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DIRICHLET FORMS FOR GENERAL WENTZELL BOUNDARY CONDITIONS, ANALYTIC SEMIGROUPS, AND COSINE OPERATOR FUNCTIONS

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Dedicated to Angelo Favini with great admiration and friendship on his 60-th birthday

Abstract. The aim of this paper is to study uniformly elliptic operators with general Wentzell boundary conditions in suitable L^p-spaces by using the approach of sesquilinear forms. We use different tools to re-prove analiticity and related results concerning the semigroups generated by the above operators. In addition, we make some complementary observations on, among other things, compactness issues and characterization of domains.

1. Introduction

Favini, G.R. Goldstein, J.A. Goldstein, Romanelli [13] investigated the heat equation in an open bounded domain Ω of $\mathbb{R}^n$ governed by the elliptic operator $A := \nabla\cdot(a\nabla)$ with general Wentzell boundary condition

$$Au + \beta\frac{\partial u}{\partial \nu} + \gamma u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where ν is the outer unit normal, and the boundary $\partial\Omega$ is C^2. Under the assumption that $a = (a_{ij}) = (\alpha\delta_{ij})$ (here δ_{ij} stands for the identity matrix in $\mathbb{R}^n$) is a diagonal $n\times n$ matrix, with real valued coefficient $0 < \alpha \in C^1(\overline{\Omega})$, $\beta, \gamma \in C^1(\partial\Omega)$, with $\beta > 0$, $\gamma \geq 0$ they have considered the realizations $\mathcal{A}_p$ of A in the spaces $\mathcal{X}^p := L^p(\overline{\Omega}, d\mu)$, $1 \leq p < \infty$, where $d\mu := dx|_\Omega \oplus \frac{\alpha d\sigma}{\beta}|_{\partial\Omega}$. The boundary condition can be rewritten as

$$\nabla\cdot(a\nabla u) + \frac{\beta}{\alpha}\big(\alpha\frac{\partial u}{\partial \nu}\big) + \gamma u = 0 \quad \text{on } \partial\Omega,$$

and $\alpha\frac{\partial u}{\partial \nu}$ is the conormal derivative of u with respect to α. We refer to [19] for a derivation of such boundary conditions.

Here, dx is the Lebesgue measure on Ω and $\frac{\alpha d\sigma}{\beta}$ denotes the natural surface measure $d\sigma$ on $\partial\Omega$ with weight $\frac{\alpha}{\beta}$. One of the main results of [13] was that the operator $\overline{\mathcal{A}_p}$, $1 < p < \infty$, generates an analytic, contraction semigroup on $\mathcal{X}^p$, and it is self-adjoint for $p = 2$.

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On the other hand, Favini, G.R. Goldstein, J.A. Goldstein and Romanelli [16] also studied a generalization of the above problem to nonlinear boundary conditions obtaining, in particular, smoothing and other optimal regularity properties of solutions. Once one specializes to consider the linear case, these are in fact optimal ultracontractive results.

Arendt, Metafune, Pallara, Romanelli [5] introduced the sesquilinear form

$$\mathcal{Q}\left(\begin{pmatrix} u \\ u|_{\partial\Omega} \end{pmatrix}, \begin{pmatrix} v \\ v|_{\partial\Omega} \end{pmatrix}\right) := \int_\Omega (a\nabla u)\cdot\overline{\nabla v}dx + \int_{\partial\Omega} \gamma u\overline{v}d\sigma \tag{1.2}$$

on $\mathcal{X}^2$ and showed that the operator associated with such a form is the same $\mathcal{A}_2$ considered above, provided that $\alpha = \beta \equiv 1$ and $\partial\Omega$ is Lipschitz. Thus, they could deduce generation and regularity results (also in $C(\overline{\Omega})$) by properties of $\mathcal{Q}$ (see also [36], in the case of the space $C[0,1]$).

In a similar order of ideas, Vogt and Voigt [35] investigated the form $\mathcal{Q}$ on a slightly different, weighted product Hilbert space that formally covers the case when the coefficient a may vanish. They proved that $\mathcal{Q}$ is Dirichlet even under weaker assumptions on a, β, γ, and $\partial\Omega$. However, the setting in [35] is very general, so that the identification of the operator associated with $\mathcal{Q}$ is not easy to perform.

The approach by operator matrices followed by Engel [9] and by Xiao and Liang [38] has recently revealed that an elliptic operator in non-divergence form with (1.1) generates a cosine operator function (hence an analytic semigroup) in the space $C[0,1]$ (see also [6]).

Likewise, Engel and Fragnelli [10] have shown that a uniformly elliptic second order operator in divergence form with (1.1) generates an analytic semigroup in the space $C(\overline{\Omega})$.

The aim of the present paper is twofold: First we use the theory of Dirichlet sesquilinear forms as presented, e.g., in [7] and [30], in order to deduce results concerning regularity and ultracontractivity of solutions that had already been obtained in [16] in a nonlinear context; second, to present some slight improvements to the known theory. We consider the operator associated with $\mathcal{Q}$ and show that the family of $\mathcal{X}^p$ semigroups, $1 < p < \infty$, to which the semigroup on $\mathcal{X}_2$ extends are analytic with an angle estimate that improves that obtained in [15]. We then extend to the uniformly elliptic case a compactness result obtained in [5], which in turn allows to apply the Perron-Frobenius theory. In addition we obtain some results on uniform convergence to a positive projection onto the subspace of constant functions or to 0, depending on the coefficients on the boundary.

Note that if a and $\partial\Omega$ are smooth enough, then the operator associated with $\mathcal{Q}$ can be described rather explicitly, cf. Theorem 3.12: as it has already been shown in [16], such an operator matrix is in fact the realization of the uniformly elliptic operator $\nabla\cdot(a\nabla)$ with general Wentzell boundary conditions (1.1). Of course, all the properties we show for semigroups and cosine operator functions actually have a counterpart for solutions to suitable first and second order Cauchy problems.

Much more can be said in the special case of a strictly positive coefficient γ on the boundary. In this framework, in a similar order of ideas as in the recent monographs [2] and [30], Section 4 supplies a treatment of many related results as estimates for ultracontractivity (particular case of those obtained in [16] but slightly sharper than those in [5]) and spectral properties.

2. General Framework

Let Ω be a bounded open domain of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$ and $1 \le p \le \infty$. Following [13] (see also [5]), we introduce the product spaces defined by
$$\mathcal{X}^p := L^p(\Omega;\mathbb{C}) \times L^p(\partial\Omega;\mathbb{C}), \quad 1 \le p \le \infty.$$
Observe that $\mathcal{X}^p$ can be identified with the space $L^p(\overline{\Omega}, d\mu)$, equipped with the norm
$$\|\cdot\|_{\mathcal{X}^p} := \Big(\|\cdot\|^p_{L^p(\Omega)} + \|\cdot\|^p_{L^p(\partial\Omega)}\Big)^{1/p} \quad \text{if} \quad 1 \le p < \infty,$$
or else
$$\|\cdot\|_{\mathcal{X}^\infty} := \|\cdot\|_{L^\infty(\Omega)} \vee \|\cdot\|_{L^\infty(\partial\Omega)}.$$
(Here $d\mu := dx|_\Omega \oplus d\sigma|_{\partial\Omega}$, with dx the Lebesgue measure on Ω and $d\sigma$ the natural surface measure on $\partial\Omega$.) Thus, the general theory of Lebesgue spaces yields
$$\mathcal{X}^p \hookrightarrow \mathcal{X}^q \quad \text{for all } 1 \le q \le p \le \infty. \tag{2.1}$$
Moreover, each $\mathcal{X}^p$ is a Banach lattice, and its positive cone is the product of the positive cones of $L^p(\Omega)$ and $L^p(\partial\Omega)$.

Let us consider $\mathcal{X}^2 = L^2(\Omega) \times L^2(\partial\Omega)$. If we equip it with the canonical inner product
$$\langle \mathfrak{u}, \mathfrak{v}\rangle_{\mathcal{X}^2} := \left\langle \begin{pmatrix} u \\ w \end{pmatrix}, \begin{pmatrix} v \\ z \end{pmatrix} \right\rangle_{\mathcal{X}^2} := \langle u, v\rangle_{L^2(\Omega)} + \langle w, z\rangle_{L^2(\partial\Omega)},$$
then $\mathcal{X}^2$ becomes a Hilbert space. We also define the linear subspace
$$\mathcal{V} := \left\{ \begin{pmatrix} u \\ w \end{pmatrix} \in H^1(\Omega) \times H^{1/2}(\partial\Omega) : w = u|_{\partial\Omega} \right\}$$
of $\mathcal{X}^2 = L^2(\Omega) \times L^2(\partial\Omega)$. We emphasize that $\mathcal{V}$ is *not* a product space.

Lemma 2.1. *The linear subspace $\mathcal{V}$ is dense in $\mathcal{X}^2$.*

Proof. It suffices to apply Lemma 5.6, with $X_1 = H^1(\Omega)$, $X_2 = L^2(\Omega)$, $Y_1 = H^{1/2}(\partial\Omega)$, and $Y_2 = L^2(\partial\Omega)$. Then the boundary trace operator L is bounded from $H^1(\Omega)$ onto $H^{1/2}(\partial\Omega)$, cf. [24, Thm. 1.8.3]. The claim follows due to the density of the imbeddings $\ker(L) = H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ and $H^{1/2}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$. $\square$

Remark 2.2. Observe that
$$\langle \mathfrak{u}, \mathfrak{v}\rangle_{\mathcal{V}} = \left\langle \begin{pmatrix} u \\ u|_{\partial\Omega} \end{pmatrix}, \begin{pmatrix} v \\ v|_{\partial\Omega} \end{pmatrix} \right\rangle_{\mathcal{V}} := \langle u, v\rangle_{H^1(\Omega)} + \langle u|_{\partial\Omega}, v|_{\partial\Omega}\rangle_{H^{1/2}(\partial\Omega)}, \tag{2.2}$$
defines an inner product on $\mathcal{V}$. With respect to it, $\mathcal{V}$ becomes a Hilbert space.

Lemma 2.3. *The norm $\|\cdot\|_{\mathcal{V}}$ on the Hilbert space $\mathcal{V}$ defined by* (2.2) *is equivalent to that defined by*
$$|||\mathfrak{u}|||^2 = \left|\left|\left| \begin{pmatrix} u \\ u|_{\partial\Omega} \end{pmatrix} \right|\right|\right|^2 := \|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\partial\Omega)}, \quad \mathfrak{u} \in \mathcal{V}.$$

Proof. Due to the boundedness of the boundary trace operator from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$, it is enough to apply the inequality
$$\|u\|_{L^{\frac{2n}{n-1}}(\Omega)} \le C(\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)}). \tag{2.3}$$
Such an inequality holds for all $u \in H^1(\Omega)$ by [26, Cor. 4.11.2], cf. also [25] for more general results in this context. $\square$

3. Main results

Throughout this paper we impose the following conditions on the coefficients a and γ.

Assumption 3.1.
(1) $a = (a_{ij})$ is a symmetric matrix of real valued $H^1_{\mathrm{loc}}(\Omega)$-functions such that the ellipticity condition
$$c_1|\xi|^2 \le \mathrm{Re} \sum_{i,j=1}^n a_{ij}(x)\xi_i\overline{\xi_j} \le C_1|\xi|^2$$
holds for suitable constants $0 < c_1, C_1$ and all $\xi \in \mathbb{C}^n$, a.e. $x \in \Omega$.
(2) $\gamma \in L^\infty(\partial\Omega)$ is real valued and such that $0 \le \gamma \le C_2$ a.e. for a suitable constant C_2.

We define a sesquilinear form $\mathcal{Q}$ with form domain $D(\mathcal{Q}) := \mathcal{V}$ on the Hilbert space $\mathcal{X}^2$ by
$$\mathcal{Q}(\mathfrak{u}, \mathfrak{v}) = \mathcal{Q}\left(\begin{pmatrix} u \\ u_{|\partial\Omega} \end{pmatrix}, \begin{pmatrix} v \\ v_{|\partial\Omega} \end{pmatrix}\right) := \int_\Omega (a\nabla u)\cdot\overline{\nabla v}dx + \int_{\partial\Omega} \gamma u\overline{v}d\sigma.$$

Our main aim is to show that $\mathcal{Q}$ (which is sesquilinear by definition, and densely defined by Lemma 2.1) is associated to a submarkovian semigroup. This is a consequence of the following.

Theorem 3.2. *The densely defined sesquilinear form $\mathcal{Q}$ is symmetric, positive, and closed, i.e.,*

- $\mathcal{Q}(\mathfrak{u}, \mathfrak{v}) = \overline{\mathcal{Q}(\mathfrak{v}, \mathfrak{u})}$ *for all* $\mathfrak{u}, \mathfrak{v} \in \mathcal{V}$,
- $\mathcal{Q}(\mathfrak{u}, \mathfrak{u}) \ge 0$ *for all* $\mathfrak{u} \in \mathcal{V}$,
- $\mathcal{V}$ *is complete for the form norm*
$$\begin{aligned}\|\mathfrak{u}\|_{\mathcal{Q}} &:= \sqrt{\mathcal{Q}(\mathfrak{u}, \mathfrak{u}) + \|\mathfrak{u}\|^2_{\mathcal{X}^2}} \\ &= \left(\int_\Omega (a\nabla u)\cdot\overline{\nabla u}dx + \int_\Omega |u|^2dx + \int_{\partial\Omega}(1+\gamma)|u|^2d\sigma\right)^{1/2}.\end{aligned}$$

Proof. The fact that the coefficients a and γ are real-valued yields that $\mathcal{Q}$ is symmetric.

To show the positivity of $\mathcal{Q}$, take $\mathfrak{u} = \begin{pmatrix} u \\ u_{|\partial\Omega} \end{pmatrix} \in \mathcal{V}$ and observe that
$$\mathcal{Q}(\mathfrak{u}, \mathfrak{u}) = \int_\Omega (a\nabla u) \cdot \overline{\nabla u}dx + \int_{\partial\Omega} \gamma|u|^2d\sigma \ge 0,$$
due to the positivity of a and γ.

Further, for all $\mathfrak{u} = \begin{pmatrix} u \\ u_{|\partial\Omega} \end{pmatrix} \in \mathcal{V}$ one has by assumption
$$\|\mathfrak{u}\|^2_{\mathcal{Q}} \ge c_1\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\partial\Omega)}$$
and
$$\|\mathfrak{u}\|^2_{\mathcal{Q}} \le C_1\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} + (1 + C_2)\|u\|^2_{L^2(\partial\Omega)}.$$
Taking into account Lemma 2.3, we conclude that the form norm of $\mathcal{Q}$ is equivalent to $\|\cdot\|_{\mathcal{V}}$, with respect to which the space $\mathcal{V}$ is complete. Thus, $\mathcal{Q}$ is closed. □

Remark 3.3. (1) Observe that the form $\mathcal{Q}$ is, in fact, also continuous, i.e.,

$$|\mathcal{Q}(\mathfrak{u},\mathfrak{v})| \leq M\|\mathfrak{u}\|_{\mathcal{V}}\|\mathfrak{v}\|_{\mathcal{V}}$$

for all $\mathfrak{u},\mathfrak{v} \in D(\mathcal{Q})$, where $M := C_1 \vee C_2 < \infty$.

To see this, take into account Remark 2.2 and Lemma 2.3. By the Cauchy–Schwartz inequality and the hypotheses on a one has

$$\Big|\sum_{i,j=1}^n a_{ij}(x)\xi_i\overline{\eta_j}\Big| \leq \Big(\sum_{i,j=1}^n a_{ij}(x)\xi_i\overline{\xi_j}\Big)^{1/2}\Big(\sum_{i,j=1}^n a_{ij}(x)\eta_i\overline{\eta_j}\Big)^{1/2}$$

for all $\xi,\eta \in \mathbb{C}^n$ and a.e. $x \in \Omega$. It follows that

$$\begin{aligned}
|\mathcal{Q}(\mathfrak{u},\mathfrak{v})| &= \left|\int_\Omega (a\nabla u)\cdot\overline{\nabla v}dx + \int_{\partial\Omega}\gamma u\overline{v}d\sigma\right| \\
&\leq \int_\Omega |(a\nabla u)\cdot\overline{\nabla v}|dx + \int_{\partial\Omega}|\gamma u\overline{v}|d\sigma \\
&\leq \int_\Omega \big((a\nabla u)\cdot\overline{\nabla u}\big)^{1/2}\big((a\nabla v)\cdot\overline{\nabla v}\big)^{1/2}dx + \int_{\partial\Omega}|\gamma u\overline{v}|d\sigma \\
&\leq \int_\Omega C_1^{1/2}|\nabla u|C_1^{1/2}|\nabla v|dx + \int_{\partial\Omega}C_2|u||v|d\sigma \\
&= C_1\langle|\nabla u|,|\nabla v|\rangle_{L^2(\Omega)} + C_2\langle|u|,|v|\rangle_{L^2(\partial\Omega)} \\
&\leq C_1\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + C_2\|u\|_{L^2(\partial\Omega)}\|v\|_{L^2(\partial\Omega)} \\
&\leq M\big(\|u\|_{H^1(\Omega)} + \|u\|_{L^2(\partial\Omega)}\big)\big(\|v\|_{H^1(\Omega)} + \|v\|_{L^2(\partial\Omega)}\big) \\
&= M\|\mathfrak{u}\|_{\mathcal{V}}\|\mathfrak{v}\|_{\mathcal{V}}.
\end{aligned}$$

(2) The form $\mathcal{Q}$ is also local, i.e., $\mathcal{Q}(\mathfrak{u},\mathfrak{v}) = 0$ for all $\mathfrak{u},\mathfrak{v} \in \mathcal{V}$ such that $\operatorname{supp}(\mathfrak{u}) \cap \operatorname{supp}(\mathfrak{v})$ is a μ-null set.

Thus, one can associate to $\mathcal{Q}$ an operator $\mathcal{A}_2$ on $\mathcal{X}^2$, given by

$$\begin{aligned}
D(\mathcal{A}_2) &:= \left\{\mathfrak{u} \in \mathcal{V} : \exists\mathfrak{z} \in \mathcal{X}^2 \text{ s.t. } \mathcal{Q}(\mathfrak{u},\mathfrak{v}) = \langle\mathfrak{z},\mathfrak{v}\rangle_{\mathcal{X}^2}\ \forall\mathfrak{v} \in \mathcal{V}\right\}, \\
\mathcal{A}_2\mathfrak{u} &:= -\mathfrak{z}.
\end{aligned}$$

By [7, Thm. 1.2.1], such an operator is self-adjoint and dissipative. In fact, the following holds.

Proposition 3.4. *The operator $\mathcal{A}_2$ associated with $\mathcal{Q}$ generates a cosine operator function with associated phase space $\mathcal{V} \times \mathcal{X}^2$.*

Proof. Due to the bounded perturbation theorem for cosine operator functions (see Lemma 5.3), the claim follows if we show that $\mathcal{A}_2 + I_{\mathcal{X}^2}$ generates a cosine operator function with the same associated phase space. Define now the form

$$\tilde{\mathcal{Q}}(\mathfrak{u},\mathfrak{v}) := \mathcal{Q}(\mathfrak{u},\mathfrak{v}) + \langle\mathfrak{u},\mathfrak{v}\rangle_{\mathcal{X}^2}, \quad \mathfrak{u},\mathfrak{v} \in \mathcal{V}.$$

on $\mathcal{X}^2$. It is apparent that $\tilde{\mathcal{Q}}$ is sesquilinear, densely defined, symmetric, and closed. Moreover, it is bounded below by 1, and the associated operator is exactly $\mathcal{A}_2 + I_{\mathcal{X}^2}$. Thus, the claim follows by [4, Prop. 7.1.3]. □

Remark 3.5. Favini, G.R. Goldstein, J.A. Goldstein, Romanelli [13, Theorem 3.1] showed that if Ω has boundary C^2, $a \in C^1(\overline{\Omega})$, $a > 0$ in Ω, and $\Gamma = \{z \in \partial\Omega :$

$a(z) > 0\} \neq \emptyset$, then the closure of the realization on $L^2(\Omega, dx) \oplus L^2(\Gamma, \frac{a\,d\sigma}{\beta})$ of the operator $A := \nabla \cdot (a\nabla)$ with general Wentzell boundary condition

$$Au + \beta \frac{\partial u}{\partial \nu} + \gamma u = 0 \quad \text{on} \quad \Gamma$$

is self-adjoint and dissipative, hence it generates a cosine operator function. There β, γ were assumed to be in $C^1(\partial\Omega)$, with $\beta > 0$ and $\gamma \geq 0$ on $\partial\Omega$. Recently, Favini, G.R. Goldstein, J.A. Goldstein, Obrecht, and Romanelli [15] have extended this result to the case of a more general elliptic operator of the type

$$A := \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial}{\partial x_j} \right)$$

with general Wentzell boundary condition given by

$$Au + \beta \frac{\partial_a u}{\partial \nu} + \gamma u = 0 \text{ on } \Gamma.$$

Here $\frac{\partial_a u}{\partial \nu}$ denotes the conormal derivative of u with respect to $a = (a_{ij})$.

Since $\mathcal{A}_2$ is self-adjoint and dissipative, it also generates a strongly continuous semigrop $\mathcal{T}_2$ that is contractive and analytic of angle $\frac{\pi}{2}$. In fact, much more can be said about $\mathcal{T}_2$.

Theorem 3.6. *The semigroup $\mathcal{T}_2$ on $\mathcal{X}^2$ associated with $\mathcal{Q}$ is sub-Markovian, i.e., it is real, positive, and contractive on $\mathcal{X}^\infty$.*

Proof. By [30, Prop. 2.5, Thm. 2.7, and Cor. 2.17], we need to check that the following criteria are verified:

- $\mathfrak{u} \in \mathcal{V} \Rightarrow \overline{\mathfrak{u}} \in \mathcal{V}$ and $\mathcal{Q}(\operatorname{Re}\mathfrak{u}, \operatorname{Im}\mathfrak{u}) \in \mathbb{R}$,
- $\mathfrak{u} \in \mathcal{V}$, $\mathfrak{u}$ real-valued $\Rightarrow |\mathfrak{u}| \in \mathcal{V}$ and $\mathcal{Q}(|\mathfrak{u}|, |\mathfrak{u}|) \leq \mathcal{Q}(\mathfrak{u}, \mathfrak{u})$,
- $0 \leq \mathfrak{u} \in \mathcal{V} \Rightarrow 1 \wedge \mathfrak{u} \in \mathcal{V}$ and $\mathcal{Q}(1 \wedge \mathfrak{u}, (\mathfrak{u} - 1)^+) \geq 0$.

It is clear that $\overline{u} \in H^1(\Omega)$ if $u \in H^1(\Omega)$, hence if $\mathfrak{u} = \begin{pmatrix} u \\ u|_{\partial\Omega} \end{pmatrix} \in \mathcal{V}$, then also $\overline{\mathfrak{u}} = \begin{pmatrix} \overline{u} \\ \overline{u}|_{\partial\Omega} \end{pmatrix} \in \mathcal{V}$. Moreover, $\mathcal{Q}(\operatorname{Re}\mathfrak{u}, \operatorname{Im}\mathfrak{u})$ is the sum of two integrals. Since all the integrated functions are real-valued, it follows that $\mathcal{Q}(\operatorname{Re}\mathfrak{u}, \operatorname{Im}\mathfrak{u}) \in \mathbb{R}$.

To check the second condition let $u \in H^1(\Omega)$. Then $|u||_{\partial\Omega} = |u|_{\partial\Omega}|$. Moreover $\nabla|u| = (\operatorname{sign} u)\nabla u$ (see [17, § 7.6]). Hence

$$Q(|u|, |u|) = \int_\Omega (a\nabla u) \cdot \overline{\nabla u}\, dx + \int_{\partial\Omega} \gamma |u|^2\, d\sigma = Q(u, u).$$

Finally, as in the point (d) in the proof of [5, Thm. 2.3] we see that if $0 \leq u \in H^1(\Omega)$, then $1 \wedge u \in H^1(\Omega)$ and $\nabla \cdot (1 \wedge u) = 1_{\{u<1\}} \nabla u$, while $\nabla\left((u-1)^+\right) = 1_{\{u>1\}} \nabla u$, i.e., $\nabla \cdot (1 \wedge u)$ and $\nabla\left((u-1)^+\right)$ are disjointly supported. Further, if $\mathfrak{u} = \begin{pmatrix} u \\ u|_{\partial\Omega} \end{pmatrix} \in \mathcal{V}$, then

$$1 \wedge \mathfrak{u} = \begin{pmatrix} 1 \wedge u \\ (1 \wedge u)|_{\partial\Omega} \end{pmatrix} \in \mathcal{V} \quad \text{and} \quad (\mathfrak{u} - 1)^+ = \begin{pmatrix} (u-1)^+ \\ ((u-1)^+)|_{\partial\Omega} \end{pmatrix}.$$

It follows that if $(x, z) \in \operatorname{supp}(\mathfrak{u} \wedge 1) \cap \operatorname{supp}\left((\mathfrak{u} - 1)^+\right)$, then necessarily $u(z) = 1$, and the claim follows by positivity of the coefficient γ. $\square$

Remark 3.7. Let $\gamma, \tilde{\gamma}$ be functions on Ω and $\partial\Omega$ satisfying the Assumptions 3.1. Denote by $\mathcal{Q}_\gamma$, $\mathcal{Q}_{\tilde{\gamma}}$ the form $\mathcal{Q}$ with coefficients γ and $\tilde{\gamma}$, respectively, and by $\mathcal{T}_\gamma$, $\mathcal{T}_{\tilde{\gamma}}$ the associated sub-Markovian $\mathcal{T}_2$-semigroups. Then $D(\mathcal{Q}_\gamma) = D(\mathcal{Q}_{\tilde{\gamma}}) = \mathcal{V}$. Also, $\mathcal{V}$ is an ideal of itself by [30, Prop. 2.20]. Assume now that $\gamma(z) \le \tilde{\gamma}(z)$ for a.e. $z \in \partial\Omega$. A direct computation shows that

$$\mathcal{Q}_\gamma(\mathfrak{u}, \mathfrak{v}) \le \mathcal{Q}_{\tilde{\gamma}}(\mathfrak{u}, \mathfrak{v})$$

for all $0 \le \mathfrak{u}, \mathfrak{v} \in \mathcal{V}$, and it then follows from [30, Thm. 2.24] that $\mathcal{T}_{\tilde{\gamma}}$ is dominated by $\mathcal{T}_\gamma$ in the sense of positive semigroups, i.e.,

$$|\mathcal{T}_{\tilde{\gamma}}(t)\mathfrak{f}| \le \mathcal{T}_\gamma(t)|\mathfrak{f}| \quad \text{for all } \mathfrak{f} \in \mathcal{X}^2,\ t \ge 0.$$

The more general case of non-positive γ will be treated later on in this section (see Corollaries 3.14 and 3.21).

Lemma 3.8. *The semigroup $\mathcal{T}_2$ on $\mathcal{X}^2$ associated with $\mathcal{Q}$ is ultracontractive, i.e., it satisfies the estimate*

$$\|\mathcal{T}_2(t)\mathfrak{f}\|_{\mathcal{X}^\infty} \le M_\mu t^{-\frac{\mu}{4}} \|\mathfrak{f}\|_{\mathcal{X}^2} \quad \textit{for all } t \in (0,1],\ \mathfrak{f} \in \mathcal{X}^2, \tag{3.1}$$

for

$$\mu \in \begin{cases} [2n-2, \infty), & \textit{if } n \ge 3, \\ (2, \infty), & \textit{if } n = 2, \\ [1, \infty), & \textit{if } n = 1, \end{cases}$$

and some constant M_μ.

Proof. By [7, Cor. 2.4.3] it suffices to show that

$$\|\mathfrak{u}\|^2_{\mathcal{X}^{\frac{2\mu}{\mu-2}}} \le M_\mu \|\mathfrak{u}\|^2_{\mathcal{Q}} \tag{3.2}$$

for some $\mu > 2$ and some constant M_μ.

Take $n \ge 3$ and recall that by the usual Sobolev imbedding theorem we obtain

$$\begin{aligned} \|u\|_{L^{\frac{2\mu}{\mu-2}}(\Omega)} &\le M_1 \|u\|_{L^{\frac{2n}{n-2}}(\Omega)} \\ &\le M_2 \left(\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}, \quad u \in H^1(\Omega), \end{aligned} \tag{3.3}$$

where we have set $\mu = 2n - 2$, cf. [17, (7.30)]. On the other hand, by [29, Theorem 2.4.2] there holds

$$\|u\|_{L^{\frac{2n-2}{n-2}}(\partial\Omega)} \le M_3 \left(\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}, \quad u \in H^1(\Omega);$$

or rather,

$$\|u\|_{L^{\frac{2\mu}{\mu-2}}(\partial\Omega)} \le M_3 \left(\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}, \quad u \in H^1(\Omega). \tag{3.4}$$

Combining (3.3) and (3.4) yields the claimed inequality for $\mu = 2n - 2$, due to the Assumption 3.1.(1). Taking into account (2.1) yields (3.2) for $\mu > 2n - 2$.

If $n \le 2$, then again by [17, (7.30)] and by [29, Theorem 2.4.6] the inequalities (3.3) and (3.4) prevail for arbitrary μ, (3.2) holds again for $\mu > 2$ and the claim follows.

Consider finally the case $n = 1$, $\mu \in [1, 2]$. In this case it is more convenient to use a criterion based on a Nash-type inequality. In fact, by [7, Cor. 2.4.7] it suffices to show that for all $0 \le \mathfrak{u} \in \mathcal{V}$ there holds

$$\|\mathfrak{u}\|_{\mathcal{X}^2} \le M_\mu \|\mathfrak{u}\|_{\mathcal{Q}}^{\frac{\mu}{\mu+2}} \cdot \|\mathfrak{u}\|_{\mathcal{X}^1}^{\frac{2}{\mu+2}}, \tag{3.5}$$

for all $\mu \ge 1$ and some constant M_μ. Recall the inequality

$$\|u\|_{L^{\frac{2}{2-3\tau}}(0,1)} \le M_4 \left(\|u'\|_{L^2(0,1)} + \|u\|_{L^1(0,1)}\right)^\tau \cdot \|u\|_{L^1(0,1)}^{1-\tau}, \tag{3.6}$$

which is valid for all $\tau \in [0, \frac{2}{3}]$ and some constant M_4, cf. [26, Thm. 1.4.8.1]. Take now $\mu \in [1, 2]$ and set $\tau := \frac{\mu}{\mu+2}$, so that $\frac{2}{2-3\tau} = \frac{2\mu+4}{4-\mu} \ge 2$. It follows by (3.6) that

$$\|u\|_{L^2(0,1)} \le M_5 \left(\|u'\|_{L^2(0,1)} + \|u\|_{L^2(0,1)}\right)^{\frac{\mu}{\mu+2}} \cdot \|u\|_{L^1(0,1)}^{\frac{2}{\mu+2}}. \tag{3.7}$$

Finally, observe that in the case $n = 1$ we have $L^p(\partial\Omega) = \mathbb{C}^2$, $1 \le p \le \infty$, so that all the norms on $L^p(\partial\Omega)$ are equivalent. This and (3.7) yield (3.5). $\square$

Remark 3.9. Following Varopoulos ([34, § 0.1], cf. also [2, § 7.3.2]) the number

$$\dim(\mathcal{T}_2) := \inf\{\mu > 0 : (3.2) \text{ is valid for some } M_\mu\}$$

is sometimes called the dimension of the semigroup $\mathcal{T}_2$. Hence, we have shown that

$$\dim(\mathcal{T}_2) \le \begin{cases} 2n-2, & \text{if } n \ge 2, \\ 1, & \text{if } n = 1. \end{cases}$$

Observe that this improves an analogous result in [5], where in the case $n = 1$ it has only been shown that $\dim(\mathcal{T}_2) \le 2$. Moreover, under slightly stronger assumptions on the coefficients a, γ, it was shown in [16] that the dimension of $\mathcal{T}_2$ is always n.

Due to the boundedness of Ω, the following holds by [7, Thm. 1.4.1, Thm. 2.1.4, and Thm. 2.1.5] and [30, Thm. 3.13].

Corollary 3.10. *The semigroup $\mathcal{T}_2$ extends to a family of compact, contractive, real, positive one-parameter semigroups $\mathcal{T}_p$ on $\mathcal{X}^p$, $1 \le p \le \infty$. Such semigroups are strongly continuous if $p \in [1, \infty)$, and analytic of angle $\frac{\pi}{2} - \arctan \frac{|p-2|}{2\sqrt{p-1}}$ for $p \in (1, \infty)$.*

Moreover, the spectrum of $\mathcal{A}_p$ is independent of p, where $\mathcal{A}_p$ denotes the generator of $\mathcal{T}_p$. All the eigenfunctions of $\mathcal{A}_2$ are of class $\mathcal{X}^\infty$.

Remark 3.11. (1) As a direct consequence of the ultracontractivity of $\mathcal{T}_2$, it follows by [7, Lemma 2.1.2] that $\mathcal{T}_2$ has an integral kernel $\mathcal{K}$ such that

$$0 \le \mathcal{K}(t, \mathbf{x}, \mathbf{y}) \le M_\mu^2 t^{-\mu/2} \quad \text{for all } t > 0, \quad \text{a.e. } \mathbf{x}, \mathbf{y} \in \overline{\Omega}, \tag{3.8}$$

where μ and M_μ are the same parameters that appear in (3.1).

(2) Since the operator $\mathcal{A}$ is self-adjoint and dissipative, its spectrum is contained in the negative halfline. Moreover, by the above corollary the spectral bounds of all operators $\mathcal{A}_p$, $p \in [1, \infty)$, agree. Since the growth bound of a positive semigroup on an L^p-space agrees with the spectral bound of its generator (see [37, Thm. 1]), we conclude that $s(\mathcal{A})$ is the common growth bound of all the semigroups $\mathcal{T}_p$, $p \in [1, \infty)$. Observe that, as an application of the abstract spectral theory for so-called one-sided coupled operator matrices developed by K.-J. Engel in [8], such a spectral bound can in several concrete cases be explicitely computed (see [23, § 9] for a one-dimensional example).

We still need to identify the operator associated with $\mathcal{Q}$.

Theorem 3.12. *Let $\partial\Omega \in C^\infty$ and $a_{ij} \in C^\infty(\overline{\Omega})$, $1 \le i,j \le n$. Then the operator $\mathcal{A}_2$ associated with the Dirichlet form $\mathcal{Q}$ is given by*

$$D(\mathcal{A}_2) = \left\{ \begin{pmatrix} u \\ w \end{pmatrix} \in H^{\frac{3}{2}}(\Omega) \times H^1(\partial\Omega) : w = u|_{\partial\Omega} \text{ and } \nabla\cdot(a\nabla u) \in L^2(\Omega) \right\},$$

$$\mathcal{A}_2 = \begin{pmatrix} \nabla\cdot(a\nabla) & 0 \\ -\langle a\nabla, \nu\rangle & -\gamma I \end{pmatrix}.$$

Here $\langle a\nabla u, \nu\rangle$ denotes the conormal derivative of u with respect to a, which is well defined (in the sense of traces) as an element of $L^2(\partial\Omega)$ for $u \in H^{\frac{3}{2}}(\Omega)$ due to the regularity of a, cf. [24, § 2.7].

Proof. Observe first that we only need to prove the claim for $\gamma \equiv 0$, since

$$\begin{pmatrix} \nabla\cdot(a\nabla) & 0 \\ -\langle a\nabla, \nu\rangle & -\gamma I \end{pmatrix} = \begin{pmatrix} \nabla\cdot(a\nabla) & 0 \\ -\langle a\nabla, \nu\rangle & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\gamma I \end{pmatrix},$$

where the second addend on the right-hand side is a bounded operator that does not affect the domain of the first one.

The operator associated with $\mathcal{Q}$ is, by definition,

$$\begin{aligned} D(\mathcal{B}) &:= \left\{ \mathfrak{u} \in \mathcal{V} : \exists \mathfrak{z} \in \mathcal{X}^2 \text{ s.t. } \mathcal{Q}(\mathfrak{u},\mathfrak{v}) = \langle \mathfrak{z}, \mathfrak{v}\rangle_{\mathcal{X}^2} \ \forall \mathfrak{v} \in \mathcal{V} \right\}, \\ \mathcal{B}\mathfrak{u} &:= -\mathfrak{z}. \end{aligned}$$

To see that $\mathcal{A}_2 \subset \mathcal{B}$, observe first that $D(\mathcal{A}_2) \subset \mathcal{V}$. Take $\mathfrak{u} = \begin{pmatrix} u \\ u|_{\partial\Omega} \end{pmatrix}$ and $\mathfrak{v} = \begin{pmatrix} v \\ v|_{\partial\Omega} \end{pmatrix}$ in $\mathcal{V}$ and apply the Gauss–Green formula to obtain

$$\begin{aligned} \mathcal{Q}(\mathfrak{u},\mathfrak{v}) &= \int_\Omega (a\nabla u)\cdot\overline{\nabla v}dx \\ &= -\int_\Omega \nabla\cdot(a\nabla u)\bar{v}dx + \int_{\partial\Omega} \langle a\nabla u, \nu\rangle \bar{v} d\sigma \\ &= \left\langle \begin{pmatrix} -\nabla\cdot(a\nabla u) \\ \langle a\nabla u, \nu\rangle \end{pmatrix}, \begin{pmatrix} v \\ v|_{\partial\Omega} \end{pmatrix} \right\rangle_{\mathcal{X}^2} = \langle -\mathcal{A}_2\mathfrak{u}, \mathfrak{v}\rangle_{\mathcal{X}^2}. \end{aligned}$$

Conversely, let $\mathfrak{u} = \begin{pmatrix} u \\ u|_{\partial\Omega} \end{pmatrix} \in D(\mathcal{B})$ and repeat the above computation to obtain that $\nabla\cdot(a\nabla u)$ is well defined as an element of $L^2(\Omega)$ and that u has a conormal derivative (in the sense of traces) in $L^2(\partial\Omega)$. Thus, by [24, Thm. 2.7.4] we deduce that $u \in H^{\frac{3}{2}}(\Omega)$. Since the second entry of the vector $\mathfrak{u}$ is the trace of a function of class $H^{\frac{3}{2}}(\Omega)$, it follows by usual boundary regularity results that $u_{|\partial\Omega} \in H^1(\partial\Omega)$, and hence $\mathfrak{u} \in D(\mathcal{A}_2)$. □

Remark 3.13. The strong regularity assumption on $\partial\Omega$ and on the coefficient a in Theorem 3.12 is solely necessary in order to apply the results in [24] on the regularity of solutions to a Neumann-type problem.

We are now in the position to discuss the complete second order operators.

Corollary 3.14. *Under the assumptions of Theorem 3.12, let $b \in \big(L^2(\Omega)\big)^n$, $c \in L^2(\Omega)$, and $\tilde\gamma \in L^2(\partial\Omega)$. Then the operator $\tilde{\mathcal{A}}_2$ defined by*
$$\tilde{\mathcal{A}}_2 := \begin{pmatrix} \nabla\cdot(a\nabla) + b\cdot\nabla + cI & 0 \\ -\langle a\nabla, \nu\rangle & -\tilde\gamma I \end{pmatrix}$$
with domain $D(\tilde{\mathcal{A}}_2) := D(\mathcal{A}_2)$ generates a cosine operator function with associated phase space $\mathcal{V} \times \mathcal{X}^2$, hence also an analytic semigroup of angle $\frac{\pi}{2}$ on $\mathcal{X}^2$.

Proof. Write $\tilde{\mathcal{A}}_2$ as
$$\tilde{\mathcal{A}}_2 = \mathcal{A}_2 + \mathcal{B}_2,$$
where
$$\mathcal{B}_2 := \begin{pmatrix} b\cdot\nabla + cI & 0 \\ 0 & (\gamma - \tilde\gamma)I \end{pmatrix}.$$
Since by assumption $b\cdot\nabla + cI$ is bounded from $H^1(\Omega)$ to $L^2(\Omega)$ and $(\gamma - \tilde\gamma)I$ is bounded on $L^2(\partial\Omega)$ it is clear that $\mathcal{B}_2$ is bounded from $\mathcal{V}$ to $\mathcal{X}^2$. Now the claim follows by Lemma 5.3. □

In the introduction we have claimed that the operator matrix associated with the sesquilinear form $\mathcal{Q}$, i.e., $\mathcal{A}_2$, is in fact a realization of a second order elliptic operator in divergence form with general Wentzell boundary conditions. Recalling that $D(\mathcal{A}_2^2)$ is a core for $\mathcal{A}_2$, this is made clear in the following.

Corollary 3.15. *Under the assumptions of Theorem 3.12, let $b \in \big(L^2(\Omega)\big)^n$, $c \in L^2(\Omega)$, and $\tilde\gamma \in L^2(\partial\Omega)$. Then for all u such that $\begin{pmatrix} u \\ u|_{\partial\Omega} \end{pmatrix}$ is in the domain of $\tilde{\mathcal{A}}_2^2$ there holds*
$$\begin{gathered} \nabla\cdot(a(z)\nabla u(z)) + b(z)\cdot\nabla u(z) + c(z)u(z) \\ +\langle a(z)\nabla u(z), \nu(z)\rangle + \tilde\gamma(z)u(z) = 0 \end{gathered} \qquad \text{for all } z \in \partial\Omega. \tag{3.9}$$

This is equivalent to the notion of of weak solution given in [13].

Proof. Take u such that $\begin{pmatrix} u \\ u|_{\partial\Omega} \end{pmatrix} =: \mathfrak{u} \in D(\tilde{\mathcal{A}}_2^2)$. Then by definition
$$\begin{pmatrix} \nabla\cdot(a\nabla u) + b\cdot\nabla u + cu \\ -\langle a\nabla u, \nu\rangle - \tilde\gamma u|_{\partial\Omega} \end{pmatrix} := \begin{pmatrix} \tilde u \\ \tilde w \end{pmatrix} = \tilde{\mathcal{A}}_2\mathfrak{u} \in D(\tilde{\mathcal{A}}_2) = D(\mathcal{A}_2),$$
and by Theorem 3.12 there holds $\tilde u|_{\partial\Omega} = \tilde w$. This yields (3.9). □

Remark 3.16. *Take $u \in L^2(\Omega)$ such that $\begin{pmatrix} u \\ u|_{\partial\Omega} \end{pmatrix}$ is in the domain of $\tilde{\mathcal{A}}_2^2$. By the above corollary u is a function in $H^{3/2}(\Omega)$ such that the homogeneous boundary condition* (3.9) *holds. Such boundary condition is expressed by means of a boundary differential operator of order 2, hence by usual boundary regularity results we obtain that $u \in H^3(\Omega)$. By induction one can in fact see that*
$$C_c^\infty(\overline\Omega) \times C^\infty(\partial\Omega) \subset D(\tilde{\mathcal{A}}_2^\infty) \subset C^\infty(\overline\Omega) \times C^\infty(\partial\Omega).$$

Since the semigroup $\tilde{\mathcal{T}}_2$ generated by $\tilde{\mathcal{A}}_2$ is analytic, its smoothing effect yields
$$\tilde{\mathcal{T}}_2(t)\left(\mathcal{X}^2\right) \subset D(\tilde{\mathcal{A}}_2^\infty) \subset \begin{cases} D(\tilde{\mathcal{A}}_2^2) \\ C^\infty(\overline\Omega) \times C^\infty(\partial\Omega) \end{cases}$$
for all $t > 0$.

Thus, in view of Corollary 3.15 and Remark 3.16 we finally conclude that the initial value problem for the heat equation with general Wentzell boundary conditions is well-posed.

Corollary 3.17. *Under the assumptions of Theorem 3.12, let $b \in \big(L^2(\Omega)\big)^n$, $c \in L^2(\Omega)$, and $\tilde\gamma \in L^2(\partial\Omega)$. Then for all $f \in L^2(\Omega)$ and all $g \in L^2(\partial\Omega)$ the initial-boundary value problem*

$$
\begin{gathered}
\dot u(t,x) = \nabla\cdot(a\nabla u(t,x)) + b(x)\cdot\nabla u(t,x) + c(x)u(t,x), \quad t>0,\ x\in\Omega,\\
\nabla\cdot(a(z)\nabla u(t,z)) + b(z)\cdot\nabla u(t,z) + c(z)u(t,z)\\
+\langle a(z)\nabla u(t,z),\nu(z)\rangle + \tilde\gamma(z)u(t,z) = 0, \quad t>0,\ z\in\partial\Omega,\\
u(0,x) = f(x), \quad x\in\Omega,\\
u(0,z) = g(z), \quad z\in\partial\Omega,
\end{gathered}
$$

admits a unique classical solution u, and $u(t,\cdot)$ is of class C^∞ for all $t>0$.

The well-posedness of the wave equation with general Wentzell boundary conditions follows immediately from the self-adjointness result of [14], as explained in Goldstein's book [21]. We state it now for completeness. A similar result in the one-dimensional case but on all $\mathcal{X}_p$ spaces, $1 \le p < \infty$, has been obtained in [27].

Corollary 3.18. *Under the assumptions of Theorem 3.12, let further $b \in \big(L^2(\Omega)\big)^n$, $c \in L^2(\Omega)$, and $\tilde\gamma \in L^2(\partial\Omega)$. Then for all $f \in H^2(\Omega)$ and $g \in H^1(\Omega)$ the second order initial-boundary value problem with dynamical boundary conditions*

$$
\begin{gathered}
\ddot u(t,x) = \nabla\cdot(a\nabla u(t,x)) + b(x)\cdot\nabla u(x) + c(x)u(x), \quad t\ge 0,\ x\in\Omega,\\
\ddot u(t,z) = -\langle a(z)\nabla u(t,z),\nu(z)\rangle - \tilde\gamma(z)u(t,z), \quad t\ge 0,\ z\in\partial\Omega,\\
u(0,x) = f(x), \quad \dot u(0,x) = g(x), \quad x\in\Omega.
\end{gathered}
$$

admits a unique classical solution u. If further $f,g \in C_c^\infty(\overline\Omega)$, then $u(t,\cdot)$ is of class C^∞ for all $t \ge 0$, and in fact it satisfies the general Wentzell boundary conditions

$$
\begin{gathered}
\nabla\cdot(a(z)\nabla u(t,z)) + b(z)\cdot\nabla u(t,z) + c(z)u(t,z)\\
+\langle a(z)\nabla u(t,z),\nu(z)\rangle + \tilde\gamma(z)u(t,z) = 0, \quad t\ge 0,\ z\in\partial\Omega.
\end{gathered}
\tag{3.10}
$$

Let us finally identify the generators of the semigroups $\mathcal{T}_p$ on $\mathcal{X}^p$. We thus answer a question that was adressed in [18, § 7.6].

Theorem 3.19. *Let $\partial\Omega \in C^\infty$ and $a_{ij} \in C^\infty(\overline\Omega)$, $1 \le i,j \le n$. Then for all $p \in [1,\infty]$ the generator $\mathcal{A}_p$ of the semigroup $\mathcal{T}_p$ is given by*

$$
\begin{aligned}
D(\mathcal{A}_p) = \Big\{ \begin{pmatrix} u \\ w \end{pmatrix} \in W^{2-\frac1p,p}(\Omega) \times W^{\frac32-\frac1p,p}(\partial\Omega) :&\\
w = u|_{\partial\Omega} \text{ and } \nabla\cdot(a\nabla u) \in L^p(\Omega)\Big\}&,\\
\mathcal{A}_p = \begin{pmatrix} \nabla\cdot(a\nabla) & 0 \\ -\langle a\nabla,\nu\rangle & -\gamma I \end{pmatrix}&.
\end{aligned}
$$

Proof. Let us prove the claim for $p>2$. We have already remarked that $\mathcal{X}^p \hookrightarrow \mathcal{X}^q$ for all $1 \le q \le p \le \infty$. Moreover, it follows by the ultracontractivity of $\mathcal{T}_2$ that $\mathcal{X}^p$ is invariant under $\mathcal{T}_2(t)$ for all $p>2$ and $t>0$. Thus, by [11, Prop. II.2.3] the generator of $\mathcal{T}_p$ is the part of $\mathcal{A}_2$ in $\mathcal{X}^p$. The claimed expressions of $\mathcal{A}_p$ and $D(\mathcal{A}_p)$ then follow as consequences of usual results on traces, cf. [1, Thm. 7.53].

Take now some p with $1 \le p < 2$. By [7, Thm. 1.4.1] one has that the adjoint semigroup of $(\mathcal{T}_p(t))_{t\ge 0}$ on $\mathcal{X}_p$ is actually $(\mathcal{T}_q(t))_{t\ge 0}$ on $\mathcal{X}_q$, where $p^{-1}+q^{-1}=1$. Set

$$\mathcal{D}_p = \Big\{ \begin{pmatrix} u \\ w \end{pmatrix} \in W^{2-\frac{1}{p},p}(\Omega) \times W^{\frac{3}{2}-\frac{1}{p},p}(\partial\Omega) : \\ w = u|_{\partial\Omega} \text{ and } \nabla\cdot(a\nabla u) \in L^p(\Omega) \Big\}.$$

Consider the operator $\mathcal{A}_p$ whose action on $\mathcal{D}_p$ is given by

$$\mathcal{A}_p \mathfrak{u} = \begin{pmatrix} \nabla\cdot(a\nabla)u \\ -\langle a\nabla u, \nu\rangle - \gamma u \end{pmatrix}.$$

Reasoning as in the proof of Theorem 3.12 one can see that its adjoint is actually $\mathcal{A}_q$, $p^{-1}+q^{-1}=1$. Then, since the generator of the pre-adjoint semigroup $(\mathcal{T}_p(t))_{t\ge 0}$ on $\mathcal{X}_p$ of $(\mathcal{T}_q(t))_{t\ge 0}$ on $\mathcal{X}_q$ is the pre-adjoint operator $\mathcal{A}_p$ of $\mathcal{A}_q$ it follows that $\mathcal{A}_p$ with domain $D(\mathcal{A}_p) = \mathcal{D}_p$ generates the C_0-semigroup $(\mathcal{T}_p(t))_{t\ge 0}$ on $\mathcal{X}_p$, and the claim follows. □

Remark 3.20. A semigroup T on a Banach lattice X is called *Markovian* if it is real, positive, and $T(t)1 = 1$ for all $t \ge 0$. One thus sees that a semigroup is Markovian if and only if it is real, positive, and its generator A satisfies $A1 = 0$. It follows from Theorem 3.19 that for all $p \in [1,\infty]$, $\mathcal{A}_p 1 = 0$ if and only if $\gamma \equiv 0$.

Corollary 3.21. *Fix $p \in [1,\infty)$. Under the assumptions of Theorem 3.19, let further $b \in (L^\infty(\Omega))^n$, $c \in L^\infty(\Omega)$, and $\tilde\gamma \in L^\infty(\partial\Omega)$. Then the operator $\tilde{\mathcal{A}}_p$ defined by*

$$\tilde{\mathcal{A}}_p := \begin{pmatrix} \nabla\cdot(a\nabla) + b\cdot\nabla + cI & 0 \\ -\langle a\nabla, \nu\rangle & -\tilde\gamma I \end{pmatrix}$$

with domain $D(\tilde{\mathcal{A}}_p) := D(\mathcal{A}_p)$ generates an analytic semigroup on $\mathcal{X}^p$.

Proof. Write $\tilde{\mathcal{A}}_p$ as

$$\tilde{\mathcal{A}}_p = \mathcal{A}_p + \mathcal{B}_p,$$

where

$$\mathcal{B}_p := \begin{pmatrix} b\cdot\nabla + cI & 0 \\ 0 & (\gamma - \tilde\gamma)I \end{pmatrix}.$$

Observe that by assumption $b\cdot\nabla + cI$ and $(\gamma-\tilde\gamma)I$ are compact operators from $W^{1+\epsilon,p}(\Omega)$ to $L^p(\Omega)$ and from $W^{\epsilon,p}(\partial\Omega)$ to $L^p(\partial\Omega)$, respectively, for all $\epsilon > 0$. Hence it is clear that $\mathcal{B}_p$ is a relatively $\mathcal{A}_p$-compact perturbation, and the claim follows by [11, Cor. 2.17]. □

Remark 3.22. Our approach based on positive forms and sub-Markovian semigroups also allows to tackle some nonautonomous Cauchy problems, at least in the one-dimensional case. In fact, we have shown in Corollary 3.10 and Theorem 3.19 that the operator matrix $\mathcal{A}_p$ generates for all $p \in [1,\infty)$ a strongly continuous semigroup of contractions. This essentially follows from the Assumptions 3.1 on the coefficients a and γ. If we allow for more general, time-dependent coefficients, we are led to introduce a family of operators on $\mathcal{X}^p$ defined by

$$\mathcal{A}_p(t) := \begin{pmatrix} \nabla\cdot(a(t)\nabla) & 0 \\ -\langle a(t)\nabla, \nu\rangle & -\gamma(t)I \end{pmatrix}, \quad t \ge s,$$

for fixed $s \in \mathbb{R}$, with joint domain $D(\mathcal{A}_p(t)) := D(\mathcal{A}_p)$. We restrict ourselves to the one-dimensional case and, instead of the Assumptions 3.1, we impose the following.

(1) $a(t)$ is a real valued $C^\infty[0,1]$-function such that $c_1 \le a(t,x) \le C_1$ holds for suitable constants $0 < c_1, C_1$, and all $t \ge s$, $x \in [0,1]$.
(2) $\gamma_1(t), \gamma_2(t)$ are real numbers such that $0 \le \gamma_i(t) \le C_2$ for a suitable constant C_2 and each $t \ge s$, $i = 1,2$.
(3) The mapping $t \mapsto a(t,\cdot)$ is continuously differentiable.
(4) The mappings $t \mapsto \gamma_i(t)$ are continuously differentiable, $i = 1,2$.

We can now consider the temporally inhomogeneous problem

$$\begin{gathered}\dot{u}(t,x) = (au')'(t,x), \quad t \ge s,\ x \in (0,1),\\ \dot{u}(t,j) = (-1)^{j+1} a(t,j) u'(t,j) - \gamma_j(t) u(t,j) = 0, \quad t \ge s,\ j = 1,2,\\ u(s,x) = f(x), \quad x \in [0,1],\end{gathered}$$

and rewrite it in an abstract form as

$$\begin{aligned}\dot{\mathfrak{u}}(t) &= \mathcal{A}_p(t)\mathfrak{u}(t), \quad t \ge s,\\ \mathfrak{u}(s) &= \mathfrak{f} \in \mathcal{X}^p,\end{aligned} \tag{3.11}$$

for some $s \ge 0$. Due to the above assumptions on the coefficients, all the operators $\mathcal{A}_p(t)$, $t \ge 0$, have joint domains and further the family $(\mathcal{A}_p(t))_{t \ge 0}$ is stable in the sense of [32, Def. 4.3.1]. We can thus apply the Kato-Tanabe theory of hyperbolic nonautonomous problems and, by virtue of the results in [32, § 4.4], deduce well-posedness (in a suitable sense) for (nACP). We refer to [22] and [32] for more details on the theory of fundamental solutions to nonautonomous Cauchy problems. Asymptotic issues could also be investigated (e.g., by means of the methods recently surveyed in [31]), but this goes beyond the scope of this paper.

Proposition 3.23. *Let Ω be connected. Then the positive semigroup $\mathcal{T}_p$ on $\mathcal{X}_p$, $p \in [1,\infty)$, is irreducible.*

Proof. Since the irreducibility of $\mathcal{T}_2$ is inherited by all semigroups $\mathcal{T}_p$ on $\mathcal{X}_p$, $1 \le p < \infty$ (a consequence of Corollary 3.10 and [2, Theorem 7.2.2]), we only consider the case $p = 2$ and show that [30, Cor. 2.11] applies. Indeed, by Lemma 3.6 and Remark 3.3, the form $\mathcal{Q}$ is densely defined, positive, continuous, closed, and local.

We have to prove that if U is an open subset of $\Omega \times \partial\Omega$ such that

$$(\chi_U \cdot \mathfrak{f}) \in \mathcal{V} \quad \text{for all } \mathfrak{f} \in \mathcal{V}, \tag{3.12}$$

then either $\mu(U) = 0$ or $\mu\big((\Omega \times \partial\Omega) \setminus U\big) = 0$. Here μ denotes the measure introduced at the beginning of Section 2, i.e., the direct sum of the Lebesgue measure λ on Ω and the Hausdorff measure σ on $\partial\Omega$.

Take then $U = V \times W$ open subset of $\Omega \times \partial\Omega$ such that (3.12) holds. By definition

$$\chi_U = \begin{pmatrix} \chi_V \\ \chi_W \end{pmatrix}. \tag{3.13}$$

Now V is an open subset of Ω and by (3.12) we deduce that

$$(\chi_V \cdot f) \in H^1(\Omega) \quad \text{for all } f \in H^1(\Omega).$$

Recall now that the Laplace operator with Neumann boundary conditions (whose associated form has domain $H^1(\Omega)$) is irreducible, cf. [30, Thm. 4.4.5]. Hence, again by [30, Cor. 2.11] we deduce that $\lambda(V) = 0$ or $\lambda(\Omega \setminus V) = 0$. Since V is an open set, $\lambda(V) = 0$ can only happen if V is empty, so that also $U = \emptyset$ and $\mu(U) = 0$.

Let us now consider the case of $\lambda(\Omega\setminus V)=0$. Observe that the constant function $\mathfrak{f}=1$ belongs to $\mathcal{V}$, so that by (3.12), (3.13), and the definition of $\mathcal{V}$ we deduce that $\chi_V\in H^1(\Omega)$ and χ_W is its trace on $\partial\Omega$. On the other hand, since $\lambda(\Omega\setminus V)=0$ one sees that the characteristic function χ_V is identically 1 a.e. in Ω. Consequently, the trace of χ_V is identically 1 σ-a.e. on $\partial\Omega$. Summing up, χ_W turns out to be identically 1 σ-a.e. on $\partial\Omega$, so that $W=\partial\Omega$ up to a set of σ-measure zero. We conclude that the claim follows since $\mu\big((\Omega\times\partial\Omega)\setminus U\big)=\lambda(\Omega\setminus V)+\sigma(\partial\Omega\setminus W)=0$. $\square$

Due to the irreducibility of the semigroup, we can then apply known results (see, e.g., [2, § 3.5.1]) and draw the following conclusion.

Corollary 3.24. *Let Ω be connected. Then there exists a strictly positive rank-1 projection $\mathcal{P}$ such that*

$$\|e^{-s(\mathcal{A})}\mathcal{T}_p(t)-\mathcal{P}\|\le Me^{-\varepsilon t},\quad t\ge 0,$$

for some constant $M\ge 0$, $\epsilon>0$.

If in particular the coefficient $\gamma\equiv 0$, then we have seen in Remark 3.20 that 0 is the spectral bound of all generators $\mathcal{A}_p$, $p\in[1,\infty)$, and by [28, C-IV.2.10 and C-III.3.5.(d)] we obtain the following.

Corollary 3.25. *Let Ω be connected and $\gamma\equiv 0$. Then for the semigroup $\mathcal{T}_p$ on $\mathcal{X}_p$, $p\in[1,\infty)$ the following assertions hold.*

(1) *The limit $\mathcal{P}\mathfrak{f}:=\lim_{t\to\infty}\mathcal{T}_p(t)\mathfrak{f}$ exists for every $\mathfrak{f}\in\mathcal{X}_p$.*
(2) *$\mathcal{P}$ is a strictly positive projection onto $\ker\mathcal{A}$, the one-dimensional subspace of $\mathcal{X}_p$ spanned by the constant 1 function χ.*
(3) *There exists $M\ge 1$ such that*

$$\|\mathcal{T}_p(t)-\mathcal{P}\|\le Me^{\lambda_2 t},\quad t\ge 0,$$

where λ_2 is the largest nonzero eigenvalue of the generator A.

Again, we stress that the second largest eigenvalue of $\mathcal{A}$ can be explicitly computed in some concrete cases, cf. [23, § 9], thus obtaining an estimate for the semigroup's rate of convergence (in norm!) toward a projection.

More results on the asymptotic behaviour of the semigroups $\mathcal{T}_p$ will be obtained in the next section.

4. The case $\gamma\not\equiv 0$

Throughout this section we impose the following conditions.

Assumption 4.1. *The coefficient γ does not identically vanish on the boundary of each connected component of Ω.*

Under the Assumption 4.1, the properties of the cosine operator function and of the semigroups associated with $\mathcal{Q}$ are essentially improved. The main reason is the following.

Lemma 4.2. *Under the Assumptions 3.1 and 4.1, the operator $\mathcal{A}_p$ is invertible for all $p\in[1,\infty]$.*

Proof. Assume without loss of generality that Ω is connected, and let $\mathfrak{u} = \begin{pmatrix} u \\ u|_{\partial\Omega} \end{pmatrix} \in D(\mathcal{A}_2)$ such that $\mathcal{A}_2\mathfrak{u} = 0$. We obtain by definition of $\mathcal{A}_2$ that $\sigma(\partial\Omega \setminus \partial V) > 0$.

$$\begin{aligned} 0 &= \mathcal{Q}(\mathfrak{u}, \mathfrak{u}) \\ &= \int_\Omega (a\nabla u)\cdot \nabla u dx + \int_{\partial\Omega} \gamma |u|^2 d\sigma \\ &\geq c_1 \int_\Omega |\nabla u|^2 dx + \int_{\partial\Omega} \gamma |u|^2 d\sigma. \end{aligned}$$

It follows by the assumptions on γ that $u \in H^1(\Omega)$ is constant in Ω and vanishes somewhere on $\partial\Omega$. Hence, $\mathfrak{u} \equiv 0$, and 0 is not an eigenvalue of $\mathcal{A}_2$. Since $\mathcal{T}_2$ is compact and analytic, $\mathcal{A}_2$ has compact resolvent. Therefore $\mathcal{A}_2$ is invertible, and the same holds for all $\mathcal{A}_p$, since $\sigma(\mathcal{A}_p) = \sigma(\mathcal{A}_2)$ for all $p \in [1, \infty]$. □

We already know by Remark 3.11.(2) that the growth bound of $\mathcal{T}_p$ is given by the spectral bound $s(\mathcal{A}_2)$. By the above lemma such a spectral bound is strictly negative, and we obtain the following.

Corollary 4.3. *Under the Assumptions 3.1 and 4.1, the semigroup $\mathcal{T}_p$ is uniformly exponentially stable for all $p \in [1, \infty]$.*

Corollary 4.4. *Under the Assumptions 3.1 and 4.1, the cosine operator function generated by $\mathcal{A}_2$ is contractive, together with the associated sine operator function. Moreover, the solutions to the second order abstract Cauchy problem*

$$\begin{aligned} \ddot{\mathfrak{u}}(t) &= \mathcal{A}_2\mathfrak{u}(t), \quad t \in \mathbb{R}, \\ \mathfrak{u}(0) &= \mathfrak{f} \in \mathcal{V}, \\ \dot{\mathfrak{u}}(0) &= \mathfrak{g} \in \mathcal{X}^2, \end{aligned} \tag{4.1}$$

are almost periodic.

Proof. By Theorem 3.2 and Lemma 4.2, the operator $\mathcal{A}_2$ is self-adjoint and strictly negative definite. It follows by [20, Lemma 3.1] that the reduction matrix associated with $\mathcal{A}_2$ generates a group of isometries, hence both the cosine operator function $(\mathcal{C}(t))_{t\in\mathbb{R}}$ generated by $\mathcal{A}_2$ and the associated sine operator function $(\mathcal{S}(t))_{t\in\mathbb{R}}$ are contractive. More precisely, by (5.1) the classical solutions to (4.1) are given by

$$\mathfrak{u}(t) = \mathcal{C}(t)\mathfrak{f} + \mathcal{S}(t)\mathfrak{g}, \quad t \in \mathbb{R}.$$

Moreover, $\mathcal{A}_2$ has compact resolvent by Corollary 3.10, hence by Lemma 5.5 the solutions to (4.1) are almost periodic. □

Even more can be said if we replace the Assumption 4.1 by the following stronger version.

Assumption 4.5. The coefficient γ is strictly positive, i.e., there holds $c_2 \leq \gamma$ $d\sigma$-a.e. for some constant $c_2 > 0$.

We can now sharpen Lemma 3.8 and obtain the following, cf. also [5, Prop. 2.6].

Proposition 4.6. *Under the Assumptions 3.1 and 4.5 the semigroup $\mathcal{T}_2$ on $\mathcal{X}^2$ associated with $\mathcal{Q}$ satisfies the estimate*

$$\|\mathcal{T}_2(t)\mathfrak{f}\|_{\mathcal{X}^\infty} \leq M_\mu t^{-\frac{\mu}{4}} \|\mathfrak{f}\|_{\mathcal{X}^2} \quad \text{for all } t > 0,\ \mathfrak{f} \in \mathcal{X}^2, \tag{4.2}$$

for

$$\mu \in \begin{cases} [2n-2,\infty), & \text{if } n \geq 3,\\ (2,\infty), & \text{if } n = 2,\\ [1,\infty), & \text{if } n = 1, \end{cases}$$

and some constant M_μ*.*

Proof. Take $\mathfrak{u} = \begin{pmatrix} u \\ u|_{\partial\Omega} \end{pmatrix} \in \mathcal{V}$. Observe that plugging (2.3) into (3.3) and (3.4) one obtains

$$\|u\|_{L^{\frac{2\mu}{\mu-2}}(\Omega)} \leq N_1 \left(\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)} \right), \tag{4.3}$$

$$\|u\|_{L^{\frac{2\mu}{\mu-2}}(\partial\Omega)} \leq N_2 \left(\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)} \right) \tag{4.4}$$

for suitable constants N_1, N_2, where $\mu \in [2n-2,\infty)$ if $n \geq 3$, and $\mu \in (2,\infty)$ if $n \leq 2$. On the other hand there holds

$$\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\partial\Omega)} \leq N_3 \mathcal{Q}(\mathfrak{u},\mathfrak{u}),$$

where $N_3 := (c_1 \vee c_2)^{-1}$. The claim then follows by [7, Thm. 2.4.2] for $n \geq 2$, and by [7, Thm. 2.4.6] for $n = 1$. □

Combining the uniform exponential stability and the ultracontractivity of $\mathcal{T}_2$ we finally derive the following $L^2 - L^\infty$ stability estimate.

Corollary 4.7. *Under the Assumptions 3.1 and 4.5 the semigroup* $\mathcal{T}_2$ *on* $\mathcal{X}^2$ *associated with* $\mathcal{Q}$ *satisfies the estimate*

$$\|\mathcal{T}_2(t)\mathfrak{f}\|_{\mathcal{X}^\infty} \leq M_\mu \Big(\frac{1 - ts(\mathcal{A}_2)}{t}\Big)^{\mu/4} e^{ts(\mathcal{A}_2)} \|\mathfrak{f}\|_{\mathcal{X}^2} \quad \text{for all } t > 0,\ \mathfrak{f} \in \mathcal{X}^2,$$

where μ, M_μ *are as in Proposition 4.6.*

Proof. The claim is a direct consequence of Remark 3.11.(2), Proposition 4.6, and [30, Lemma 6.5]. □

We can reformulate the above results aas follows.

Corollary 4.8. *The semigroup* $\mathcal{T}_2$ *satisfies the estimate*

$$\|\mathcal{T}_2(t)\mathfrak{f}\|_{\mathcal{X}^\infty} \leq e^{\kappa(t)} \|\mathfrak{f}\|_{\mathcal{X}^2} \quad \text{for all } t > 0,\ \mathfrak{f} \in \mathcal{X}^2, \tag{4.5}$$

where κ *is a function related to the estimate* (4.2) *and such that*

$$\kappa(\varepsilon) \sim C - \frac{n-1}{2} \log \varepsilon \quad \text{as } \varepsilon \to 0^+ \tag{4.6}$$

for some constant $C > 0$*, if* $n \geq 3$*.*

Proof. By Proposition 4.6, it follows that the estimate (4.5) holds with

$$\kappa(t) := \log M_\mu - \frac{\mu}{4} \log t.$$

for all $\mu \in [2n-2,\infty)$ if $n \geq 3$, $\mu \in (2,\infty)$ if $n = 2$, or $\mu \in [1,\infty)$ if $n = 1$. If in particular $n \geq 3$, then (4.6) holds for some constant $C > 0$. □

Note that this is a special case of the nonlinear result in [16] specialized to the linear case. In [16] it is shown that an estimate analogous to the above one holds with $\|f\|_{\mathcal{X}_2}$ replaced by $\|f\|_{\mathcal{X}_1}$, which is a much stronger result.

A direct computation shows that for μ in the ranges defined above κ is a continuous, monotonically decreasing function on $(0,\infty)$. We thus apply [7, Thm. 2.2.3] and finally derive the logarithmic Sobolev inequality

$$\int_{\overline{\Omega}} \mathfrak{f}^2 \log \mathfrak{f} d\mu \leq \varepsilon \mathcal{Q}(\mathfrak{f},\mathfrak{f}) + \kappa(\varepsilon)\|\mathfrak{f}\|^2_{\mathcal{X}^2} + \|\mathfrak{f}\|^2_{\mathcal{X}^2} \log \|\mathfrak{f}\|_{\mathcal{X}^2}, \tag{4.7}$$

which is valid for all $0 \leq \mathfrak{f} \in \mathcal{V} \cap \mathcal{X}^\infty$ and all $\epsilon > 0$.

Remark 4.9. (1) We have already seen in Remark 3.11.1) that $\mathcal{T}_2$ has a bounded, positive integral kernel. If we assume γ to be strictly positive, we can derive from Corollary 4.7 the alternative estimate

$$\mathcal{K}(t,\mathbf{x},\mathbf{y}) \leq M_\mu^2 \left(\frac{1 - ts(\mathcal{A}_2)}{t}\right)^{\frac{\mu}{2}} e^{2ts(\mathcal{A}_2)} \quad \text{for all } t > 0, \quad \text{a.e. } \mathbf{x},\mathbf{y} \in \overline{\Omega},$$

on the upper bound of the integral kernel, cf. [7, § 2.1]. Here μ and M_μ are the same parameters that appear in Proposition 4.6.

(2) The logarithmic Sobolev inequality (4.7) for κ of the form $\kappa(\varepsilon) = C - \frac{n}{4}\log\varepsilon$ is typical for uniformly elliptic operators with Dirichlet boundary conditions on connected domains of $\mathbb{R}^n$, cf. [7, § 2.3]. Now, our $\mathcal{A}_2$ may be regarded as a differential operator on $\overline{\Omega}$, where Ω is an n-dimensional bounded open domain and $\partial\Omega$ is an $(n-1)$-dimensional manifold. The results of [16] give the best estimate of the form (4.5) near $t = 0$; in fact, they agree with the best estimate for the linear heat equation.

Davies has developed (see [7, § 3.2] and references therein) a method that makes use of such logarithmic Sobolev inequalities for sesquilinear forms and allows to deduce that the associated semigroups admit Gaussian estimates *with respect to a suitable metric*, cf. [7, Thm. 3.2.7]. In view of Remark 4.9.(2), this means in our context that a certain mild form of domination of $\mathcal{T}_2$ by the Gaussian semigroup (in $\mathbb{R}^{2n-2}$, if $n \geq 2$, or in $\mathbb{R}$, if $n = 1$) holds − where, again, the metric we endow $\mathbb{R}^{2n-2}$ or $\mathbb{R}$ with is a suitable one that needs not be equivalent to the Euclidean metric.

We point out that Gaussian estimates are a key argument for discussing several issues, including the p-independence of the angle of analyticity of the semigroups on L^p, L^1-analyticity, and the boundedness of the H^∞-calculus of their generators in L^p. We refer the reader to [30, § 7.1], [2, § 7.4], and references therein.

5. Appendix: A remainder of the theory of cosine operator functions

We summarize a few generalities from the theory of cosine operator functions as presented, e.g. in [12] or [4, § 3.14].

Definition 5.1. *Let X be a Banach space. A strongly continuous function $C : \mathbb{R} \to \mathcal{L}(X)$ is called a* cosine operator function *if it satisfies the D'Alembert functional relations*

$$C(t+s) + C(t-s) = 2C(t)C(s), \quad t,s \in \mathbb{R},$$
$$C(0) = I_X.$$

Further, the operator A on X defined by

$$Ax := \lim_{t\to 0} \frac{2}{t^2}(C(t)x - x), \quad D(A) := \big\{x \in X : \lim_{t\to 0} \frac{2}{t^2}(C(t)x - x) \text{ exists}\big\},$$

is called the generator of $(C(t))_{t\in\mathbb{R}}$. *We define the associated* sine operator function $(S(t))_{t\in\mathbb{R}}$ *by*

$$S(t)x := \int_0^t C(s)x ds, \quad t \in \mathbb{R},\ x \in X.$$

Lemma 5.2. *Let A be a closed operator on a Banach space X. Then the operator A generates a cosine operator function $(C(t))_{t\in\mathbb{R}}$ on X, with associated sine operator function $(S(t))_{t\in\mathbb{R}}$, if and only if there exists a Banach space V, with $[D(A)] \hookrightarrow V \hookrightarrow X$, such that the operator matrix*

$$\mathbf{A} := \begin{pmatrix} 0 & I_V \\ A & 0 \end{pmatrix}, \quad D(\mathbf{A}) := D(A) \times V,$$

generates a C_0-semigroup $(e^{t\mathbf{A}})_{t\geq 0}$ in $V \times X$, and in this case there holds

$$e^{t\mathbf{A}} = \begin{pmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{pmatrix}, \quad t \geq 0. \tag{5.1}$$

If such a space V exists, then it is unique. The (unique) product space $\mathbf{X} = V \times X$ is called phase space associated with $(C(t))_{t\in\mathbb{R}}$ *(or* with A*).*

Lemma 5.3. *Let A generate a cosine operator function with associated phase space $V \times X$. Then also $A+B$ generates a cosine operator function with associated phase space $V \times X$, provided B is an operator that is bounded from V to X.*

Concerning regularity, it is known that cosine operator functions have in general no smoothing effect (see [33, Prop. 4.1]). However, the following can be deduced by (5.1) and the fact that a semigroup leaves invariant the domains of all of its generator's powers.

Lemma 5.4. *Let A generate a cosine operator function $(C(t))_{t\in\mathbb{R}}$ with associated sine operator function $(S(t))_{t\in\mathbb{R}}$. Consider the solution to the second order abstract Cauchy problem*

$$\begin{gathered} \ddot{u}(t) = Au(t), \quad t \in \mathbb{R}, \\ u(0) = f, \quad \dot{u}(0) = g, \end{gathered}$$

which is given by $u(t) = C(t)f + S(t)g$, $t \in \mathbb{R}$. Then $u(t) \in D(A^k)$ for all $t \in \mathbb{R}$, provided that $f, g \in D(A^{2k})$, $k \in \mathbb{N}$.

It is known that cosine and sine operator functions cannot be stable – i.e., one cannot expect the decay of the norm of a solution to a second order abstract Cauchy problem. Hence, one is usually interested in boundedness and almost periodicity of such solutions. The following results are due to Arendt and Batty, cf. [3, Cor. 5.6].

Lemma 5.5. *Let A generate a bounded cosine operator function $(C(t))_{t\in\mathbb{R}}$ with associated sine operator function $(S(t))_{t\in\mathbb{R}}$ on a Banach space X. If A has compact resolvent, then $(C(t))_{t\in\mathbb{R}}$ is almost periodic. If further A is invertible, then also $(S(t))_{t\in\mathbb{R}}$ is almost periodic.*

A technical lemma.

Lemma 5.6. *Let X_1, X_2, Y_1, Y_2 be Banach spaces, such that $X_1 \subset X_2$, and Y_1 is dense in Y_2. Consider a surjective operator $L \in \mathcal{L}(X_1, Y_1)$ such that $\ker(L)$ is dense in X_2. Then*

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in X_1 \times Y_1 : Lx = y \right\}$$

is dense in $X_2 \times Y_2$.

Proof. Let $x \in X_2$, $y \in Y_2$, $\epsilon > 0$. Take $z \in Y_1$ such that $\|y - z\|_{Y_2} < \epsilon$. The surjectivity of L ensures that there exists $u \in X_1$ such that $Lu = z$. Take $\tilde{u}, \tilde{x} \in \ker(L)$ such that $\|u - \tilde{u}\|_{X_2} < \epsilon$ and $\|x - \tilde{x}\|_{X_2} < \epsilon$. Let $w := \tilde{x} + u - \tilde{u} \in X_1$. Then

$$\begin{aligned} &\| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} w \\ z \end{pmatrix} \|_{X_2 \times Y_2} \\ &\leq \| \begin{pmatrix} x - \tilde{x} \\ 0 \end{pmatrix} \|_{X_2 \times Y_2} + \| \begin{pmatrix} u - \tilde{u} \\ 0 \end{pmatrix} \|_{X_2 \times Y_2} + \| \begin{pmatrix} 0 \\ y - z \end{pmatrix} \|_{X_2 \times Y_2} < 3\epsilon. \end{aligned}$$

Since $L(w) = L(u) = z$, we obtain $\begin{pmatrix} w \\ z \end{pmatrix} \in X_1 \times Y_1$. □

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