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A STABILITY THEOREM FOR CONVERGENCE OF A LYAPOUNOV FUNCTION ALONG TRAJECTORIES OF NONEXPANSIVE SEMIGROUPS

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Abstract. It is known that a regularly Lyapounov function for a semigroup of contractions on a Hilbert space converges to its minimum along the trajectories of the semigroup. In this paper we show that this Lyapounov function nearly converges to its minimum along trajectories of the semigroup generated by a small bounded perturbation of the semigroup generator.

1. Introduction

Let K be a closed convex subset of a real Hilbert space H and let $\{S(t)\}_{t\geq 0}$ be a semigroup of contractions on K generated by a maximal monotone operator A on H. The study of convergence of a trajectory $S(t)x$ as $t \to \infty$ has called the attention of several mathematicians; see, for example, [4, 5, 6, 7, 8, 10, 11, 13, 14, 15]. In general $S(t)x$ does not converge strongly or even weakly as $t \to \infty$, and convergence requires additional conditions.

The fact that $S(t)x$ does not converge even weakly, in general, as $t \to \infty$, together with the use of Lyapounov functions to determine the asymptotic behaviour of semigroups, inspired us to consider a Lyapounov function $f : H \to \mathbb{R} \cup \{\infty\}$ and then study convergence of $f(S(t)x)$ as $t \to \infty$.

In [9], we show that, in general in infinite dimensional space, $f(S(t)x) \not\to \min(f)$ as $t \to \infty$, even if f strictly decreases along the trajectories of $\{S(t)\}_{t\geq 0}$. If for a semigroup $\{S(t)\}_{t\geq 0}$, f decreases along the trajectories at a particular rate, then we call f regularly Lyapounov for the semigroup $\{S(t)\}_{t\geq 0}$, and we have $f(S(t)x) \to \min(f)$ as $t \to \infty$. Further, under some mild conditions on A and f, we construct a complete metric space $(\mathcal{A}^1, d)$ of the bounded perturbations of the generator A such that f is Lyapounov for all the semigroups generated by these perturbations. We show that there is a very large subset $\mathcal{F}^1$ of $\mathcal{A}^1$ such that f is regularly Lyapounov for all the semigroups generated by the maximal monotone operators in $\mathcal{F}^1$. In particular, the Lyapounov function f for $\{S(t)\}_{t\geq 0}$, generated by A, converges along the trajectories of a class of semigroups generated by most bounded perturbations of A in $\mathcal{A}^1$.

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Since a regularly Lyapounov function f for a nonexpansive semigroup on a Hilbert space converges to its minimum along the trajectories of the semigroup a natural question arises about the behaviour of f along the trajectories of the semigroup $\{S^1(t)\}_{t\geq 0}$ generated by a small bounded perturbation of the semigroup generator, not assumed to be in $\mathcal{A}^1$. So the main question becomes: is $f(S^1(t)x) - \min(f)$ small for large t if the perturbation is small? Theorem 1 and 2 of this paper give an answer to this question. The idea came from the work of Reich and Zaslavski; in [18] they investigated a similar problem for a continuous convex function f on a Banach space along the trajectories given by an everywhere defined bounded vector field on a Banach space. For more recent results on continuous descent methods see [1, 2, 3]. In [16, 17, 18] one considers the semigroup generators to be bounded vector fields, whereas here and in [9] we consider them to be maximal monotone operators. Hence we are not in a position to use [16, 17, 18] as their suppositions do not hold. On the other hand, you might hope that this paper would generalise [18]. However, it does not because in [18] one considers single trajectories and does not assume the semigroup to be nonexpansive or even Lipschitz continuous. Moreover, because of the properties of maximal monotone operators in Hilbert space, one needs less stringent conditions on f to obtain a positive result.

Reich and Zaslavski, in [17, 18], considered a bounded below, convex, continuous function f on a Banach space X. With f, they associated a complete metric space of bounded vector fields $V : X \to X$ such that the right hand directional derivative of f at x in the direction of Vx is non-positive. In [16] and [17], they considered two gradient like iterative processes defined by these vector fields. Under some assumptions on V and f, they showed that for most of the vector fields in this complete metric space, both iterative processes generate a sequence $\langle x_n\rangle_{n=1}^{\infty}$ such that $f(x_n) \to \inf(f)$ as $n \to \infty$. In [18], they gave the continuous version of the same problem. They showed that if there are trajectories given by these vector fields then f converges to its infimum along these trajectories for most vector fields. They defined the notion of regular vector fields and showed that the function f converges to its infimum along the sequences/trajectories given by these regular vector fields. Under the assumptions that f is coercive and Lipschitz on bounded subsets of the Banach space, they showed that for regular vector fields the function f remains close to its infimum along finite horizon perturbed trajectories.

2. Preliminaries and Notation

Throughout this paper H stands for a real Hilbert space. Let A be a maximal monotone operator on H such that $A^{-1}\{0\} \neq \emptyset$ and let $\{S(t)\}_{t\geq 0}$ be the semigroup of contractions generated by A on $K = \overline{D(A)}$. Usually $-A$ is called the generator of $\{S(t)\}_{t\geq 0}$ but we find it more convenient to say A is the generator of $\{S(t)\}_{t\geq 0}$ in the same sense as Pazy [12]. Since in a Hilbert space there is one to one correspondence between the maximal monotone operators and the semigroups of contractions [12] we will switch frequently between semigroups and the maximal monotone operators generating them.

Definition 2.1. Suppose A is a maximal monotone operator and $\{S(t)\}_{t\geq 0}$ the semigroup it generates. Assume $A^{-1}\{0\}$ is nonempty. Let f be a proper l.s.c. function from H to $\mathbb{R} \cup \{+\infty\}$ such that $K = \overline{D(A)} \subseteq \operatorname{Dom} f)$ and suppose there

exists $x_0 \in A^{-1}\{0\}$ such that $f(x_0) = \min(f) := \min\{f(x) : x \in H\}$. We say f is Lyapounov for $\{S(t)\}_{t\geq 0}$ if
$$f(S(t)x) \leq f(x) \quad \forall\, x \in K, \quad t \geq 0,$$
and strictly Lyapounov if
$$f(S(t)x) < f(x) \quad \forall\, x \in D(A) \setminus A^{-1}\{0\}, \quad t > 0.$$

Definition 2.2. A Lyapounov function f for a semigroup $\{S(t)\}_{t\geq 0}$ is called regularly Lyapounov for $\{S(t)\}_{t\geq 0}$ if for each positive integer n there exists a positive number $\delta(n)$ (depending on n) such that for every x in D_n, where $D_n = \{x \in D(A) : \|x\| \leq n,\ f(x) > \min(f) + \frac{1}{n}\}$, there exists $\alpha(x) > 0$ such that
$$f(x) - f(S(t)x) \geq t\delta(n) \quad \forall\, t \in [0, \alpha(x))\,.$$

The idea of regularity that we use was essentially already given in [18, page 4], and it had previously been given in [17, page 1005], in the study of discrete descent methods. In [17, 18], they considered a bounded below, convex, continuous function f on a Banach space X, and with f they associated a bounded vector field $V : X \to X$. By a regular vector field they mean, for any natural number n there exists a positive number $\delta(n)$ such that for each $x \in X$ satisfying $\|x\| \leq n$ and $f(x) \geq \inf(f) + \frac{1}{n}$,
$$f^\circ(x, Vx) \leq -\delta(n), \tag{2.1}$$
where $f^\circ(x, Vx)$ denotes the right hand directional derivative of f at x in the direction of Vx. That means if there are trajectories governed by a regular vector field then the function f decreases along these trajectories at a particular rate. In Definition 2, the function f need not be convex and continuous and we are considering the decrease of f along trajectories of a semigroup generated by a maximal monotone operator. In [9, Proposition 4] we assumed f to be convex and continuous on $D(A)$ and obtained an analogue of (2.1) from Definition 2.2.

Through out this paper, for any real $r > 0$ by D_r we mean $D_r = \{x \in D(A) : \|x\| \leq r, f(x) > \min(f) + \frac{1}{r}\}$. Set $D_{\epsilon^{-1}}(x_0) = \{x \in D(A) : \|x - x_0\| < \epsilon^{-1}, f(x) > \min(f) + \frac{\epsilon}{2}\}$.

3. Stability Theorems

We begin with a simple geometrical result that for a small perturbation of a closed convex set the elements of minimal norm in perturbed and unperturbed sets are not very far from each other.

Lemma 3.1. *Let K be a closed convex subset of H and e be a given vector in H. Let $K' = K + e$. Let y° and $(y')^\circ$ be the elements of minimal norm in K and K' respectively. Then*
$$\|(y')^\circ - y^\circ\| \leq 2\|e\|\,.$$

Proof. Let $(y')^\circ = y + e$, for some y in K. We note that the nearest point projection P_K of origin and $-e$ in K are y° and y. Since P_K is a contraction
$$\|y^\circ - y\| = \|P_K(0) - P_K(-e)\| \leq \|e\|\,.$$
Hence
$$\|(y')^\circ - y^\circ\| = \|y + e - y^\circ\| \leq \|y - y^\circ\| + \|e\| \leq 2\|e\|\,.$$
$\square$

The following hypotheses will be assumed when specified.

- (A1) A is a maximal monotone operator on H and $A^{-1}\{0\}$ is nonempty.
- (A2) $f : H \to \mathbb{R}$ is bounded below and Lipschitzian on bounded subsets of H, $x_0 \in A^{-1}\{0\}$ satisfies $f(x_0) = \min\{f(x) : x \in H\} = \min(f)$, and f is regularly Lyapounov for the semigroup $\{S(t)\}_{t\geq 0}$ generated by A.
- (A3) $D(A)$ is a convex subset of H.

Our next result tells that, under the assumption (A1) and (A2), the function f behaves almost like a regularly Lyapounov function for the semigroup $\{S^1(t)\}_{t\geq 0}$ generated by a small perturbation of A. Note f is not even a Lyapounov function for $\{S^1(t)\}_{t\geq 0}$.

Lemma 3.2. *Let A and f satisfy (A1) and (A2). For all $\epsilon > 0$, there exist positive numbers $\overline{\delta}$, δ such that for all A' satisfying:*

1. *A' is single valued,*
2. *$D(A) \subseteq D(A')$,*
3. *A' is bounded on bounded subsets of $D(A)$,*
4. *$A'x_0 = 0$,*
5. *$A^1 = A + A'$ is maximal monotone,*
6. $\sup_{\{x\in D(A):\|x-x_0\|\leq\epsilon^{-1}\}} \|A'x\| < \delta$,

and for every $x \in D_{\epsilon^{-1}}(x_0)$, where recall $D_{\epsilon^{-1}}(x_0) = \{x \in D(A) : \|x - x_0\| < \epsilon^{-1}, f(x) > \min(f) + \frac{\epsilon}{2}\}$, there exists $\overline{\alpha} > 0$ such that

$$f(x) - f(S^1(t)x) \geq t\overline{\delta} \quad \forall\, t \in [0, \overline{\alpha})\,,$$

where $\{S^1(t)\}_{t\geq 0}$ is the semigroup generated by A^1.

Proof. Let $\epsilon > 0$ be given, and set M to be a positive integer greater than $\max\{\epsilon^{-1} + \|x_0\|, 2\epsilon^{-1}\}$. Since f is Lipschitzian on bounded subsets of H, there exists $L > 0$ such that

$$|f(x) - f(y)| \leq L\|x - y\| \quad \forall\, x, y \in \{x \in H : \|x\| \leq M\}\,. \tag{3.1}$$

Since f is a regularly Lyapounov function for the semigroup $\{S(t)\}_{t\geq 0}$, there exists $\delta_1 > 0$ such that for every $x \in D_M$ there exists $\alpha_1(x) > 0$ such that

$$f(S(t)x) - f(x) \leq -t\delta_1 \quad \forall\, t \in [0, \alpha_1(x))\,. \tag{3.2}$$

Let

$$\overline{\delta} = \frac{\delta_1}{2}, \tag{3.3}$$

and

$$\delta = \frac{\delta_1}{8L}. \tag{3.4}$$

Let A' satisfy 1-5 and

$$\sup_{\{x\in D(A):\|x-x_0\|\leq\epsilon^{-1}\}} \|A'x\| < \delta\,, \tag{3.5}$$

and let $\{S^1(t)\}_{t\geq 0}$ be the semigroup generated by A^1. Let $x \in D_{\epsilon^{-1}}(x_0)$ be given. We note that $\|S(t)x\|$ is bounded for all $t \geq 0$ as

$$\|S(t)x\| \leq \|S(t)x - x_0\| + \|x_0\| \leq \|x - x_0\| + \|x_0\| \leq \epsilon^{-1} + \|x_0\| \leq M\,, \tag{3.6}$$

and similarly

$$\|S^1(t)x\| \leq M \quad \forall\, t \geq 0, \forall\, x \in D_{\epsilon^{-1}}(x_0)\,. \tag{3.7}$$

Also, we note that $x \in D_M$. Therefore, by (3.2) there exists $\alpha_1(x) > 0$ such that

$$f(S(t)x) - f(x) \le -t\delta_1 \quad \forall\, t \in [0, \alpha_1(x))\,. \tag{3.8}$$

Since $\lim_{t\to 0+} \frac{S(t)x-x}{t} = -A^\circ x$, there exists $\alpha_2(x) > 0$ such that

$$\|\frac{S(t)x - x}{t} + A^\circ x\| < \frac{\delta_1}{8L} \quad \forall\, t \in (0, \alpha_2(x))\,. \tag{3.9}$$

Similarly, since $\lim_{t\to 0+} \frac{S^1(t)x-x}{t} = -(A^1)^\circ x$, there exists $\alpha_3(x) > 0$ such that

$$\|\frac{S^1(t)x - x}{t} + (A^1)^\circ x\| < \frac{\delta_1}{8L} \quad \forall\, t \in (0, \alpha_3(x))\,. \tag{3.10}$$

Choose $\overline{\alpha} = \min(\alpha_1(x), \alpha_2(x), \alpha_3(x))$. Then for $t \in (0, \overline{\alpha})$,

$$\begin{aligned}
&f(S^1(t)x) - f(x)\\
&= f(S^1(t)x) - f(S(t)x) + f(S(t)x) - f(x)\\
&\le L\|S^1(t)x - S(t)x\| + f(S(t)x) - f(x) \quad (\text{ by (3.6), (3.7) and (3.1) })\\
&\le L\|S^1(t)x - S(t)x\| - t\delta_1 \quad (\text{by (3.8)})\\
&= Lt\|\frac{S^1(t)x - x}{t} + (A^1)^\circ x - (A^1)^\circ x + A^\circ x - A^\circ x - \frac{S(t)x - x}{t}\| - t\delta_1\\
&\le Lt\left(\|\frac{S^1(t)x - x}{t} + (A^1)^\circ x\| + \|(A^1)^\circ x - A^\circ x\| + \|A^\circ x + \frac{S(t)x - x}{t}\|\right) - t\delta_1\\
&\le tL\frac{\delta_1}{8L} + tL\|(A^1)^\circ x - A^\circ x\| + tL\frac{\delta_1}{8L} - t\delta_1 \quad (\text{by (3.9) and (3.10)})\\
&= tL\|(A^1)^\circ x - A^\circ x\| - \frac{3}{4}t\delta_1\\
&\le tL2\|A'x\| - \frac{3}{4}t\delta_1 \quad (\text{by Lemma 3.1 })\\
&< tL2\delta - \frac{3}{4}t\delta_1 \quad (\text{by (3.5)})\\
&= tL2\frac{\delta_1}{8L} - \frac{3}{4}t\delta_1 \quad (\text{by (3.4)})\\
&= -\frac{t\delta_1}{2} = -t\overline{\delta} \quad (\text{by (3.3)}).
\end{aligned}$$

□

By using Lemma 3.2, in the next result, we show that the function f may not converge along the perturbed trajectory $S^1(t)x$ but stays close to $\min(f)$ if the perturbation is small, assuming $x \in D(A)$.

Lemma 3.3. *Let A and f satisfy (A1) and (A2). For all $\epsilon > 0$, there exists $\delta > 0$ such that for all A' satisfying:*

1. *A' is single valued,*
2. *$D(A) \subseteq D(A')$,*
3. *A' is bounded on bounded subsets of $D(A)$,*
4. *$A'x_0 = 0$,*
5. *$A^1 = A + A'$ is maximal monotone,*
6. *$\sup_{\{x\in D(A):\|x-x_0\|\le\epsilon^{-1}\}} \|A'x\| < \delta$,*

and for every $x \in \{x \in D(A) : \|x - x_0\| \le \epsilon^{-1}\}$ *there exists* $T \ge 0$ *such that*

$$f(S^1(t)x) \le \min(f) + \frac{\epsilon}{2} \quad \forall\, t \ge T\,,$$

where $\{S^1(t)\}_{t\ge 0}$ *is the semigroup generated by* A^1*.*

Proof. Let $\epsilon > 0$ be given. For convenience we write $M = \epsilon^{-1} + \|x_0\|$. By Lemma 3.2 there exist positive numbers δ, $\overline{\delta}$ such that for all A' satisfying 1 - 5 and

$$\sup_{\{x\in D(A):\|x-x_0\|\le\epsilon^{-1}\}} \|A'x\| < \delta\,,$$

and for every $x \in D_{\epsilon^{-1}}(x_0)$ there exists $\overline{\alpha} > 0$ such that

$$f(x) - f(S^1(t)x) \ge t\overline{\delta} \quad \forall\, t \in [0, \overline{\alpha})\,, \tag{3.11}$$

where $\{S^1(t)\}_{t\ge 0}$ is the semigroup generated by A^1. Let $x \in \{x \in D(A) : \|x-x_0\| \le \epsilon^{-1}\}$. Note $\|S^1(t)x\|$ is bounded, as

$$\|S^1(t)x\| \le \|S^1(t)x - x_0\| + \|x_0\| \le \|x - x_0\| + \|x_0\| \le \epsilon^{-1} + \|x_0\| = M\,. \tag{3.12}$$

Also,

$$\|S^1(t)x - x_0\| \le \|x - x_0\| \le \epsilon^{-1}\,. \tag{3.13}$$

Since f is Lipschitzian on bounded subsets of H, there exists $L > 0$ such that

$$|f(x) - f(y)| \le L\|x - y\| \quad \forall\, x, y \in \{x \in H : \|x\| \le M\}\,. \tag{3.14}$$

We note that $t \to f(S^1(t)x)$ is Lipschitz continuous on $[0,\infty)$ as for every $t_1, t_2 \in [0,\infty)$ we have

$$\begin{aligned} |f(S^1(t_1)x) - f(S^1(t_2)x)| &\le L\|S^1(t_1)x - S^1(t_2)x\| \quad \text{(by (3.12) and (3.14))}\\ &\le L|t_1 - t_2|\|(A^1)^\circ x\|\,. \end{aligned}$$

Firstly, we claim that there exists $T \ge 0$ such that

$$f(S^1(T)x) \le \min(f) + \frac{\epsilon}{2}\,.$$

Assume the contrary. Then

$$f(S^1(t)x) > \min(f) + \frac{\epsilon}{2} \quad \forall\, t \ge 0\,.$$

Therefore by (3.13), $S^1(t)x \in D_{\epsilon^{-1}}(x_0)\,\forall\, t \ge 0$. Let $V = \{T : f(x) - f(S^1(\tau)x) \ge \tau\overline{\delta}\ \forall\, \tau \in [0,T]\}$. Then V is a nonempty subinterval of $[0,\infty)$. We claim V is open and closed in $[0,\infty)$. To see V is open in $[0,\infty)$ let $T \in V$ be given. Since $S^1(T)x \in D_{\epsilon^{-1}}(x_0)$ there exists $\alpha' > 0$ such that

$$f(S^1(T)x) - f(S^1(t)S^1(T)x) \ge t\overline{\delta} \quad \forall\, t \in [0, \alpha']\,. \tag{3.15}$$

Also $T \in V$ implies

$$f(x) - f(S^1(T)x) \ge T\overline{\delta}\,. \tag{3.16}$$

Adding the inequalities (3.15) and (3.16) we get

$$f(x) - f(S^1(t+T)x) \ge (t+T)\overline{\delta} \quad \forall\, t \in [0, \alpha']\,.$$

Thus $[0, T+\alpha'] \subseteq V$, and V is open. To see that V is closed in $[0,\infty)$, let $\langle t_n\rangle_{n=1}^{\infty}$ be a sequence in V and, let $t_n \nearrow t$ as $n \to \infty$, $t > 0$. Since $t_n \in V$, for every n, we have

$$f(x) - f(S^1(t_n)x) \ge t_n\overline{\delta}\,.$$

Since $t \to f(S^1(t)x)$ is continuous on $[0, \infty)$, letting n go to infinity we get

$$f(x) - f(S^1(t)x) \geq t\overline{\delta}\,.$$

Hence $t \in V$. Now V is a nonempty open and closed subinterval of $[0, \infty)$, and therefore $V = [0, \infty)$. Hence for every $t \in [0, \infty)$, $f(x) - f(S^1(t)x) \geq t\overline{\delta}$. Therefore by taking the limit as $t \to \infty$, we get $\lim_{t\to\infty} f(S^1(t)x) = -\infty$, contradicting the fact that f is bounded below. Hence our assumption is wrong, proving the claim.

Secondly, we show that for every $t \geq T$

$$f(S^1(t)x) \leq \min(f) + \frac{\epsilon}{2}\,.$$

Assume, to obtain a contradiction that there exists $T_1 > T$ such that

$$f(S^1(T_1)x) > \min(f) + \frac{\epsilon}{2}\,.$$

Since $t \to f(S^1(t)x)$ is continuous on $[0, \infty)$ there exists $T_2 \in (T, T_1)$ such that

$$f(S^1(T_2)x) = \min(f) + \frac{\epsilon}{2}\,, \tag{3.17}$$

and

$$f(S^1(\tau)x) > \min(f) + \frac{\epsilon}{2} \quad \forall\tau \in (T_2, T_1]\,. \tag{3.18}$$

Since $t \to f(S^1(t)x)$ is Lipschitz continuous on $[0, \infty)$, it is differentiable a.e. and

$$f(S^1(T_1)x) = f(S^1(T_2)x) + \int_{T_2}^{T_1} \frac{d}{dt} f(S^1(t)x)dt\,. \tag{3.19}$$

Let $T_3 \in (T_2, T_1)$ be such that $f(S^1(t)x)$ is differentiable at T_3. By (3.13) and (3.18), $S^1(T_3)x \in D_{\epsilon^{-1}}(x_0)$. Therefore by (3.11) there exists $\alpha_3 > 0$ such that

$$f(S^1(T_3)x) - f(S^1(T_3 + t)x) \geq t\overline{\delta} \quad \forall t \in (0, \alpha_3)\,,$$

which in turn implies

$$\frac{d}{dt} f(S^1(T_3)x) \leq -\overline{\delta}\,. \tag{3.20}$$

Using (3.20) in (3.19) we get

$$f(S^1(T_1)x) \leq f(S^1(T_2)x) - \overline{\delta}(T_1 - T_2) = \min(f) + \frac{\epsilon}{2} - \overline{\delta}(T_1 - T_2)\,,$$

which contradicts our assumption. □

Let us recall from [12] that an operator A is maximal $\mathcal{M}(\omega)$ if and only if $A + \omega I$ is a maximal monotone operator.

Remark 3.4. We note that if we assume f to be Lipschitzian on K, rather than on bounded sets, and A to be a maximal $\mathcal{M}(\omega)$ operator, in (A1) and (A2), then the conclusion of Lemma 3.2 holds. We are unable to extend the conclusion of Lemma 3.3 for a maximal $\mathcal{M}(\omega)$ operator even if we assume perturbed trajectories to be bounded.

Finally we establish the first stability theorem, extending Lemma 3.3 by not assuming $x \in D(A)$.

Theorem 3.5. *Let A and f satisfy (A1) and (A2). For all $\epsilon > 0$, there exists $\delta > 0$ such that for all A' satisfying:*

(1) *A' is single valued,*
(2) *$D(A) \subseteq D(A')$,*

(3) *A' is bounded on bounded subsets of $D(A)$,*
(4) $A'x_0 = 0$,
(5) *$A^1 = A + A'$ is maximal monotone,*
(6) $\sup_{\{x\in D(A):\|x-x_0\|\le\epsilon^{-1}\}} \|A'x\| < \delta$,

and for every $x \in \{x \in \overline{D(A)} : \|x - x_0\| < \epsilon^{-1}\}$, there exists $T \ge 0$ such that

$$f(S^1(t)x) \le \min(f) + \epsilon \quad \forall\, t \ge T\,,$$

where $\{S^1(t)\}_{t\ge0}$ is the semigroup generated by A^1.

Proof. Let $\epsilon > 0$ be given, and set $M = \epsilon^{-1} + \|x_0\|$. Let $x \in \{x \in \overline{D(A)} : \|x-x_0\| < \epsilon^{-1}\}$ then there exists a sequence $\langle x_k\rangle_{k=1}^{\infty}$ in $\{x \in D(A) : \|x - x_0\| \le \epsilon^{-1}\}$ such that $x_k \to x$ as $k \to \infty$. Since f is Lipschitzian on bounded subsets of H, there exists $L > 0$ such that

$$|f(x) - f(y)| \le L\|x - y\| \quad \forall\, x, y \in \{x \in H : \|x\| \le M\}\,. \tag{3.21}$$

Since $x_k \to x$ there exists a positive integer k' such that

$$\|x_{k'} - x\| < \frac{1}{L+1}.\frac{\epsilon}{2}\,. \tag{3.22}$$

By Lemma 3.3, noting $x_{k'} \in D(A)$, and $\|x_{k'} - x_0\| \le \epsilon^{-1}$, there exists $\delta > 0$ such that for all A' satisfying 1-6 of this theorem, there exists $T \ge 0$ such that

$$f(S^1(t)x_{k'}) \le \min(f) + \frac{\epsilon}{2} \quad \forall\, t \ge T\,, \tag{3.23}$$

where $\{S^1(t)\}_{t\ge0}$ is the semigroup generated by A^1. Note for every k, and all $t \ge 0$, $\|S^1(t)x_k\|$ and $\|S^1(t)x\|$ are bounded as

$$\|S^1(t)x_k\| \le \|S^1(t)x_k - x_0\| + \|x_0\| \le \|x_k - x_0\| + \|x_0\| \le \epsilon^{-1} + \|x_0\| = M\,, \tag{3.24}$$

and

$$\|S^1(t)x\| \le \|S^1(t)x - x_0\| + \|x_0\| \le \|x - x_0\| + \|x_0\| \le \epsilon^{-1} + \|x_0\| = M\,. \tag{3.25}$$

Thus by (3.24), (3.25), (3.21) and (3.22), for all $t \ge 0$,

$$\begin{aligned} |f(S^1(t)x_{k'}) - f(S^1(t)x)| &< L\|S^1(t)x_{k'} - S^1(t)x\| \\ &\le L\|x_{k'} - x\| \quad (\text{as } S^1 \text{ is a contraction}) \\ &< \frac{L}{L+1}.\frac{\epsilon}{2} \\ &< \frac{\epsilon}{2} \end{aligned} \tag{3.26}$$

Combining (3.26) and (3.23) we get $\forall\, t \ge T$

$$\begin{aligned} f(S^1(t)x) &\le |f(S^1(t)x) - f(S^1(t)x_{k'})| + f(S^1(t)x_{k'}) \\ &< \frac{\epsilon}{2} + \min(f) + \frac{\epsilon}{2} \\ &= \min(f) + \epsilon\,. \end{aligned}$$

□

Instead of assuming $A+A'$ to be maximal monotone, one can choose A' satisfying additional conditions such that $A+A'$ is maximal monotone operator. In Theorem 2, we use our perturbation result of [9] to replace (5) of Theorem 3.5 by some suitable conditions.

Theorem 3.6. *Let A and f satisfy (A1)-(A3). Then for all $\epsilon > 0$, there exists $\delta > 0$ such that for all A' satisfying:*

1. *A' is single valued, hemicontinuous and monotone on $D(A)$,*
2. *$D(A) \subseteq D(A')$,*
3. *A' is bounded on bounded subsets of $D(A)$,*
4. *$A'x_0 = 0$,*
5. $\sup_{\{x \in D(A): \|x - x_0\| \leq \epsilon^{-1}\}} \|A'x\| < \delta$,

we have $A^1 = A + A'$ maximal monotone, and the semigroup $\{S^1(t)\}_{t \geq 0}$ generated by A^1 has the property that for every $x \in \{x \in \overline{D(A)} : \|x - x_0\| < \epsilon^{-1}\}$ there exists $T \geq 0$ such that

$$f(S^1(t)x) \leq \min(f) + \epsilon \quad \forall\, t \geq T\,.$$

Proof. By [9, Proposition 4], 1,2,3 and (A3), imply that $A + A'$ is maximal monotone. By Theorem 3.5 we have our conclusion. □

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References

[1] S. Aizicovici, S. Reich and A. J. Zaslavski; *Convergence theorems for continuous descent methods*, J. Evol. Equ. **4** (2004), 139-156.
[2] S. Aizicovici, S. Reich and A. J. Zaslavski; *Convergence results for a class of abstract continuous descent methods*, Electron. J. Differential Equations **2004** no. 45 (2004), 1-13.
[3] S. Aizicovici, S. Reich and A. J. Zaslavski; *Most continuous descent methods converge*, Arch. Math. (Basel) **85** (2005), 268-277.
[4] J. B. Baillon; *Un exemple concernant le comportement asymptotique de la solution du problème $du/dt + \partial\phi(u) \ni 0$*, J. Functional Anal. **28** (1978), 369-376.
[5] J. B. Baillon, R. E. Bruck and S. Reich; *On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces*, Houston J. Math. **4** (1978), 1-9.
[6] H. Brezis; *Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert*, North-Holland, Amsterdam, 1973.
[7] H. Brezis; *Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations*, Contributions to Nonlinear Functional Analysis (E. H. Zarantonello, ed.), Academic Press, New York, 1971, 101-156.
[8] R. E. Bruck; *Asymptotic convergence of nonlinear contraction semigroups in Hilbert space*, J. Functional Analysis **18** (1975), 15-26.
[9] R. Choudhary; *Generic convergence of a convex Lyapounov function along trajectories of nonexpansive semigroups in Hilbert space*, J. Nonlinear Convex Anal. **7** (2006), no. 2, 245-268.
[10] C. M. Dafermos and M. Slemrod; *Asymptotic behavior of nonlinear contraction semigroups*, J. Functional Analysis **13** (1973), 97-106.
[11] O. Nevanlinna and S. Reich; *Strong convergence of contraction semigroups and of iterative methods for accretive operators in Banach spaces*, Israel J. Math. **32** (1979), 44-58.
[12] A. Pazy; *Semigroups of nonlinear contractions in Hilbert spaces*, C.I.M.E. Varenna, 1970, Ed. Cremonese (1971), 343-430.
[13] A. Pazy; *On the asymptotic behavior of semigroups of nonlinear contractions in Hilbert space*, J. Functional Analysis **27** (1978), 292-307.
[14] A. Pazy; *Strong convergence of semigroups of nonlinear contractions in Hilbert space*, J. Analyse Math. **34** (1978), 1-35.
[15] S. Reich; *Almost convergence and nonlinear ergodic theorems*, J. Approx. Theory **24** (1978), 269-272.
[16] S. Reich and A. J. Zaslavski; *Generic convergence of descent methods in Banach spaces*, Math. Oper. Res. **25** (2000), 231-242.

[17] S. Reich and A. J. Zaslavski; *The set of divergent descent methods in a Banach space is σ-porous*, SIAM J. Optim. **11** (2001), 1003-1018.

[18] S. Reich and A. J. Zaslavski; *Two convergence results for continuous descent methods*, Electron. J. Differential Equations **2003** no. 24 (2003), 1-11.

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