilon R_\varepsilon$ and for sufficiently small $\eta > 0$, $-F(r,u) \ge -F(r,-1) = 0$ ($u \in [-1,-1+\eta]$), we obtain the estimate

$$\begin{aligned} J_\varepsilon(u_\varepsilon, B_1(0)\backslash B_{\overline{r}_\varepsilon + R_\varepsilon\varepsilon}(0)) &\ge -\int_{\overline{r}_\varepsilon+\varepsilon R_\varepsilon}^1 r^{N-1}F(r,u_\varepsilon)dr \\ &\ge -\int_{\overline{r}_\varepsilon+\varepsilon R_\varepsilon}^1 r^{N-1}F(r,-1)dr = 0. \end{aligned} \tag{3.13}$$

Thus we obtain

$$J(u_\varepsilon) \ge A + \omega_{N-1}H(r')\beta_1\varepsilon + O(\eta\varepsilon) + o(\varepsilon). \tag{3.14}$$

Next we give an upper bound for $J_\varepsilon(u_\varepsilon)$. Consider the function

$$\overline{w}_\varepsilon(r) := \begin{cases} 1 & r \in [0, \bar{t} - \varepsilon] \\ 1 - \frac{\eta}{\varepsilon}(r - \bar{t} + \varepsilon) & r \in [\bar{t} - \varepsilon, \bar{t}] \\ V\big(h(\bar{t})\frac{r-\bar{t}}{\varepsilon}\big) & r \in [\bar{t}, \bar{t} + \varepsilon R'] \\ -1 - \frac{\eta}{\varepsilon}(r - \bar{t} - \varepsilon R' - \varepsilon) & r \in [\bar{t} + \varepsilon R', \bar{t} + \varepsilon R' + \varepsilon] \\ -1 & r \in [\bar{t} + \varepsilon R' + \varepsilon, 1], \end{cases}$$

where $R' > 0$ is the number satisfying $V(h(\bar{t})R') = -1 + \eta$. Then

$$J_\varepsilon(u_\varepsilon) \le J_\varepsilon(\overline{w}_\varepsilon) \le A + \omega_{N-1}H(\bar{t})\beta_1\varepsilon + O(\eta\varepsilon) + o(\varepsilon). \tag{3.15}$$

By (3.14) and (3.15) we have a contradiction. The proof of Theorem 1.1 is complete. The more complicate case, can be shown by a similar method (see Remark below). $\square$

Remark. We briefly show the more complicate case, that is, when a is the function as in Figure 2. More precisely we set $I_1 := [r_1, r_2]$ and $I_2 := [r_3, r_4]$ and we assume $a > 0$ on $[0, r_1) \cup (r_4, 1]$ and $a < 0$ on (r_3, r_4).

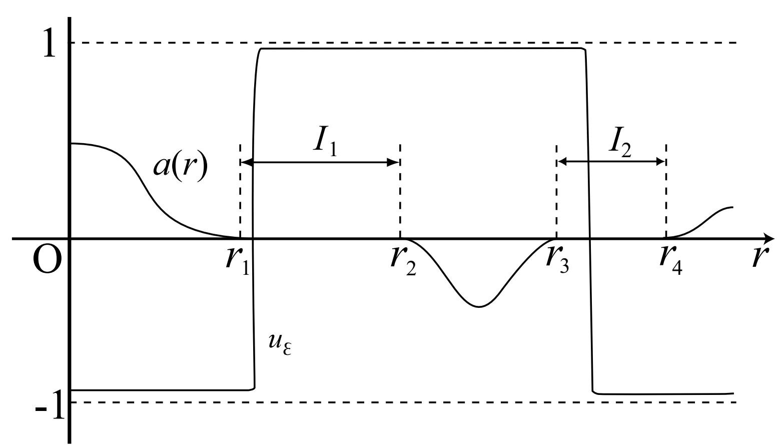

FIGURE 2. Special case of coefficient $a(t)$

Let $\eta > 0$ and $\theta > 0$ be small numbers. As in Part 1, we can find pairs of numbers $(\overline{r}_{1,\varepsilon}, \overline{r}_{2,\varepsilon})$ and $(R_{1,\varepsilon}, R_{\varepsilon,2})$ satisfying $\overline{r}_{1,\varepsilon} \in (r_1 - \theta, r_2 + \theta)$, $\overline{r}_{2,\varepsilon} \in (r_3 - \theta, r_4 + \theta)$,

$\sup_\varepsilon |R_{1,\varepsilon}| < \infty$, $\sup_\varepsilon |R_{2,\varepsilon}| < \infty$ and

$$\begin{gathered}
u_\varepsilon(r) < -1+\eta \quad \text{for } 0 < r < \overline{r}_{1,\varepsilon} \\
u_\varepsilon(\overline{r}_{1,\varepsilon}) = -1+\eta \\
u_\varepsilon(\overline{r}_{1,\varepsilon} + \varepsilon R_{1,\varepsilon}) = 1-\eta \\
u_\varepsilon(r) > 1-\eta \quad \text{for } \overline{r}_{1,\varepsilon} + \varepsilon R_{1,\varepsilon} < r < \overline{r}_{2,\varepsilon} \\
u_\varepsilon(\overline{r}_{2,\varepsilon}) = 1-\eta \\
u_\varepsilon(\overline{r}_{2,\varepsilon} + \varepsilon R_{2,\varepsilon}) = -1+\eta \\
u_\varepsilon(r) < -1+\eta \quad \text{for } \overline{r}_{2,\varepsilon} + \varepsilon R_{2,\varepsilon} < r < 1
\end{gathered}$$

We assume that $\overline{r}_{1,\varepsilon_j} \to \overline{r}_1 \in I_1$ and that $\overline{r}_{2,\varepsilon_j} \to \overline{r}_2 \in I_2$ for some sequence $\{\varepsilon_j\}$ which converges to 0 as $j \to \infty$. In this case it is easy to show that the energy of global minimizer $J(u_\varepsilon)$ is estimated as follows

$$J_{\varepsilon_j}(u_{\varepsilon_j}) \geq J_{\varepsilon_j}(u_{\varepsilon_j}, B_{r_2-\varepsilon}(0)) + \varepsilon_j \omega_{N-1} H(\overline{r}_2)\beta_1 + B + O(\varepsilon_j\eta) + o(\varepsilon_j), \tag{3.16}$$

where $B = -\int_{r_2}^{r_3} r^{N-1}F(r,1)dr$.

Let us assume the result does not hold. Then $H(\overline{r}_1) > \min_{s\in I_1} H(s)$ or $H(\overline{r}_2) > \min_{s\in I_2}$ hold. We assume $H(\overline{r}_1) = \min_{s\in I_1}$ and $H(\overline{r}_2) > \min_{s\in I_2} H(s)$. We also assume $r_1 = \overline{r}_1$. We note that if $H(\overline{r}_1) > \min_{s\in I_1} H(s)$ or $\overline{r}_1 \in \mathrm{int} I_1$, the proof is more easy.

Let we take $\tilde{r}_2 \in \mathrm{int} I_2$ such that $H(\overline{r}_2) > H(\tilde{r}_2) > \min_{s\in I_2} H(s)$ and consider the function

$$\tilde{u}_\varepsilon(r) := \begin{cases}
u_\varepsilon(r) \quad \text{on } [0, r_2-\varepsilon) \\
1 + \frac{\eta}{\varepsilon}(r-r_2) \quad \text{on } [r_2-\varepsilon, r_2] \\
1 \quad \text{on } [r_2, \tilde{r}_2-\varepsilon] \\
1 - \frac{\eta}{\varepsilon}(r-\tilde{r}_2+\varepsilon) \quad \text{on } [\tilde{r}_2-\varepsilon, \tilde{r}_2] \\
V\left(h(\tilde{r}_2)\frac{r-\tilde{r}_2}{\varepsilon}\right) \quad \text{on } [\tilde{r}_2, \tilde{r}_2+\varepsilon R''] \\
-1 - \frac{\eta}{\varepsilon}(r-\tilde{r}_2-\varepsilon R''-\varepsilon) \quad \text{on } [\tilde{r}_2+\varepsilon R'', \tilde{r}_2+\varepsilon R''+\varepsilon] \\
-1 \quad \text{on } [\tilde{r}_2+\varepsilon R''+\varepsilon, 1],
\end{cases}$$

where V is the unique solution of (3.2) and R'' is the unique value such that $V(h(r_1)R'') = -1+\eta$.

Since u_ε is global minimizer, we can estimate the energy of $J_\varepsilon(\tilde{u}_\varepsilon)$ as follows

$$J_\varepsilon(u_\varepsilon) \leq J_\varepsilon(\tilde{u}_\varepsilon) \leq J_\varepsilon(u_\varepsilon, B_{r_2-\varepsilon}(0)) + \varepsilon\omega_{N-1}H(\tilde{r}_2)\beta_1 + B + O(\varepsilon\eta) + o(\varepsilon). \tag{3.17}$$

Then we have a contradiction from (3.16) and (3.17) by taking $\varepsilon = \varepsilon_j$ and sufficiently large j.

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