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BMO ESTIMATES NEAR THE BOUNDARY FOR SOLUTIONS OF ELLIPTIC SYSTEMS

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Dedicated to all the civilian victims of the catalogue of horror in the Middle East

Abstract. In this paper we show that the scale of Sobolev-Campanato spaces $\mathcal{L}^{p,\lambda,s}$ contain the general BMO-Triebel-Lizorkin spaces $F^s_{\infty,p}$ as special cases, so that the conjecture by Triebel regarding estimates for solutions of scalar regular elliptic boundary value problems in $F^s_{\infty,p}$ spaces (solved in the case $p = 2$ in a previous work) is completely solved now.

Also we prove that the method used for the scalar case works for systems, and we give a priori estimates near the boundary for solutions of regular elliptic systems in the general spaces $\mathcal{L}^{p,\lambda,s}$ containing BMO, $F^s_{\infty,p}$, and Morrey-Campanato spaces $\mathcal{L}^{2,\lambda}$ as special cases. This result extends the work by the author in the scalar case.

1. Introduction

The aim of this paper is to give the regularity for solutions of regular elliptic systems in the John and Nirenberg space BMO and more generaly in Morrey-Campanato spaces $\mathcal{L}^{2,\lambda}$ and their local versions bmo and $l^{2,\lambda}$. So, this paper is the continuation of [9] where we got the regularity for solutions of a scalar regular elliptic boundary value probem in $\mathcal{L}^{p,\lambda,s}(\Omega)$ spaces containing $\dot{F}^s_{\infty,p}$, BMO, $\mathcal{L}^{2,\lambda}$, and their local versions as special cases.

Firstly, we mention the well known work for variational systems of Campanato [5] who obtained some results concerning local and global (under Dirichlet boundary conditions) regularity for solutions $u \in H^1_0(\Omega, \mathbb{R}^N)$ of second order linear strongly elliptic systems of the form

$$\sum_{i,j=1}^{n} \int_\Omega \langle A_{ij}(x).D_j u | D_i\phi\rangle dx = \sum_{i=1}^{n} \int_\Omega \langle f_i(x) | D_i\phi\rangle dx$$

for any $\phi \in C_0^\infty(\Omega, \mathbb{R}^N)$. He showed that if $f \in BMO(\Omega, \mathbb{R}^{nN})$ then $Du \in BMO(\Omega, \mathbb{R}^{nN})$ provided the cofficients A_{ij} are Hölder continuous in $\overline{\Omega}$ and $\partial\Omega$ is Hölder differentiable, and he got the a-priori estimate

$$\|Du\|_{BMO} \le C\|f\|_{BMO}$$

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An inspection of Campanato's proof gave a refinement of this result for (nonregular) elliptic systems with coefficients just belonging to the class of small multipliers of $BMO(\Omega)$, cf. [1].

In this paper we deal with non-variational and inhomogeneous systems. For instance, let us take the classical regular second order elliptic system

$$\begin{gathered} Au = f \quad \text{in } \Omega \\ u|_{\Omega} = \varphi \quad \text{on } \partial\Omega \,, \end{gathered} \tag{1.1}$$

where

- Ω is a regular bounded open set of $\mathbb{R}^n$.
- $A = \sum_{|\alpha|\leq 2} a_\alpha(x) D_x^\alpha$, $a_\alpha(x)$ is the $N \times N$ matrix $(a_\alpha^{ij}(x))_{i,j=1\dots N}$ with smooth coefficients on $\overline{\Omega}$, here $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, $|\alpha| = \alpha_1 + \dots + \alpha_n$ is the length of α, and $D_x^\alpha = D_{x_1}^{\alpha_1} \dots D_{x_n}^{\alpha_n}$ is the derivation, with $D_{x_j}^{\alpha_j} = \frac{1}{i^{\alpha_j}} \frac{\partial^{\alpha_j}}{\partial x_j^{\alpha_j}}$.
- u, f, φ are vector-valued functions in $\mathbb{R}^N$.

We prove in this paper that under the proper ellipticity of A, if $f \in BMO(\Omega, \mathbb{R}^N)$ and $\varphi \in BMO^{3/2}(\partial\Omega, \mathbb{R}^N)$ then the solution u of the system (1.1) belongs to $BMO^2(\Omega, \mathbb{R}^N)$, that is u, Du *and* $D^2u \in BMO$, and we give the estimate

$$\|u\|_{BMO} + \|Du\|_{BMO} + \|D^2u\|_{BMO} \leq C\{\|f\|_{BMO} + \|\varphi\|_{BMO^{3/2}}\}$$

In addition, this work generalizes the result of the above example to the elliptic systems in the sense of Douglis and Nirenberg [6] and to the general spaces $\mathcal{L}^{p,\lambda,s}$ defined in [7] and [8].

In [10] we showed that the Sobolev-Campanato spaces $\mathcal{L}^{p,\lambda,s}$ contain $BMO, \mathcal{L}^{2,\lambda}$ and their local versions $bmo, l^{2,\lambda}$ as special cases. In this paper we give a generalization of some results of [10] by showing that $\mathcal{L}^{p,\lambda,s}$ spaces contain the general BMO-Triebel-Lizorkin spaces $F^s_{\infty,p}$ as special cases cf. Theorem 3.1. We want to attract attention in this paper that, with the result of Theorem 3.1 in mind and the a priori estimates of [9] relative to the scalar case, Triebel's conjecture [16, section 4.3.4] previously solved in the case $p = 2$ in [9] is completely solved now in the general spaces $F^s_{\infty,p}$.

We show that the method used for the scalar case in [9] can be adapted for elliptic systems. The plan of the paper is as follows: in the first section we give the main definitions and results (Theorems 1.5 and 1.6), section 2 contains an index theorem for a system of ordinary differential equations needed in the proof. In section 3 we identify the space $\mathcal{L}^{p,\lambda,s}$ for $\lambda = n$ and we give a partial result on the topological dual of $\mathring{F}^s_{\infty,p}$ (the closure of Schwartz class $\mathcal{S}(\mathbb{R}^n)$ in BMO-Triebel-Lizorkin spaces $F^s_{\infty,p}$), and next we recall some results proved in previous papers concerning intermediate derivatives, compactness, interpolation and traces. Finally, section 4 deals with the proof of the main result: we follow one Peetre's method used in the scalar case [9]. This method was described in the case of Sobolev spaces H^s for a class of degenerate elliptic systems in [3], and consists in doing a partial Fourier transform with respect to the tangential direction on the system of equations, and reducing the problem to an isomorphism theorem for a system of ordinary differential operators. Thereby we estimate the "almost tangential derivatives" of the solution in some vector-valued L^p−spaces, built on $\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1})$ in the sense of Bochner's integrals

(Proposition 4.1). Next, we make use of an interpolation lemma to estimate the normal derivatives of the solution.

From these BMO estimates we can get again the classical L^p estimates [2] via an interpolation theorem due to Stampacchia [14].

To make the paper self contained, we recall the definitions of the spaces $\mathcal{L}^{p,\lambda,s}$. For this we need a Littlewood-Paley partition of unity: Denote $x = (t, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $\xi = (\tau, \xi')$ its dual variable.

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\varphi \geq 0$ and φ equal to 1 for $|\xi| \leq 1$, 0 for $|\xi| \geq 2$. Putting $\theta(\xi) = \varphi(\xi) - \varphi(2\xi)$, we have $\operatorname{supp}\theta \subset \{\frac{1}{2} \leq |\xi| \leq 2\}$. For $j \in \mathbb{Z}$ we set

$$\dot{\Delta}_j u = \theta(2^{-j} D_x)u,\ \dot{\Delta}'_j u = \theta(2^{-j} D_{x'})u$$

which means that $\dot{\Delta}_j$ and $\dot{\Delta}'_j$ are the convolution operators with symbols $\theta(2^{-j}\xi)$ and $\theta(0, 2^{-j}\xi')$; and denoting $\psi_j(\xi) = \sum_{k\leq j} \theta(2^{-k}\xi)(= \varphi(2^{-j}\xi)$ for $\xi \neq 0)$ we set

$$\dot{S}_j u = \psi_j(D_x)u,\ \dot{S}'_j u = \psi_j(D_{x'})u$$

with the same meaning as above. If $j \geq 1$ we set also

$$[c]l\Delta_j u = \dot{\Delta}_j u,\ \Delta'_j u = \dot{\Delta}'_j u,$$
$$S_j u = \varphi(2^{-j} D_x)u,\ S'_j u = \varphi(2^{-j} D_{x'})u,$$
$$S_0 u = \Delta_0 u = \varphi(D_x)u,\ S'_0 u = \Delta'_0 u = \varphi(D_{x'})u$$

Remark 1.1. For $u \in \mathcal{S}'(\mathbb{R}^n)$ we have

$$u = \sum_{k\geq 0} \Delta_k u = \Delta_0 u + \sum_{k\geq 1} \Delta_k u = S_j u + \sum_{k\geq j+1} \Delta_k u \quad \text{for } j \in \mathbb{N}.$$

If $0 \notin \operatorname{supp}\mathcal{F}u$, then

$$u = \sum_{k\in\mathbb{Z}} \dot{\Delta}_k u = \dot{S}_j u + \sum_{k\geq j+1} \dot{\Delta}_k u \quad \text{for } j \in \mathbb{Z}.$$

If we remove the condition $0 \notin \operatorname{supp}\mathcal{F}u$, the above formula remains valid modulo polynomials, cf. [8, Lemma 2.6].

Definition 1.2. Let $s \in \mathbb{R}$, $\lambda \geq 0$ and $1 \leq p < +\infty$. The space $\mathcal{L}^{p,\lambda,s}(\mathbb{R}^n)$ denotes the set of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|u\|_{\mathcal{L}^{p,\lambda,s}(\mathbb{R}^n)} = \big\{ \sup_{J,B} \frac{1}{|B|^{\lambda/n}} \sum_{k\geq J^+} 2^{kps} \|\Delta_k u\|^p_{L^p(B)} \big\}^{1/p} < +\infty \tag{1.2}$$

where $J^+ = \max(J, 0)$, $|B|$ is the measure of B and the supremum is taken over all $J \in \mathbb{Z}$ and all balls B of $\mathbb{R}^n$ of radius 2^{-J}.

The space $\mathcal{L}^{p,\lambda,s}(\mathbb{R}^n)$ equipped with the norm (1.2) is a Banach space. If Ω is either $\mathbb{R}^n_+$ or a bounded C^∞-domain in $\mathbb{R}^n$, $\mathcal{L}^{p,\lambda,s}(\Omega)$ denotes the space of all restrictions to Ω of elements of $\mathcal{L}^{p,\lambda,s}(\mathbb{R}^n)$.

To give the homogeneous counterpart of the spaces $\mathcal{L}^{p,\lambda,s}(\mathbb{R}^n)$, we recall the notation of [16, chapter 5]. Let

$$Z(\mathbb{R}^n) = \{\varphi \in \mathcal{S}(\mathbb{R}^n); (D^\alpha \mathcal{F}\varphi)(0) = 0 \text{ for every multi-index } \alpha\}$$

$Z(\mathbb{R}^n)$ is considered as a subspace of $\mathcal{S}(\mathbb{R}^n)$ with the same topology, and $Z'(\mathbb{R}^n)$ is the topological dual of $Z(\mathbb{R}^n)$. We may identify $Z'(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$, where $\mathcal{P}$ is the set of all polynomials of $\mathbb{R}^n$ with complex coefficients. $Z'(\mathbb{R}^n)$ is interpreted as $\mathcal{S}'(\mathbb{R}^n)$ modulo polynomials.

Definition 1.3. Let $s \in \mathbb{R}$, $\lambda \geq 0$ and $1 \leq p < +\infty$. The dotted space $\dot{\mathcal{L}}^{p,\lambda,s}(\mathbb{R}^n)$ denotes the set of all $u \in Z'(\mathbb{R}^n)$ such that

$$\|u\|_{\dot{\mathcal{L}}^{p,\lambda,s}(\mathbb{R}^n)} = \Big\{ \sup_{J,B} \frac{1}{|B|^{\frac{\lambda}{n}}} \sum_{k \geq J} 2^{kps} \|\dot{\Delta}_k u\|^p_{L^p(B)} \Big\}^{1/p} < +\infty \tag{1.3}$$

where the supremum is taken over all $J \in \mathbb{Z}$ and all balls B of $\mathbb{R}^n$ of radius 2^{-J}.

If P is a polynomial of $\mathcal{P}$ and $u \in \mathcal{S}'(\mathbb{R}^n)$, it follows immediately that

$$\|u + P\|_{\dot{\mathcal{L}}^{p,\lambda,s}(\mathbb{R}^n)} = \|u\|_{\dot{\mathcal{L}}^{p,\lambda,s}(\mathbb{R}^n)}$$

This shows that the norm (1.3) is well defined. Further, the space $\dot{\mathcal{L}}^{p,\lambda,s}(\mathbb{R}^n)$ equipped with this norm is a Banach space.

Remark 1.4. The supremum in expressions (1.2) and (1.3) can be taken over all $J \in \mathbb{Z}$ and all cubes of $\mathbb{R}^n$ of sidelength 2^{-J}.

The reader can find the properties of these spaces in [7, 8, 10]. We recall that the space $\mathcal{L}^{p,\lambda,s}$ coincides with Campanato space $l^{2,\lambda} = \mathcal{L}_2^{\frac{\lambda-n}{2}}$ when $s = 0, p = 2$ and $0 \leq \lambda < n+2$, itself equals *bmo* for $\lambda = n$, cf. [10]; and we will see in Theorem 3.1 that in the case $\lambda = n$, the space $\dot{\mathcal{L}}^{p,n,s}$ coincides with the homogeneous BMO-Triebel-Lizorkin space $\dot{F}^s_{\infty,p}(1 \leq p < +\infty)$, itself equals BMO when $s = 0$ and $p = 2$.

We return to our main goal which is the estimates for solutions of regular elliptic systems in $C^\infty-$bounded open sets Ω in $\mathbb{R}^n$. This problem can be reduced, via a partition of unity, to a priori estimates for solutions of regular elliptic systems in the upper-half space $\mathbb{R}^n_+ = \{x = (t, x');\ t > 0\}$. We denote $\xi = (\tau, \xi')$ the dual variable of $x = (t, x') \in \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$.

Consider L the following differential system in the sense of Douglis and Nirenberg [6]:

$$L = L(t, x'; D_{x'}, D_t) = (L_{ij}(x; D_x))_{i,j=1,\dots,N} \tag{1.4}$$

where

$$L_{ij}(x; D_x)u(x) = L_{ij}u(x) = \sum_{|\alpha| \leq s_i + t_j} a^{ij}_\alpha(x) D^\alpha_x u(x) \tag{1.5}$$

here $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, $|\alpha| = \alpha_1 + \dots + \alpha_n$ is the length of α, and $D^\alpha_x = D^{\alpha_1}_{x_1} \dots D^{\alpha_n}_{x_n}$ is the derivation, with $D^{\alpha_j}_{x_j} = \frac{1}{i^{\alpha_j}} \frac{\partial^{\alpha_j}}{\partial x_j^{\alpha_j}}$. The coefficients a^{ij}_α are assumed to be in $C^\infty(\overline{\mathbb{R}^n_+})$ and s_i, t_j are integers which we can suppose satisfying $s_i \leq 0$ and $t_j \geq 0$. In the definition of L_{ij}, it is to be understood that if $s_i + t_j < 0$ then $L_{ij} = 0$. Let $L^0_{ij}(x; D_x)$ represent the principal part of $L_{ij}(x, D_x)$, which is the sum of the terms in $L_{ij}(x; D_x)$ which are exactly of the order $s_i + t_j$.

We suppose that L is elliptic in the following classical sense.

(E1) For any $(\tau, \xi') \in \mathbb{R}^n \setminus \{0\}$,

$$\det(L^0_{ij}(0; \xi', \tau))_{i,j=1,\dots,N} \neq 0$$

and the number $m_+(\xi')$ of the roots with positive imaginary parts of the polynomial

$$P(\tau) = \det(L^0_{ij}(0; \xi', \tau))_{i,j=1,\dots,N}$$

in the complex variable τ, is constant and equals m_+.

Now we will define the traces. For each $u \in \mathcal{L}^{p,\lambda,s+t_j}(\mathbb{R}^n_+)$ we consider t_j traces of u defined for $x' \in \mathbb{R}^{n-1}$ by:

$$\gamma_l u(x') = D_t^l u(0,x') \text{ for } l = 0,\dots,t_j-1.$$

Set $\gamma = (\gamma_0,\dots,\gamma_{t_j-1})$. We showed in [9, Theorem 3.1] that the operator γ is continuous from $\mathcal{L}^{p,\lambda,s+t_j}(\mathbb{R}^n_+)$ to $\prod_{l=0}^{t_j-1}\mathcal{L}^{p,\lambda,s+t_j-l-1/p}(\mathbb{R}^{n-1})$.

If $m_+ = 0$ there is no boundary conditions for our problem. If $m_+ > 0$, for each $i = 1,\dots,m_+$, let σ_i be an integer $\sigma_i \le -1$. Set

$$(B_{ij}(x',D_{x'})\cdot\gamma)u = \sum_{l=0}^{t_j-1} B_{ijl}(x',D_{x'})\gamma_l u,$$

for $i = 1,\dots,m_+,\ j = 1,\dots,N$; where $B_{ijl}(x',D_{x'})$ is a differential operator of degree less than or equal $\sigma_i + t_j - l$, with smooth coefficients bounded with their derivatives. If $\sigma_i + t_j - l < 0$ we put $B_{ijl} = 0$.

Denote $B\gamma$ the matrix $B(x',D_{x'})\gamma = (B_{ij}(x',D_{x'})\cdot\gamma)_{i=1,\dots,m_+;j=1,\dots,N}$. This operator is continuous from $\prod_{j=1}^N \mathcal{L}^{p,\lambda,s+t_j}(\mathbb{R}^n_+)$ into $\prod_{i=1}^{m_+}\mathcal{L}^{p,\lambda,s-\sigma_i-1/p}(\mathbb{R}^{n-1})$. Let $B^0_{ij}\cdot\gamma$ represent the principal part of $B_{ij}\cdot\gamma$, which is the sum of the terms in $B_{ij}(x',D_{x'})\cdot\gamma$ which are exactly of the order $\sigma_i + t_j - l$ in $B_{ijl}(x',D_{x'})$. We denote $B^0\gamma = B^0(x',D_{x'})\gamma = (B^0_{ij}\cdot\gamma)_{\substack{i=1,\dots,m_+\\ j=1,\dots,N}}$.

We assume that

(E2) For any $\xi' \in \mathbb{R}^{n-1}, |\xi'| = 1$, the problem

$$L^0(t,0;\xi',D_t)v(t) = 0$$
$$B^0(0,\xi')\gamma v = 0$$

has only the trivial solution $v = 0$ in $\prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+)$.

The first part of our main result is the following theorem.

Theorem 1.5. *Let s and λ be two nonnegative real numbers and $1 \le p < +\infty$. Under hypotheses (E1) and (E2), for any compact set K of $\overline{\mathbb{R}^n_+}$, there is a constant $C_K > 0$, such that for any $u \in \prod_{j=1}^N \mathcal{L}^{p,\lambda,s+t_j}(\mathbb{R}^n_+)$ with $\operatorname{supp} u \subset K$, we get*

$$\begin{aligned}\|u\|_{\prod_{j=1}^N \mathcal{L}^{p,\lambda,s+t_j}(\mathbb{R}^n_+)} \le C\big\{&\|Lu\|_{\prod_{i=1}^N \mathcal{L}^{p,\lambda,s-s_i}(\mathbb{R}^n_+)}\\ &+ \|B\gamma u\|_{\prod_{i=1}^{m_+}\mathcal{L}^{p,\lambda,s-\sigma_i-\frac{1}{p}}(\mathbb{R}^{n-1})} + \|u\|_{\prod_{j=1}^N \mathcal{L}^{p,\lambda,s+t_j-1}(\mathbb{R}^n_+)}\big\}\end{aligned}$$

This statement remains true if we replace $\mathcal{L}$ by the dotted space $\dot{\mathcal{L}}$.

If $\lambda = n$ we get the estimates in the spaces of BMO-Triebel-Lizorkin $F^s_{\infty,p}$ and $\dot{F}^s_{\infty,p}$. If $0 \le \lambda < n+2$ and $p = 2$ we get the estimates in Morrey-Campanato spaces $\mathcal{L}^{2,\lambda}$.

Now, let Ω be a $C^\infty-$bounded open set of $\mathbb{R}^n$, Γ its boundary. Let L be the differential system defined by (1.4) and (1.5), where the coefficients a^{ij}_α are belonging to $C^\infty(\overline{\Omega})$.

We assume the following hypotheses:

(H1) For any $x \in \Gamma$ and any $\xi \in \mathbb{R}^n \setminus \{0\}$, $\det(L^0_{ij}(x;\xi))_{i,j=1,\dots,N} \ne 0$ and for any $\xi_x \in \mathbb{R}^n \setminus \{0\}$ tangent to Γ at x, the number $m_+(x,\xi)$ of the roots with positive imaginary parts of the polynomial

$$P(\tau) = \det(L^0_{ij}(x;\xi_x + \tau\nu_x))_{i,j=1,\dots,N}$$

in the complex variable τ, is constant and equals m_+. Here ν_x is the inward unit normal vector to the boundary Γ at x.

To give the complementing condition we define the traces. For each $u \in \mathcal{L}^{p,\lambda,s+t_j}(\Omega)$ we consider t_j traces of u defined by

$$\gamma_l u = \frac{\partial^l u}{\partial \nu^l}\Big|_\Gamma \quad \text{for } l = 0, \dots, t_j - 1.$$

The operator $\gamma = (\gamma_0, \dots, \gamma_{t_j-1})$ is a continuous operator from $\mathcal{L}^{p,\lambda,s+t_j}(\Omega)$ to $\prod_{l=0}^{t_j-1} \mathcal{L}^{p,\lambda,s+t_j-l-1/p}(\Gamma)$. Let σ_i be an integer ≤ -1. We define the following differential operators on Γ,

$$(B_{ij}(x,D)\cdot\gamma)u = \sum_{l=0}^{t_j-1} B_{ijl}(x,D)\gamma_l u$$

where $B_{ijl}(x,D)$ is a differential operator on Γ with smooth coefficients on Γ, of degree $\leq \sigma_i + t_j - l$. If $\sigma_i + t_j - l < 0$ we put $B_{ijl} = 0$.

Denote $B\gamma = B(x,D)\gamma = (B_{ij}\cdot\gamma)_{i=1,\dots,m_+;j=1,\dots,N}$. This operator is continuous from $\prod_{j=1}^N \mathcal{L}^{p,\lambda,s+t_j}(\Omega)$ into $\prod_{i=1}^{m_+} \mathcal{L}^{p,\lambda,s-\sigma_i-1/p}(\Gamma)$. Let $B^0_{ij}\cdot\gamma$ represent the principal part of $B_{ij}\cdot\gamma$. We denote $B^0\gamma = B^0(x;D_x)\gamma = (B^0_{ij}\cdot\gamma)_{i=1,\dots,m_+;j=1,\dots,N}$.

Finally we suppose:

(H2) For any $x \in \Gamma$ and any $\xi_x \in \mathbb{R}^n \setminus \{0\}, |\xi_x| = 1$, tangent to Γ at x, the problem,

$$\begin{gathered} L^0(x;\xi_x + \nu_x D_t)v(t) = 0 \\ B^0(x;\xi_x)\gamma v = 0 \end{gathered}$$

has only the trivial solution $v = 0$ in $\prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+)$.

Theorem 1.6. *Let s and λ be two nonnegative real numbers and $1 \leq p < +\infty$. Under hypotheses (H1) and (H2), there exists a constant $C > 0$ such that for any $u \in \prod_{j=1}^N \mathcal{L}^{p,\lambda,s+t_j}(\Omega)$, we get*

$$\begin{aligned} &\|u\|_{\prod_{j=1}^N \mathcal{L}^{p,\lambda,s+t_j}(\Omega)} \\ &\leq C\big\{\|Lu\|_{\prod_{i=1}^N \mathcal{L}^{p,\lambda,s-s_i}(\Omega)} + \|B\gamma u\|_{\prod_{i=1}^{m_+} \mathcal{L}^{p,\lambda,s-\sigma_i-\frac{1}{p}}(\Gamma)} + \|u\|_{\prod_{j=1}^N \mathcal{L}^{p,\lambda,s+t_j-1}(\Omega)}\big\} \end{aligned}$$

This theorem extends the scalar case work [9]. From this theorem and the interpolation theorem of Stampacchia, cf. [14] or [12, Theorem 4.6], we get in the same manner as in [9], the classical L^p estimates for solutions of regular elliptic systems [2].

2. A system of ordinary differential equations

Let $L = (L_{ij})_{i,j=1,\dots,N}$ be a system of ordinary differential operators with constant coefficients defined on $\mathbb{R}_+$ by:

$$L_{ij}u = L_{ij}(D_t)u = \sum_{k=0}^{s_i+t_j} a_k^{ij} D_t^k u, \quad \text{for } i,j = 1, \dots, N.$$

where $a_k^{ij} \in \mathbb{C}$ and u is a function of the variable $t \in \mathbb{R}_+$.

In this section we are interested in the system $Lu(t) = f(t)$, where

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}$$

are vectors of functions defined on $\mathbb{R}_+$. The operator L is bounded from the space $\prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+)$ to $\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+)$. Let L^0_{ij} be the principal part of L_{ij}. We assume that the polynomial

$$P(\tau) = \det(L^0_{ij}(\tau))_{i,j}$$

is not vanishing on the real line $\mathbb{R}$. Let m_+ be the number of the roots of $P(\tau)$ satisfying $\operatorname{Im}\tau > 0$.

Theorem 2.1. *Under the above assumption the operator*

$$L : \prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+) \longrightarrow \prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+)$$

is a Fredholm operator and its index is equal to m_+.

The proof of this theorem is classical see for example [2, 3]. We study L on a neighborhood of 0 and next on a neighborhood of $+\infty$.

3. Some preliminary results

In [10] we established the connection between $\mathcal{L}^{p,\lambda,s}$ spaces, BMO, Campanato spaces $\mathcal{L}^{p,\lambda}$ and their local versions. Also, we showed directly that for $p = 2$ and $\lambda = n$ the space $\mathcal{L}^{2,n,s}$ coincides with the space $F^s_{\infty,2}$ itself equals $I^s(bmo)$, where I^s is the Riesz potential operator. The following theorem shows that the general BMO-Triebel-Lizorkin spaces $F^s_{\infty,p}$ are a particular case of $\mathcal{L}^{p,\lambda,s}$ spaces. The definition of the spaces $F^s_{\infty,p}$ and their homogeneous version $\dot{F}^s_{\infty,p}$ are respectively given in [16, section 2.3.4] and [16, section 5.1.4].

Theorem 3.1. *Let $s \in \mathbb{R}$ and $1 \le p < +\infty$. The space $F^s_{\infty,p}(\mathbb{R}^n)$ [respectively $\dot{F}^s_{\infty,p}(\mathbb{R}^n)$] coincides algebraically and topologically with the space $\mathcal{L}^{p,n,s}(\mathbb{R}^n)$ [respectively $\dot{\mathcal{L}}^{p,n,s}(\mathbb{R}^n)$].*

In particular, for $p = 2$ we have a result in [10].

Proof. The proof is a consequence of some results of [11]. Firstly, we remark that [11, (5.1) and (5.2)] gives the homogeneous part of the theorem. To show the inhomogeneous counterpart, we mention the following equivalent norm for $F^s_{\infty,p}$ space, cf. [11, (12.8)]

$$\|f\|_{F^s_{\infty,p}} \approx \Big\{ \sup_{J\ge 0, B(2^{-J})} \frac{1}{|B|} \int_B \sum_{k\ge J} 2^{kps} |\Delta_k f|^p dx \Big\}^{1/p} \tag{3.1}$$

the supremum is taken over all nonnegative integers J and over all cubes B of $\mathbb{R}^n$ of sidelength 2^{-J}. This equivalence yields the continuous embedding $\mathcal{L}^{p,n,s}(\mathbb{R}^n) \hookrightarrow F^s_{\infty,p}(\mathbb{R}^n)$.

Conversely, let $f \in F^s_{\infty,p}(\mathbb{R}^n)$. Let B a cube of $\mathbb{R}^n$ of sidelength 2^{-J}, with J a negative integer. We divide the cube B into 2^{-nJ} nonoverlapping cubes Q_i of sidelength equal to 1. Thus

$$\begin{aligned}\frac{1}{|B|}\int_B \sum_{k\geq 0=J^+} 2^{kps}|\Delta_k f|^p dx &\leq 2^{nJ}\sum_i \frac{1}{|Q_i|}\int_{Q_i}\sum_{k\geq 0} 2^{kps}|\Delta_k f|^p dx\\ &\leq \sup_Q \frac{1}{|Q|}\int_Q \sum_{k\geq 0} 2^{kps}|\Delta_k f|^p dx\end{aligned}$$

the supremum is taken over all cubes Q of sidelength equal to 1. The last term is obviously bounded from above by the right hand side of (3.1). Hence $f \in \mathcal{L}^{p,n,s}(\mathbb{R}^n)$ and we have the continuous embedding $F^s_{\infty,p}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^{p,n,s}(\mathbb{R}^n)$. The proof of Theorem 3.1 is complete. □

Let us denote by $\mathring{F}^s_{\infty,p}$ (respectively *cmo*) the closure of Schwartz class $\mathcal{S}(\mathbb{R}^n)$ in BMO -Triebel-Lizorkin spaces $F^s_{\infty,p}$ (respectively $bmo = F^0_{\infty,2}$). The following result shed some light on the topological dual of $\mathring{F}^s_{\infty,p}$.

Corollary 3.2. *Let $s \in \mathbb{R}, 1 \leq p < +\infty$, and $1 < p' \leq +\infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$. We have*

$$F^{-s}_{1,1}(\mathbb{R}^n) \hookrightarrow (\mathring{F}^s_{\infty,p}(\mathbb{R}^n))' \hookrightarrow F^{-s-\frac{n}{p}}_{p',p'}(\mathbb{R}^n)$$

in particular

$$F^0_{1,1}(\mathbb{R}^n) \hookrightarrow (cmo)' \hookrightarrow F^{-\frac{n}{2}}_{2,2}(\mathbb{R}^n)\,.$$

We have a similar result for the homogeneous spaces.

The proof of this corollary is a simple consequence of [8, (3.3)] and Theorem 3.1.

Now we recall some lemmas needed in the proof of Theorem 1.5. The following lemma is proved in [8].

Lemma 3.3. *Let $1 \leq p < +\infty$, and $A < 0$. If $(a_{j\nu})_{j,\nu}$ is a sequence of positive real numbers satisfying $(a_{j\nu})_j \in l^p$ for any $\nu \geq 1$, then there is a constant $C > 0$ such that*

$$\sum_{j\geq 1}\sum_{\nu\geq 1} 2^{\nu A} a_{j\nu})^p \leq C \sup_{\nu\geq 1}\sum_{j\geq 1} a^p_{j\nu}$$

holds.

We introduce the following spaces needed in the proof.

Definition 3.4. We denote $W^{t_j,p}(\mathbb{R})$ the classical Sobolev space of all $u \in L^p(\mathbb{R})$ satisfying $D^k_t u \in L^p(\mathbb{R})$ for $1 \leq k \leq t_j$. By $W^{t_j,p}(\mathbb{R};\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))$ we denote the functions u in $L^p(\mathbb{R};\mathcal{L}^{p,\lambda,s+t_j}(\mathbb{R}^{n-1}))$ satisfying $D^k_t u \in L^p(\mathbb{R};\mathcal{L}^{p,\lambda,s+t_j-k}(\mathbb{R}^{n-1}))$ for $k = 1,\dots,t_j$.

Remark 3.5. The most convenient norm of $L^p(\mathbb{R};\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))$ for this purpose is

$$\|u\|_{L^p(\mathbb{R};\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))} = \{\sup_{J,B}\frac{1}{|B|^{\frac{\lambda}{n-1}}}\sum_{k\geq J^+} 2^{ksp}\|\Delta'_k u\|^p_{L^p(\mathbb{R}\times B)}\}^{1/p}$$

where the supremum is taken over all $J \in \mathbb{Z}$ and all balls B of $\mathbb{R}^{n-1}$ with radius 2^{-J}.

Here are some results proved in [9] regarding intermediate derivatives, compactness and interpolation.

Lemma 3.6. *There exists $C > 0$ such that for any $\varepsilon > 0$ and any $u \in \mathcal{L}^{p,\lambda,s+t_j}(\mathbb{R}^n)$ [respectively $W^{t_j,p}(\mathbb{R};\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))$] we get for $k = 0,\dots,t_j-1$,*

$$\|D_t^k u\|_{\mathcal{L}^{p,\lambda,s+t_j-j}(\mathbb{R}^n)} \le C\{\varepsilon\|D_t^{t_j}u\|_{\mathcal{L}^{p,\lambda,s}(\mathbb{R}^n)} + \varepsilon^{-\frac{k}{t_j-k}}\|u\|_{\mathcal{L}^{p,\lambda,s+t_j}(\mathbb{R}^n)}\}$$

[respectively

$$\begin{aligned}&\|D_t^k u\|_{L^p(\mathbb{R};\mathcal{L}^{p,\lambda,s+t_j-j}(\mathbb{R}^{n-1}))}\\ &\le C\big\{\varepsilon\|D_t^{t_j}u\|_{L^p(\mathbb{R};\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))} + \varepsilon^{-\frac{k}{t_j-k}}\|u\|_{L^p(\mathbb{R};\mathcal{L}^{p,\lambda,s+t_j}(\mathbb{R}^{n-1}))}\big\}.]\end{aligned}$$

We have the same result when we replace $\mathcal{L}$ by $\dot{\mathcal{L}}$.

Lemma 3.7. *Let $s_1 \le s_2 < s_3$ be three real numbers, $\lambda \ge 0$ and $1 \le p < +\infty$. There exists a constant $C > 0$ such that for any $\varepsilon > 0$, and $u \in \mathcal{L}^{p,\lambda,s_3}(\mathbb{R}^n)$ [respectively $L^p(\mathbb{R};\mathcal{L}^{p,\lambda,s_3}(\mathbb{R}^{n-1}))$] we get*

$$\|u\|_{\mathcal{L}^{p,\lambda,s_2}(\mathbb{R}^n)} \le C\big\{\varepsilon\|u\|_{\mathcal{L}^{p,\lambda,s_3}(\mathbb{R}^n)} + \varepsilon^{-\frac{s_2-s_1}{s_3-s_2}}\|u\|_{\mathcal{L}^{p,\lambda,s_1}(\mathbb{R}^n)}\big\}$$

[respectively

$$\begin{aligned}&\|u\|_{L^p(\mathbb{R};\mathcal{L}^{p,\lambda,s_2}(\mathbb{R}^{n-1}))}\\ &\le C\big\{\varepsilon\|u\|_{L^p(\mathbb{R};\mathcal{L}^{p,\lambda,s_3}(\mathbb{R}^{n-1}))} + \varepsilon^{-\frac{s_2-s_1}{s_3-s_2}}\|u\|_{L^p(\mathbb{R};\mathcal{L}^{p,\lambda,s_1}(\mathbb{R}^{n-1}))}\big\}.]\end{aligned}$$

We have the same result if we replace $\mathcal{L}$ by $\dot{\mathcal{L}}$.

Lemma 3.8. *Let $\lambda \ge 0$ and $1 \le p < +\infty$. Let m be an integer ≥ 1, and s be a real $< m$. If $u \in L^p(\mathbb{R};\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))$ is such that $D_t^m u \in L^p(\mathbb{R};\mathcal{L}^{p,\lambda,s-m}(\mathbb{R}^{n-1}))$ then $u \in \mathcal{L}^{p,\lambda,s}(\mathbb{R}^n)$, and there is a constant $C > 0$ independent of u such that*

$$\|u\|_{\mathcal{L}^{p,\lambda,s}(\mathbb{R}^n)} \le C\big\{\|D_t^m u\|_{L^p(\mathbb{R};\mathcal{L}^{p,\lambda,s-m}(\mathbb{R}^{n-1}))} + \|u\|_{L^p(\mathbb{R};\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))}\big\}.$$

This statement remains true if we replace $\mathcal{L}$ by $\dot{\mathcal{L}}$ provided $-m < s < m$.

Lemma 3.9. *There exists $C_0 > 0$ such that for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, there exists $C_1 > 0$ satisfying for any $u \in \mathcal{L}^{p,\lambda,s}(\mathbb{R}^n)$ [respectively $L^p(\mathbb{R};\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))$]*

$$\|\varphi u\|_{\mathcal{L}^{p,\lambda,s}(\mathbb{R}^n)} \le C_0\|\varphi\|_{L^\infty(\mathbb{R}^n)}\|u\|_{\mathcal{L}^{p,\lambda,s}(\mathbb{R}^n)} + C_1\|u\|_{\mathcal{L}^{p,\lambda,s-1}(\mathbb{R}^n)}$$

[respectively

$$\begin{aligned}&\|\varphi u\|_{L^p(\mathbb{R};\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))}\\ &\le C_0\|\varphi\|_{L^\infty(\mathbb{R}^n)}\|u\|_{L^p(\mathbb{R};\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))} + C_1\|u\|_{L^p(\mathbb{R};\mathcal{L}^{p,\lambda,s-1}(\mathbb{R}^{n-1}))}.]\end{aligned}$$

This statement remains true if we replace $\mathcal{L}$ by $\dot{\mathcal{L}}$.

The characterization of the traces for elements of the spaces involved in this paper is given in the following theorem proved in [9].

Theorem 3.10. *Let t_j be an integer ≥ 1, $l \in \{0,1,\dots,t_j-1\}$, $s \in \mathbb{R}$, $\lambda \ge 0$ and $1 \le p < +\infty$. For $u \in W_{loc}^{t_j,p}(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))$, the series $\sum_{k\ge 0} D_t^l \Delta_k' u(0,.)$ converges in $\mathcal{S}'(\mathbb{R}^{n-1})$ and define an element $\gamma_l u$ belonging to the space $\mathcal{L}^{p,\lambda,s+t_j-l-\frac{1}{p}}(\mathbb{R}^{n-1})$. Further, the map $u \longmapsto \gamma_l u$ is a continuous operator from $W_{loc}^{t_j,p}(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))$ to $\mathcal{L}^{p,\lambda,s+t_j-l-\frac{1}{p}}(\mathbb{R}^{n-1})$ and there exists an extension operator R_l from the space $\mathcal{L}^{p,\lambda,s+t_j-l-\frac{1}{p}}(\mathbb{R}^{n-1})$ to the space $W^{t_j,p}(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))$ such that*

$$\gamma_l \circ R_l = id_{\mathcal{L}^{p,\lambda,s+t_j-l-\frac{1}{p}}(\mathbb{R}^{n-1})}.$$

In particular, if $s \geq 0$, then the operator γ_l maps $\mathcal{L}^{p,\lambda,s+t_j}(\mathbb{R}^n_+)$ into the space $\mathcal{L}^{p,\lambda,s+t_j-l-\frac{1}{p}}(\mathbb{R}^{n-1})$. We have the same results if we replace $\mathcal{L}$ by $\dot{\mathcal{L}}$.

Remark 3.11. Let $s > 0, \lambda = n$ and $l = 0$. In the case $p = 2$, Strichartz [15] showed that the trace γ_0 of functions in $I^s(BMO)(= \dot{\mathcal{L}}^{2,n,s}(\mathbb{R}^n)$ cf. [10]) must be in the homogeneous Hölder space $C^s(\mathbb{R}^n)$. In addition, he proved that γ_0 is surjective by showing that the extension operator $R_0 f(x) = \mathcal{F}^{-1}(e^{-t^2|\xi|^2}\mathcal{F}f), x = (t, x')$, maps $C^s(\mathbb{R}^n)$ into $I^s(BMO)$. In the case $1 \leq p < +\infty$, Frazier and Jawerth [11, Theorem 11.2] generalized the last result by showing that the space of traces of functions in $\dot{F}^s_{\infty,p}(\mathbb{R}^n)(= \dot{\mathcal{L}}^{p,n,s}(\mathbb{R}^n)$ cf. Theorem 3.1 above) is independent of p and coincides with $C^s(\mathbb{R}^n)$ as well.

4. Proof of Theorem 1.5

The first step in the proof is the following statement.

Proposition 4.1. *Let $s \in \mathbb{R}, \lambda \geq 0$ and $1 \leq p < +\infty$. Under hypotheses (E1) and (E2), for any compact set K of $\overline{\mathbb{R}^n_+}$, there is a constant $C_K > 0$ such that for any $u \in \prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+; \mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))$ with $\operatorname{supp} u \subset K$, we get*

$$
\begin{aligned}
\|u\|_{\prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))} \leq C_K \Big\{ & \|Lu\|_{\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))} \\
& + \|B\gamma u\|_{\prod_{i=1}^{m_+} \mathcal{L}^{p,\lambda,s-\sigma_i-\frac{1}{p}}(\mathbb{R}^{n-1})} \\
& + \|u\|_{\prod_{j=1}^N L^p(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s+t_j-1}(\mathbb{R}^{n-1}))} \Big\}
\end{aligned}
$$

We have the same result if we replace $\mathcal{L}$ by $\dot{\mathcal{L}}$.

Proof. It is classical that with the aid of Lemmas 3.6, 3.7 and 3.9, and with the freezing technique of the coefficients of L, we can restrict ourselves to proving the last proposition for the following homogeneous system of operators with constant coefficients

$$
L^0 = L^0(D_{x'}, D_t) = (L^0_{ij}(0; D_{x'}, D_t))_{i,j=1\dots N} \text{ and } B^0\gamma = B^0(0, D_{x'})\gamma
$$

where

$$
L^0_{ij}(0; D_{x'}, D_t) = \sum_{k+|\alpha'|=s_i+t_j} a^{ij}_\alpha(0) D^{\alpha'}_{x'} D^k_t.
$$

With the aid of Theorem 2.1, we prove as in [13, 3, 9] that under hypotheses (E1) and (E2), for every $\xi' \in \mathbb{R}^{n-1}\backslash\{0\}$, the operator $(L^0(\xi', D_t), B^0(0, \xi')\gamma)$ is invertible from $\prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+)$ to $\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+) \times \mathbb{C}^{m_+}$ and if $K_{\xi'}$ denotes its inverse, then the mapping $\xi' \longmapsto K_{\xi'}$ is C^∞ from $\mathbb{R}^{n-1} \setminus \{0\}$ to $\mathcal{L}(\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+) \times \mathbb{C}^{m_+}; \prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+))$ and for any multi-index α', there exists $C_{\alpha'} > 0$ such that for any ξ', $\frac{1}{2} \leq |\xi'| \leq 2$, and any $(f, g) \in \prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+) \times \mathbb{C}^{m_+}$,

$$
\|D^{\alpha'}_{\xi'} K_{\xi'}(f, g)\|_{\prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+)} \leq C_{\alpha'} \|(f, g)\|_{\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+)\times\mathbb{C}^{m_+}}. \tag{4.1}
$$

First of all, we will prove that for any integer $M \geq 1$ sufficiently large, there exists a constant $C > 0$ such that for any ball B of $\mathbb{R}^{n-1}$ of radius $2^{-J}, J \in \mathbb{Z}$, centered at $x'_0 \in \mathbb{R}^{n-1}$,

$$
\begin{aligned}
&\|u\|_{\prod_{j=1}^N L^p(B;W^{t_j,p}(\mathbb{R}_+))} \\
&\leq C\Big\{\|L^0u\|_{\prod_{i=1}^N L^p(2B;W^{-s_i,p}(\mathbb{R}_+))} + \|B^0\gamma u\|_{\prod_{i=1}^{m_+} L^p(2B)} \\
&\quad + |B|^{1/p}\sum_{\nu\geq -J+1} 2^{-2\nu M}|F_\nu|^{1-\frac{1}{p}}\Big(\|L^0u\|_{\prod_{i=1}^N L^p(F_\nu;W^{-s_i,p}(\mathbb{R}_+))} \\
&\quad + \|B^0\gamma u\|_{\prod_{i=1}^{m_+} L^p(F_\nu)}\Big)\Big\}
\end{aligned}
\tag{4.2}
$$

holds for any $u \in \prod_{j=1}^N \mathcal{S}(\mathbb{R}^{n-1}; W^{t_j,p}(\mathbb{R}_+))$ whose tangential spectrum (i.e. the support of the tangential Fourier transform of u) belongs to the annulus $\frac{1}{2} \leq |\xi'| \leq 2$, here $F_\nu = \{x' \in \mathbb{R}^{n-1};\ 2^\nu \leq |x'-x'_0| \leq 2.2^\nu\}$.

For this, we apply the operator $(L^0(\xi', D_t), B(0,\xi')\gamma)$ to the relation

$$
\mathcal{F}u(.,\xi') = \int e^{-iy'\cdot\xi'}u(.,y')dy'
$$

to obtain the system

$$
\begin{gathered}
L^0(\xi',D_t)\mathcal{F}u(.,\xi') = \mathcal{F}L^0u(.,\xi') = \int e^{-iy'\cdot\xi'}(L^0u)(.,y')dy' \\
B^0(0,\xi')\gamma\mathcal{F}u(.,\xi') = \mathcal{F}B^0\gamma u(\xi') = \int e^{-iy'\cdot\xi'}(B^0\gamma u)(y')dy'
\end{gathered}
$$

Then we apply $K_{\xi'}$ to this system,

$$
\mathcal{F}u(.,\xi') = \int e^{-iy'\cdot\xi'}K_{\xi'}(L^0u(.,y'), B^0\gamma u(y'))dy'
$$

Since $u(.,x') = \int e^{ix'\cdot\xi'}\Phi(\xi')\mathcal{F}u(.,\xi')\frac{d\xi'}{(2\pi)^{n-1}}$, $\Phi \in C_0^\infty(\mathbb{R}^{n-1})$ is equal to 1 for $\frac{1}{2} \leq |\xi'| \leq 2$ and its support belongs to an annulus, we integrate by parts with respect to ξ', then

$$
\begin{aligned}
&u(.,x') \\
&= \iint \frac{e^{i(x'-y')\cdot\xi'}}{(1+|x'-y'|^2)^M}(I-\Delta_{\xi'})^M\{\Phi(\xi')K_{\xi'}(L^0u(.,y'),B^0\gamma u(y'))\}\frac{dy'd\xi'}{(2\pi)^{n-1}}.
\end{aligned}
$$

Inequality (4.1) yields

$$
\begin{aligned}
&\|u(.,x')\|_{\prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+)} \\
&\leq C\int_{\mathbb{R}^{n-1}} \frac{1}{(1+|x'-y'|^2)^M}\|(L^0u(.,y'),B^0\gamma u(y'))\|_{\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+)\times\mathbb{C}^{m_+}}dy'\,.
\end{aligned}
$$

We integrate with respect to $x' \in B$,

$$
\begin{aligned}
&\|u\|_{L^p(B;\prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+))} \\
&\leq C\Big\{\int_{x'\in B}\Big(\int_{y'\in\mathbb{R}^{n-1}}\frac{1}{(1+|x'-y'|^2)^M} \\
&\quad \times \|(L^0u(.,y'),B^0\gamma u(y'))\|_{\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+)\times\mathbb{C}^{m_+}}dy'\Big)^p dx'\Big\}^{1/p}
\end{aligned}
$$

We decompose $\mathbb{R}^{n-1} = (2B) \cup \bigcup_{\nu \geq -J+1} F_\nu$, where $2B = \{y';\ |y' - x_0'| \leq 2.2^{-J}\}$ and $F_\nu = \{y';\ 2^\nu \leq |y' - x_0'| \leq 2.2^\nu,\ \nu \geq -J+1$. Thus

$$\begin{aligned}
&\|u\|_{L^p(B;\prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+))} \\
&\leq C\Big\{ \int_{x'\in B} \Big(\int_{2B} \frac{1}{(1+|x'-y'|^2)^M} \chi_{2B}(y') \\
&\quad \times \|(L^0u(.,y'), B^0\gamma u(y'))\|_{\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+)\times \mathbb{C}^{m_+}} dy' \Big)^p dx' \Big\}^{1/p} \\
&\quad + C\Big\{ \int_{x'\in B} \Big(\sum_{\nu\geq -J+1} \int_{F_\nu} \frac{1}{(1+|x'-y'|^2)^M} \\
&\quad \times \|(L^0u(.,y'), B^0\gamma u(y'))\|_{\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+)\times \mathbb{C}^{m_+}} dy' \Big)^p dx' \Big\}^{1/p}
\end{aligned}$$

The first term of the right hand side of the above inequality is an L^p- norm of a convolution product of a function of $L^1(\mathbb{R}^{n-1})$ (for M large) and a function of $L^p(\mathbb{R}^{n-1})$; on the other hand, for the second term we remark that for $x' \in B$ and $y' \in F_\nu, \nu \geq -J+1$, we have $|x'-y'| \sim |x_0' - y'| \sim 2^\nu$. Hence

$$\begin{aligned}
&\|u\|_{L^p(B;\prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+))} \\
&\leq C\{\|L^0u\|_{L^p(2B;\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+))} + \|B^0\gamma u\|_{\prod_{i=1}^{m_+} L^p(2B)}\} \\
&\quad + C|B|^{1/p} \sum_{\nu\geq -J+1} 2^{-2\nu M} \int_{y'\in F_\nu} \|(L^0u(.,y'), B^0\gamma u(y'))\|_{\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+)\times \mathbb{C}^{m_+}} dy' \\
&\leq C\{\|L^0u\|_{L^p(2B;\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+))} + \|B^0\gamma u\|_{\prod_{i=1}^{m_+} L^p(2B)}\} \\
&\quad + C|B|^{1/p} \sum_{\nu\geq -J+1} 2^{-2\nu M} |F_\nu|^{1-\frac{1}{p}} \\
&\quad \times \Big(\int_{y'\in F_\nu} \|(L^0u(.,y'), B^0\gamma u(y'))\|^p_{\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+)\times \mathbb{C}^{m_+}} dy' \Big)^{1/p}.
\end{aligned}$$

So that inequality (4.2) is proved. Let $u = \begin{pmatrix} u^1 \\ \vdots \\ u^N \end{pmatrix} \in \prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+; \mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))$ with $\operatorname{supp} u \subset K$, K is a compact set of $\overline{\mathbb{R}^n_+}$. For $k \in \mathbb{N}$, we set $u_k(x) = \Delta_k' u(2^{-k}x)$. If $k \geq 1$, then $u_k \in \prod_{j=1}^N \mathcal{S}(\mathbb{R}^{n-1}; W^{t_j,p}(\mathbb{R}_+))$ and its tangential spectrum (i.e. the support of its tangential Fourier transform) belongs to the annulus $\{\frac{1}{2} \leq |\xi'| \leq 2\}$. We have

$$(L^0_{ij}u^j)_k = 2^{k(s_i+t_j)} L^0_{ij} u^j_k \quad \text{and} \quad (B^0_{ij}\gamma u^j)_k = 2^{k(\sigma_i+t_j)} B^0_{ij}\gamma u^j_k\,. \tag{4.3}$$

We apply inequality (4.2) for each $u_k, k \geq 1$,

$$\begin{aligned}
&\|u_k\|_{\prod_{j=1}^N L^p(B;W^{t_j,p}(\mathbb{R}_+))} \\
&\leq C\Big\{ \|L^0u_k\|_{\prod_{i=1}^N L^p(2B;W^{-s_i,p}(\mathbb{R}_+))} + \|B^0\gamma u_k\|_{\prod_{i=1}^{m_+} L^p(2B)} \\
&\quad + |B|^{1/p} \sum_{\nu\geq -J+1} 2^{-2\nu M} |F_\nu|^{1-\frac{1}{p}} \Big(\|L^0u_k\|_{\prod_{i=1}^N L^p(F_\nu;W^{-s_i,p}(\mathbb{R}_+))} \\
&\quad + \|B^0\gamma u_k\|_{\prod_{i=1}^{m_+} L^p(F_\nu)} \Big) \Big\}
\end{aligned} \tag{4.4}$$

The operator Δ'_k commutes with the derivation and then with the constant coefficients operator L^0, so with the aid of (4.3) we have

$$
\begin{aligned}
\|u_k\|_{\prod_{j=1}^N L^p(B;W^{t_j,p}(\mathbb{R}_+))} &= \sum_{j=1}^N \|u_k^j\|_{L^p(B;W^{t_j,p}(\mathbb{R}_+))}\\
&= \sum_{j=1}^N\sum_{r=0}^{t_j} \|D_t^r u_k^j\|_{L^p(B;L^p(\mathbb{R}_+))}\\
&= \sum_{j=1}^N\sum_{r=0}^{t_j} 2^{kn/p}2^{-kr}\|\Delta'_k D_t^r u^j\|_{L^p(\mathbb{R}_+\times 2^{-k}B)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\|L^0 u_k\|_{\prod_{i=1}^N L^p(2B;W^{-s_i,p}(\mathbb{R}_+))}\\
&= \sum_{i=1}^N \|(L^0u_k)^i\|_{L^p(2B;W^{-s_i,p}(\mathbb{R}_+))}\\
&= \sum_{i=1}^N \|\sum_{j=1}^N 2^{-k(s_i+t_j)}(L^0_{ij}u^j)_k\|_{L^p(2B;W^{-s_i,p}(\mathbb{R}_+))}\\
&= \sum_{i=1}^N\sum_{r=0}^{-s_i} 2^{kn/p}\|\sum_{j=1}^N 2^{-k(s_i+t_j+r)}\Delta'_k D_t^r(L^0_{ij}u^j)\|_{L^p(\mathbb{R}_+\times 2^{-k+1}B)}
\end{aligned}
$$

and finally

$$
\begin{aligned}
&\|B^0\gamma u_k\|_{\prod_{i=1}^{m_+} L^p(2B)}\\
&= \sum_{i=1}^{m_+} \|(B^0\gamma u_k)^i\|_{L^p(2B)}\\
&= \sum_{i=1}^{m_+} \|\sum_{j=1}^N 2^{-k(\sigma_i+t_j)}(B^0_{ij}\gamma u^j)_k\|_{L^p(2B)}\\
&= \sum_{i=1}^{m_+} 2^{k(n-1)/p}\|\sum_{j=1}^N 2^{-k(\sigma_i+t_j)}\Delta'_k B^0_{ij}\gamma u^j\|_{L^p(2^{-k+1}B)}
\end{aligned}
$$

Substituting the above equalities in (4.4) gives

$$
\begin{aligned}
&\sum_{j=1}^N\sum_{r=0}^{t_j} 2^{-kr}\|\Delta'_k D_t^r u^j\|_{L^p(\mathbb{R}_+\times 2^{-k}B)}\\
&\le C\Big\{\sum_{i=1}^N\sum_{r=0}^{-s_i}\|\sum_{j=1}^N 2^{-k(s_i+t_j+r)}\Delta'_k D_t^r L^0_{ij}u^j\|_{L^p(\mathbb{R}_+\times 2^{-k+1}B)}\\
&\quad + \sum_{i=1}^{m_+}\|\sum_{j=1}^N 2^{-k(\sigma_i+t_j+1)}\Delta'_k B^0_{ij}\gamma u^j\|_{L^p(2^{-k+1}B)}\\
&\quad + |B|^{1/p}\sum_{\nu\ge -J+1} 2^{-2\nu M}|F_\nu|^{1-\frac1p}\Big[\sum_{i=1}^N\sum_{r=0}^{-s_i}\|\sum_{j=1}^N 2^{-k(s_i+t_j+r)}\Delta'_k D_t^r L^0_{ij}u^j\|_{L^p(\mathbb{R}_+\times 2^{-k}F_\nu)}
\end{aligned}
$$

$$+\sum_{i=1}^{m_+}\|\sum_{j=1}^{N}2^{-k(\sigma_i+t_j+1)}\Delta'_kB^0_{ij}\gamma u^j\|_{L^p(2^{-k}F_\nu)}\Big]\Big\}$$

Now we replace u^j by $2^{k(s+t_j)}u^j$ to get

$$\begin{aligned}
&\sum_{j=1}^{N}\sum_{r=0}^{t_j}2^{k(s+t_j-r)}\|\Delta'_kD^r_tu^j\|_{L^p(\mathbb{R}_+\times 2^{-k}B)}\\
&\le C\Big\{\sum_{i=1}^{N}\sum_{r=0}^{-s_i}2^{k(s-s_i-r)}\|\sum_{j=1}^{N}\Delta'_kD^r_tL^0_{ij}u^j\|_{L^p(\mathbb{R}_+\times 2^{-k+1}B)}\\
&\quad+\sum_{i=1}^{m_+}2^{k(s-\sigma_i-1/p)}\|\sum_{j=1}^{N}\Delta'_kB^0_{ij}\gamma u^j\|_{L^p(2^{-k+1}B)}\\
&\quad+|B|^{1/p}\sum_{\nu\ge -J+1}2^{-2\nu M}|F_\nu|^{1-\frac{1}{p}}\Big[\sum_{i=1}^{N}\sum_{r=0}^{-s_i}2^{k(s-s_i-r)}\|\sum_{j=1}^{N}\Delta'_kD^r_tL^0_{ij}u^j\|_{L^p(\mathbb{R}_+\times 2^{-k}F_\nu)}\\
&\quad+\sum_{i=1}^{m_+}2^{k(s-\sigma_i-1/p)}\|\sum_{j=1}^{N}\Delta'_kB^0_{ij}\gamma u^j\|_{L^p(2^{-k}F_\nu)}\Big]\Big\}
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{j=1}^{N}\sum_{r=0}^{t_j}2^{k(s+t_j-r)}\|\Delta'_kD^r_tu^j\|_{L^p(\mathbb{R}_+\times 2^{-k}B)}\\
&\le C\Big\{\sum_{i=1}^{N}\sum_{r=0}^{-s_i}2^{k(s-s_i-r)}\|\Delta'_kD^r_t(L^0u)^i\|_{L^p(\mathbb{R}_+\times 2^{-k+1}B)}\\
&\quad+\sum_{i=1}^{m_+}2^{k(s-\sigma_i-1/p)}\|\Delta'_k(B^0\gamma u)^i\|_{L^p(2^{-k+1}B)}\\
&\quad+|B|^{1/p}\sum_{\nu\ge -J+1}2^{-2\nu M}|F_\nu|^{1-\frac{1}{p}}\Big[\sum_{i=1}^{N}\sum_{r=0}^{-s_i}2^{k(s-s_i-r)}\|\Delta'_kD^r_t(L^0u)^i\|_{L^p(\mathbb{R}_+\times 2^{-k}F_\nu)}\\
&\quad+\sum_{i=1}^{m_+}2^{k(s-\sigma_i-1/p)}\|\Delta'_k(B^0\gamma u)^i\|_{L^p(2^{-k}F_\nu)}\Big]\Big\}
\end{aligned}$$

Set $K=J+k\in\mathbb{Z}$ and $\mu=\nu-k\in\mathbb{Z}$, then the ball $2^{-k}B$ becomes the ball B_K of $\mathbb{R}^{n-1}$ of radius 2^{-K}, the annulus $2^{-k}F_\nu$ becomes the annulus F_μ of $\mathbb{R}^{n-1}$, and we deduce

$$\begin{aligned}
&\sum_{j=1}^{N}\sum_{r=0}^{t_j}2^{k(s+t_j-r)p}\|\Delta'_kD^r_tu^j\|^p_{L^p(\mathbb{R}_+\times B_K)}\\
&\le C\Big\{\sum_{i=1}^{N}\sum_{r=0}^{-s_i}2^{k(s-s_i-r)p}\|\Delta'_kD^r_t(L^0u)^i\|^p_{L^p(\mathbb{R}_+\times 2B_K)}\\
&+\sum_{i=1}^{m_+}2^{k(s-\sigma_i-1/p)p}\|\Delta'_k(B^0\gamma u)^i\|^p_{L^p(2B_K)}
\end{aligned}$$

$$
\begin{aligned}
&+ |2^k B_K| \Big(\sum_{\mu \geq -K+1} 2^{(\mu+K)(-2M+(n-1)(1-\frac{1}{p}))} 2^{\mu \frac{\lambda}{p}} \\
&\times \Big[\sum_{i=1}^{N} \sum_{r=0}^{-s_i} \frac{2^{k(s-s_i-r)}}{|F_\mu|^{\frac{\lambda}{(n-1)p}}} \|\Delta'_k D_t^r (L^0 u)^i\|_{L^p(\mathbb{R}_+ \times F_\mu)} \\
&+ \sum_{i=1}^{m_+} \frac{2^{k(s-\sigma_i-1/p)}}{|F_\mu|^{\frac{\lambda}{(n-1)p}}} \|\Delta'_k (B^0 \gamma u)^i\|_{L^p(F_\mu)} \Big] \Big)^p \Big\}
\end{aligned}
$$

A simple calculation yields

$$
\begin{aligned}
&\sum_{j=1}^{N} \sum_{r=0}^{t_j} 2^{k(s+t_j-r)p} \|\Delta'_k D_t^r u^j\|^p_{L^p(\mathbb{R}_+ \times B_K)} \\
&\leq C \Big\{ \sum_{i=1}^{N} \sum_{r=0}^{-s_i} 2^{k(s-s_i-r)p} \|\Delta'_k D_t^r (L^0 u)^i\|^p_{L^p(\mathbb{R}_+ \times 2B_K)} \\
&\quad + \sum_{i=1}^{m_+} 2^{k(s-\sigma_i-1/p)p} \|\Delta'_k (B^0 \gamma u)^i\|^p_{L^p(2B_K)} \\
&\quad + 2^{(k-K)(-2N+n-1)p} 2^{-K\lambda} \Big(\sum_{\mu \geq 1} 2^{\mu(-2M+(n-1)(1-\frac{1}{p})+\frac{\lambda}{p})} \\
&\quad \Big[\sum_{i=1}^{N} \sum_{r=0}^{-s_i} \frac{2^{k(s-s_i-r)}}{|F_{\mu-K}|^{\frac{\lambda}{(n-1)p}}} \|\Delta'_k D_t^r (L^0 u)^i\|_{L^p(\mathbb{R}_+ \times F_{\mu-K})} \\
&\quad + \sum_{i=1}^{m_+} \frac{2^{k(s-\sigma_i-1/p)}}{|F_{\mu-K}|^{\frac{\lambda}{(n-1)p}}} \|\Delta'_k (B^0 \gamma u)^i\|_{L^p(F_{\mu-K})} \Big] \Big)^p \Big\}
\end{aligned}
$$

Set $A_M = -2M + (n-1)(1-\frac{1}{p}) + \frac{\lambda}{p}$. Multiply by $1/|B_K|^{\frac{\lambda}{n-1}}$ and sum over j, $k \geq \max(K^+, 1)$,

$$
\begin{aligned}
&\frac{1}{|B_K|^{\frac{\lambda}{n-1}}} \sum_{k \geq \max(K^+,1)} \sum_{j=1}^{N} \sum_{r=0}^{t_j} 2^{k(s+t_j-r)p} \|\Delta'_k D_t^r u^j\|^p_{L^p(\mathbb{R}_+ \times B_K)} \\
&\leq C \Big\{ \frac{1}{|B_K|^{\frac{\lambda}{n-1}}} \sum_{k \geq K^+} \sum_{i=1}^{N} \sum_{r=0}^{-s_i} 2^{k(s-s_i-r)p} \|\Delta'_k D_t^r (L^0 u)^i\|^p_{L^p(\mathbb{R}_+ \times 2B_K)} \\
&\quad + \frac{1}{|B_K|^{\frac{\lambda}{n-1}}} \sum_{k \geq K^+} \sum_{i=1}^{m_+} 2^{k(s-\sigma_i-1/p)p} \|\Delta'_k (B^0 \gamma u)^i\|^p_{L^p(2B_K)} \\
&\quad + \sum_{k \geq K^+} \Big(\sum_{\mu \geq 1} 2^{\mu A_M} \Big[\sum_{i=1}^{N} \sum_{r=0}^{-s_i} \frac{2^{k(s-s_i-r)}}{|F_{\mu-K}|^{\frac{\lambda}{(n-1)p}}} \|\Delta'_k D_t^r (L^0 u)^i\|_{L^p(\mathbb{R}_+ \times F_{\mu-K})} \\
&\quad + \sum_{i=1}^{m_+} \frac{2^{k(s-\sigma_i-1/p)}}{|F_{\mu-K}|^{\frac{\lambda}{(n-1)p}}} \|\Delta'_k (B^0 \gamma u)^i\|_{L^p(F_{\mu-K})} \Big] \Big)^p \Big\}
\end{aligned}
$$

Now we use Lemma 3.3 for the last sum $k \geq K^+$,

$$
\frac{1}{|B_K|^{\frac{\lambda}{n-1}}} \sum_{k \geq \max(K^+,1)} \sum_{j=1}^{N} \sum_{r=0}^{t_j} 2^{k(s+t_j-r)p} \|\Delta'_k D_t^r u^j\|^p_{L^p(\mathbb{R}_+ \times B_K)}
$$

$$
\begin{aligned}
&\le C\Big\{\frac{1}{|B_K|^{\frac{\lambda}{n-1}}}\sum_{k\ge K^+}\sum_{i=1}^{N}\sum_{r=0}^{-s_i}2^{k(s-s_i-r)p}\|\Delta'_k D_t^r(L^0u)^i\|^p_{L^p(\mathbb{R}_+\times 2B_K)}\\
&\quad+\frac{1}{|B_K|^{\frac{\lambda}{n-1}}}\sum_{k\ge K^+}\sum_{i=1}^{m_+}2^{k(s-\sigma_i-1/p)p}\|\Delta'_k(B^0\gamma u)^i\|^p_{L^p(2B_K)}\\
&\quad+\sup_{\mu\ge1}\sum_{k\ge K^+}\Big[\sum_{i=1}^{N}\sum_{r=0}^{-s_i}\frac{2^{k(s-s_i-r)p}}{|F_{\mu-K}|^{\frac{\lambda}{n-1}}}\|\Delta'_k D_t^r(L^0u)^i\|^p_{L^p(\mathbb{R}_+\times F_{\mu-K})}\\
&\quad+\sum_{i=1}^{m_+}\frac{2^{k(s-\sigma_i-1/p)p}}{|F_{\mu-K}|^{\frac{\lambda}{n-1}}}\|\Delta'_k(B^0\gamma u)^i\|^p_{L^p(F_{\mu-K})}\Big]\Big\}
\end{aligned}
$$

On the left hand side of the above inequality we add the terms associated to $k=0$, and since $F_{\mu-K}\subset B_{K-\mu-1}$ we deduce

$$
\begin{aligned}
&\frac{1}{|B_K|^{\frac{\lambda}{n-1}}}\sum_{k\ge K^+}\sum_{j=1}^{N}\sum_{r=0}^{t_j}2^{k(s+t_j-r)p}\|\Delta'_k D_t^r u^j\|^p_{L^p(\mathbb{R}_+\times B_K)}\\
&\le C\Big\{\frac{1}{|B_K|^{\frac{\lambda}{n-1}}}\sum_{k\ge K^+}\sum_{i=1}^{N}\sum_{r=0}^{-s_i}2^{k(s-s_i-r)p}\|\Delta'_k D_t^r(L^0u)^i\|^p_{L^p(\mathbb{R}_+\times 2B_K)}\\
&\quad+\frac{1}{|B_K|^{\frac{\lambda}{n-1}}}\sum_{k\ge K^+}\sum_{i=1}^{m_+}2^{k(s-\sigma_i-1/p)p}\|\Delta'_k(B^0\gamma u)^i\|^p_{L^p(2B_K)}\\
&\quad+\sup_{\mu\ge1}\sum_{k\ge (K-\mu-1)^+}\sum_{i=1}^{N}\Big[\sum_{r=0}^{-s_i}\frac{2^{k(s-s_i-r)p}}{|F_{\mu-K}|^{\frac{\lambda}{n-1}}}\|\Delta'_k D_t^r(L^0u)^i\|^p_{L^p(\mathbb{R}_+\times B_{K-\mu-1})}\\
&\quad+\sum_{i=1}^{m_+}\frac{2^{k(s-\sigma_i-1/p)p}}{|F_{\mu-K}|^{\frac{\lambda}{n-1}}}\|\Delta'_k(B^0\gamma u)^i\|^p_{L^p(B_{K-\mu-1})}\Big]\Big\}+R_0^K,
\end{aligned}
$$

where

$$
R_0^K=\frac{1}{|B_K|^{\frac{\lambda}{n-1}}}\sum_{j=1}^{N}\sum_{r=0}^{t_j}\|\Delta'_0 D_t^r u^j\|^p_{L^p(\mathbb{R}_+\times B_K)}.
$$

Taking the supremum over K and B_K yields

$$
\begin{aligned}
&\sum_{j=1}^{N}\sum_{r=0}^{t_j}\|D_t^r u^j\|^p_{L^p(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s+t_j-r}(\mathbb{R}^{n-1}))}\\
&\le C\Big\{\sum_{i=1}^{N}\sum_{r=0}^{-s_i}\|D_t^r(L^0u)^i\|^p_{L^p(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s-s_i-r}(\mathbb{R}^{n-1}))}\\
&\quad+\sum_{i=1}^{m_+}\|(B^0\gamma u)^i\|^p_{\mathcal{L}^{p,\lambda,s-\sigma_i-1/p}(\mathbb{R}^{n-1})}\Big\}+R_0,
\end{aligned}
$$

where

$$
R_0=\sup_{K,B_K}\frac{1}{|B_K|^{\frac{\lambda}{n-1}}}\sum_{j=1}^{N}\sum_{r=0}^{t_j}\|\Delta'_0 D_t^r u^j\|^p_{L^p(\mathbb{R}_+\times B_K)}
$$

Finally

$$\begin{aligned}\|u\|^p_{\prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))} \le C\Big\{&\|L^0u\|^p_{\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))}\\ &+\|B^0\gamma u\|^p_{\prod_{i=1}^{m_+}\mathcal{L}^{p,\lambda,s-\sigma_i-\frac{1}{p}}(\mathbb{R}^{n-1})}\Big\}+R_0\end{aligned} \tag{4.5}$$

To estimate from above the remainder term R_0, we write

$$R_0^K=\frac{1}{|B_K|^{\frac{\lambda}{n-1}}}\sum_{j=1}^N\sum_{r=0}^{t_j-1}\|\Delta_0' D_t^r u^j\|^p_{L^p(\mathbb{R}_+\times B_K)}+\frac{1}{|B_K|^{\frac{\lambda}{n-1}}}\sum_{j=1}^N\|\Delta_0' D_t^{t_j}u^j\|^p_{L^p(\mathbb{R}_+\times B_K)} \tag{4.6}$$

For the first term in R_0^K, we use Lemma 3.6 to get a constant $C>0$ such that for any $\varepsilon>0$,

$$\begin{aligned}&\frac{1}{|B_K|^{\frac{\lambda}{n-1}}}\sum_{j=1}^N\sum_{r=0}^{t_j-1}\|\Delta_0' D_t^r u^j\|^p_{L^p(\mathbb{R}_+\times B_K)}\\ &\le\sum_{j=1}^N\sum_{r=0}^{t_j-1}\|D_t^r u^j\|^p_{L^p(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s+t_j-r-1}(\mathbb{R}^{n-1}))}\\ &\le C\sum_{j=1}^N\Big\{\varepsilon^p\|D_t^{t_j}u^j\|^p_{L^p(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))}+\sum_{r=0}^{t_j-1}\varepsilon^{-\frac{rp}{t_j-r}}\|u^j\|^p_{L^p(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s+t_j-1}(\mathbb{R}^{n-1}))}\Big\}\\ &\le C\sum_{j=1}^N\Big\{\varepsilon^p\|u^j\|^p_{W^{t_j,p}(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))}+C'_\varepsilon\|u^j\|^p_{L^p(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s+t_j-1}(\mathbb{R}^{n-1}))}\Big\}.\end{aligned} \tag{4.7}$$

To estimate the second term of R_0^K we return to the equation $L^0u=f$. For each $i=1,\dots,N$, $\sum_{j=1}^N L_{ij}^0u^j=f^i$, so $\sum_{j=1}^N\sum_{r+|\alpha'|=s_i+t_j}a^{ij}_{r,\alpha'}(0)D_{x'}^{\alpha'}D_t^ru^j=f^i$, here $f^i=(L^0u)^i$. Thus

$$\sum_{j=1}^N a^{ij}_{s_i+t_j,\alpha'}(0)D_t^{s_i+t_j}u^j=f^i-\sum_{j=1}^N\sum_{\substack{r+|\alpha'|=s_i+t_j\\ 0\le r\le s_i+t_j-1}}a^{ij}_{r,\alpha'}(0)D_{x'}^{\alpha'}D_t^ru^j$$

Applying $D_t^{-s_i}$ to the both sides gives

$$\sum_{j=1}^N a^{ij}_{s_i+t_j,\alpha'}(0)D_t^{t_j}u^j=D_t^{-s_i}f^i-\sum_{j=1}^N\sum_{\substack{r'+|\alpha'|=t_j\\ -s_i\le r'\le t_j-1}}a^{ij}_{r'+s_i,\alpha'}(0)D_{x'}^{\alpha'}D_t^{r'}u^j \tag{4.8}$$

The ellipticity condition gives that the constant matrix $A=(a^{ij}_{s_i+t_j,\alpha'}(0))_{i,j}$ is invertible. Let us denote D_t^Tu the vector $(D_t^{t_j}u^j)_j$, $D_t^{-S}f$ the vector $(D_t^{-s_i}f^i)_i$ and V the vector $(v^i)_i$ where $v^i=\sum_{j=1}^N\sum_{\substack{r'+|\alpha'|=t_j\\ -s_i\le r'\le t_j-1}}a^{ij}_{r'+s_i,\alpha'}(0)D_{x'}^{\alpha'}D_t^{r'}u^j$. From (4.8) we obtain

$$D_t^Tu=A^{-1}D_t^{-S}f-A^{-1}V \tag{4.9}$$

and then

$$\Delta_0' D_t^Tu=A^{-1}\Delta_0' D_t^{-S}f-A^{-1}\Delta_0' V$$

For the second term of R_0^K, we write

$$
\begin{aligned}
&\frac{1}{|B_K|^{\frac{\lambda}{n-1}}}\sum_{j=1}^{N}\|\Delta_0' D_t^{t_j}u^j\|^p_{L^p(\mathbb{R}_+\times B_K)}\\
&= \frac{1}{|B_K|^{\frac{\lambda}{n-1}}}\|\Delta_0' D_t^{T}u\|^p_{\prod_{j=1}^N L^p(\mathbb{R}_+\times B_K)}\\
&\leq C\{\frac{1}{|B_K|^{\frac{\lambda}{n-1}}}\|\Delta_0' D_t^{-S}f\|^p_{\prod_{i=1}^N L^p(\mathbb{R}_+\times B_K)} + \frac{1}{|B_K|^{\frac{\lambda}{n-1}}}\|\Delta_0' V\|^p_{\prod_{j=1}^N L^p(\mathbb{R}_+\times B_K)}\}\\
&\leq C\{\|D_t^{-S}f\|^p_{\prod_{i=1}^N L^p(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))} + \|V\|^p_{\prod_{j=1}^N L^p(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s-1}(\mathbb{R}^{n-1}))}\}\\
&\leq C\{\|f\|^p_{\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))} + \|V\|^p_{\prod_{j=1}^N L^p(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s-1}(\mathbb{R}^{n-1}))}\}
\end{aligned}
\tag{4.10}
$$

Now

$$
\begin{aligned}
\|V\|^p_{\prod_{j=1}^N L^p(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s-1}(\mathbb{R}^{n-1}))} &\leq C\sum_{j=1}^{N}\sum_{\substack{r'+|\alpha'|=t_j\\ 0\leq r'\leq t_j-1}}\|D_{x'}^{\alpha'}D_t^{r'}u^j\|^p_{L^p(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s-1}(\mathbb{R}^{n-1}))}\\
&\leq C\sum_{j=1}^{N}\sum_{0\leq r'\leq t_j-1}\|D_t^{r'}u^j\|^p_{L^p(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s+t_j-r'-1}(\mathbb{R}^{n-1}))}
\end{aligned}
$$

In the same way as for (4.7) there is a constant $C>0$ such that for any $\varepsilon>0$,

$$
\begin{aligned}
&\|V\|^p_{\prod_{j=1}^N L^p(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s-1}(\mathbb{R}^{n-1}))}\\
&\leq C\sum_{j=1}^{N}\Big\{\varepsilon^p\|D_t^{t_j}u^j\|^p_{L^p(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))} + \sum_{r'=0}^{t_j-1}\varepsilon^{-\frac{r'p}{t_j-r}}\|u^j\|^p_{L^p(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s+t_j-1}(\mathbb{R}^{n-1}))}\Big\}\\
&\leq C\sum_{j=1}^{N}\Big\{\varepsilon^p\|u^j\|^p_{W^{t_j,p}(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))} + C'_\varepsilon\|u^j\|^p_{L^p(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s+t_j-1}(\mathbb{R}^{n-1}))}\Big\}.
\end{aligned}
\tag{4.11}
$$

Finally (4.6)–(4.11) give

$$
\begin{aligned}
R_0 \leq C\{&\|f\|^p_{\prod_{i=1}^N W^{-s_i,p}(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))} + \varepsilon^p\|u\|^p_{\prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))}\\
&+ C'_\varepsilon\|u\|^p_{\prod_{j=1}^N L^p(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s+t_j-1}(\mathbb{R}^{n-1}))}\}
\end{aligned}
$$

Substituting the above inequality in (4.5) and choosing $\varepsilon>0$ arbitrarily small we get Proposition 4.1 for the system $(L^0,B^0\gamma)$. $\square$

To complete the proof of Theorem 1.5, we have to estimate the normal derivatives of the solution.

Lemma 4.2. *Let s and λ be two real numbers ≥ 0 and $1\leq p<+\infty$. For any compact set K of $\overline{\mathbb{R}^n_+}$, there exists a constant $C_K>0$ such that for any $u\in\prod_{j=1}^N \mathcal{L}^{p,\lambda,s+t_j}(\mathbb{R}^n_+)$ with* $\operatorname{supp} u\subset K$ *we get*

$$
\|u\|_{\prod_{j=1}^N \mathcal{L}^{p,\lambda,s+t_j}(\mathbb{R}^n_+)} \leq C_K\{\|Lu\|_{\prod_{i=1}^N \mathcal{L}^{p,\lambda,s-s_i}(\mathbb{R}^n_+)} + \|u\|_{\prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))}\}
\tag{4.12}
$$

Proof. As in the proof of Proposition 4.1 we can restrict ourselves to the operator L^0. Firstly let us take $0 \le s < 1$. We have

$$\|u^j\|_{\mathcal{L}^{p,\lambda,s+t_j}(\mathbb{R}^n_+)} = \|D_t^{t_j} u^j\|_{\mathcal{L}^{p,\lambda,s}(\mathbb{R}^n_+)} + \sum_{\substack{k+|\alpha'|\le t_j\\ 0\le k\le t_j-1}} \|D_{x'}^{\alpha'} D_t^k u^j\|_{\mathcal{L}^{p,\lambda,s}(\mathbb{R}^n_+)}$$

The interpolation lemma 3.8 gives for $k+|\alpha'| \le t_j$, $0 \le k \le t_j - 1$,

$$\begin{aligned}
&\|D_{x'}^{\alpha'} D_t^k u^j\|_{\mathcal{L}^{p,\lambda,s}(\mathbb{R}^n_+)}\\
&\le C\Big\{\|D_{x'}^{\alpha'} D_t^k u^j\|_{L^p(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))} + \|D_{x'}^{\alpha'} D_t^{k+1} u^j\|_{L^p(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s-1}(\mathbb{R}^{n-1}))}\Big\}\\
&\le C\Big\{\|D_t^k u^j\|_{L^p(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s+t_j-k}(\mathbb{R}^{n-1}))} + \|D_t^{k+1} u^j\|_{L^p(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s+t_j-k-1}(\mathbb{R}^{n-1}))}\Big\}\\
&\le C\|u^j\|_{W^{t_j,p}(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))}
\end{aligned} \tag{4.13}$$

To estimate $\|D_t^{t_j} u^j\|_{\mathcal{L}^{p,\lambda,s}(\mathbb{R}^n_+)}$, we return again to the equation $L^0 u = f$, (4.9), to get

$$D_t^T u = A^{-1} D_t^{-S} f - A^{-1} V$$

and then with the aid of (4.13),

$$\begin{aligned}
&\|D_t^T u\|_{\prod_{j=1}^N \mathcal{L}^{p,\lambda,s}(\mathbb{R}^n_+)}\\
&= \sum_{j=1}^N \|D_t^{t_j} u^j\|_{\mathcal{L}^{p,\lambda,s}(\mathbb{R}^n_+)}\\
&\le C\Big\{\sum_{i=1}^N \|D_t^{-s_i}(L^0 u)^i\|_{\mathcal{L}^{p,\lambda,s}(\mathbb{R}^n_+)} + \sum_{\substack{k+|\alpha'|=t_j\\ 0\le k\le t_j-1}} \|D_{x'}^{\alpha'} D_t^k u^j\|_{\mathcal{L}^{p,\lambda,s}(\mathbb{R}^n_+)}\Big\}\\
&\le C\Big\{\sum_{i=1}^N \|(L^0 u)^i\|_{\mathcal{L}^{p,\lambda,s-s_i}(\mathbb{R}^n_+)} + \sum_{j=1}^N \|u^j\|_{W^{t_j,p}(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))}\Big\}
\end{aligned} \tag{4.14}$$

The lemma is proved for $0 \le s < 1$. For the general case $s \ge 0$, we write $s = q+r$ with $q \in \mathbb{N}$ and $0 \le r < 1$, and we do an induction on q.This is true for the case $q = 0$. Assuming that the estimation (4.12) is true for any q, we show that it holds for $q+1$. Let K be a compact set of $\overline{\mathbb{R}^n_+}$ and $u \in \prod_{j=1}^N \mathcal{L}^{p,\lambda,s+1+t_j}(\mathbb{R}^n_+)$ with $\operatorname{supp} u \subset K$.

We remark that $u^j \in \mathcal{L}^{p,\lambda,s+1+t_j}(\mathbb{R}^n_+)$ if and only if $u^j \in \mathcal{L}^{p,\lambda,s+t_j}(\mathbb{R}^n_+)$ and $D_{x_k} u^j \in \mathcal{L}^{p,\lambda,s+t_j}(\mathbb{R}^n_+)$ for any $1 \le k \le n-1$, and $D_t^{t_j} u^j \in \mathcal{L}^{p,\lambda,s+1}(\mathbb{R}^n_+)$. Moreover we have

$$\begin{aligned}
&\|u\|_{\prod_{j=1}^N \mathcal{L}^{p,\lambda,s+1+t_j}(\mathbb{R}^n_+)}\\
&\le C\Big\{\|u\|_{\prod_{j=1}^N \mathcal{L}^{p,\lambda,s+t_j}(\mathbb{R}^n_+)}\\
&\quad + \sum_{k=1}^{n-1} \|D_{x_k} u\|_{\prod_{j=1}^N \mathcal{L}^{p,\lambda,s+t_j}(\mathbb{R}^n_+)} + \|D_t^T u\|_{\prod_{j=1}^N \mathcal{L}^{p,\lambda,s+1}(\mathbb{R}^n_+)}\Big\}
\end{aligned} \tag{4.15}$$

where $D_t^T u$ is defined in (4.9). By the induction hypothesis,

$$\|u\|_{\prod_{j=1}^N \mathcal{L}^{p,\lambda,s+t_j}(\mathbb{R}^n_+)} \le C_K\{\|L^0 u\|_{\prod_{i=1}^N \mathcal{L}^{p,\lambda,s-s_i}(\mathbb{R}^n_+)} + \|u\|_{\prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s}(\mathbb{R}^{n-1}))}\} \tag{4.16}$$

We substitute in (4.16) u by $D_{x_k}u$, $1 \le k \le n-1$, and the coefficients of L^0 are constant, so

$$\begin{aligned} &\|D_{x_k}u\|_{\prod_{j=1}^N \mathcal{L}^{p,\lambda,s+t_j}(\mathbb{R}^n_+)} \\ &\le C_K\Big\{\|L^0u\|_{\prod_{i=1}^N \mathcal{L}^{p,\lambda,s-s_i+1}(\mathbb{R}^n_+)} + \|u\|_{\prod_{j=1}^N W^{t_j,p}(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s+1}(\mathbb{R}^{n-1}))}\Big\} \end{aligned} \tag{4.17}$$

In the same way as for (4.14), with $s+1$ instead of s, we return to the equation $L^0u = f$ to get

$$\begin{aligned} &\|D_t^T u\|_{\prod_{j=1}^N \mathcal{L}^{p,\lambda,s+1}(\mathbb{R}^n_+)} \\ &\le C\Big\{\sum_{i=1}^N \|(L^0u)^i\|_{\mathcal{L}^{p,\lambda,s-s_i+1}(\mathbb{R}^n_+)} + \sum_{j=1}^N \|u^j\|_{W^{t_j,p}(\mathbb{R}_+;\mathcal{L}^{p,\lambda,s+1}(\mathbb{R}^{n-1}))}\Big\} \end{aligned} \tag{4.18}$$

Finally we substitute inequalities (4.16), (4.17) and (4.18) in (4.15) to get (4.12) for $s+1 = q+1+r$. $\square$

Note that Theorem 1.5 is a consequence of Proposition 4.1 and Lemma 4.2.

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