Electronic Journal of Differential Equations, Vol. 2006(2006), No. 128, pp. 1–23.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

EXISTENCE OF SOLUTIONS FOR THE ONE-PHASE AND THE MULTI-LAYER FREE-BOUNDARY PROBLEMS WITH THE P-LAPLACIAN OPERATOR

IDRISSA LY, DIARAF SECK

Abstract. By considering the p-laplacian operator, we show the existence of a solution to the exterior (resp interior) free boundary problem with non constant Bernoulli free boundary condition. In the second part of this article, we study the existence of solutions to the two-layer shape optimization problem. From a monotonicity result, we show the existence of classical solutions to the two-layer Bernoulli free-boundary problem with nonlinear joining conditions. Also we extend the existence result to the multi-layer case.

1. Introduction

In part I, we study the exterior and interior free-boundary problem with non-constant Bernoulli boundary condition. Given K a $\mathcal{C}^2$-regular bounded domain in $\mathbb{R}^N$ and a positive continuous function g, such that $g(x) > \alpha > 0$ for all $x \in R$, we find for the exterior problem a domain Ω and a function u_Ω such that

$$\begin{aligned} -\Delta_p u_\Omega &= 0 \quad \text{in } \Omega\backslash K,\ 1 < p < \infty \\ u_\Omega &= 0 \quad \text{on } \partial\Omega \\ u_\Omega &= 1 \quad \text{on } \partial K \\ -\frac{\partial u_\Omega}{\partial \nu_e} &= g(x) \quad \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

Here Δ_p denotes the p-Laplace operator, i.e. $\Delta_p u := \operatorname{div}(\|\nabla u\|^{p-2}\nabla u)$ and ν_e is the normal exterior unit of Ω. And for the interior problem, we look for a domain Ω and a function u_Ω such that

$$\begin{aligned} -\Delta_p u_\Omega &= 0 \quad \text{in } K\backslash\bar{\Omega},\ 1 < p < \infty \\ u_\Omega &= 1 \quad \text{on } \partial\Omega \\ u_\Omega &= 0 \quad \text{on } \partial K \\ \frac{\partial u_\Omega}{\partial \nu_i} &= g(x) \quad \text{on } \partial\Omega \end{aligned} \tag{1.2}$$

2000 *Mathematics Subject Classification.* 35R35.
Key words and phrases. Bernoulli free boundary problem; starshaped domain; shape optimization; shape derivative; monotonicity.

Submitted March 6, 2006. Published October 11, 2006.

where ν_i is the normal interior unit of Ω. The problem arises when a fluid flows in porous medium around an obstacle. In certain industrial problems such as shape optimization, galvanization, we seek to find level lines of the potential function with prescribed pressure.

Inspired by the pioneering work of Beurling, where the notion of sub and supersolutions in geometrical case is used, Henrot and Shahgohlian [16] studied this problem . They proved that when $K \subset \mathbb{R}^N$ is a bounded and convex domain a generalization of [14, 15] to the case of the non-constant Bernoulli boundary condition.

By combining a variational approach and a sequential method, we establish an existence result by generalizing the problems studied in [25, 26] to the case of the non-constant Bernoulli boundary condition.

The structure of Part I is as follows: In the first part, we present the main result which generalizes results in [25, 26], to the case of non-constant Bernoulli boundary condition. In the second section, we give auxiliary results. The third part deals with the study of the shape optimization problem and the existence of Lagrange multiplier λ_Ω for the exterior (respectively interior) case. First, we study the existence result for the shape optimization problem for the exterior case: Find

$$\min\{J_1(w), w \in \mathcal{O}^1_\epsilon\},$$

where $\mathcal{O}^1_\epsilon = \{w \supset K : w$ is an open set satisfying the ϵ-cone property, $\int_w \frac{g^p}{c^p}(x)dx = V_0\}$, where V_0 is a given positive value. The functional J_1 is

$$J_1(w) := \frac{1}{p}\int_{w\backslash K} \|\nabla u_w\|^p dx,$$

where u_w is a solution to the Dirichlet problem

$$\begin{aligned} -\Delta_p u_w &= 0 \quad \text{in } w\backslash K,\ 1 < p < \infty \\ u_w &= 0 \quad \text{on} \quad \partial w \\ u_w &= 1 \quad \text{on} \quad \partial K. \end{aligned} \tag{1.3}$$

Second, we study the existence result for the shape optimization problem. For the interior case: Find

$$\min\{J_2(w), w \in \mathcal{O}^2_\epsilon\},$$

where $\mathcal{O}^2_\epsilon = \{w \subset K : w$ is an open set satisfying the ϵ-cone property, $\int_w \frac{g^p}{c^p}(x)dx = W_0\}$, where W_0 is a given positive value. The functional J_2 is

$$J_2(w) := \frac{1}{p}\int_{K\backslash \bar{w}} \|\nabla u_w\|^p dx,$$

where u_w is a solution to the Dirichlet problem

$$\begin{aligned} -\Delta_p u_w &= 0 \quad \text{in } K\backslash \bar{w},\ 1 < p < \infty \\ u_w &= 1 \quad \text{on} \quad \partial w \\ u_w &= 0 \quad \text{on} \quad \partial K. \end{aligned} \tag{1.4}$$

Next, we obtain an optimality condition for the exterior (respectively interior) case

$$-\frac{\partial u}{\partial \nu_e} = (\frac{p}{1-p}\lambda_\Omega)^{1/p} \quad (\text{respectively } \frac{\partial u}{\partial \nu_i} = (\frac{p}{p-1}\lambda_\Omega)^{1/p}) \quad \text{on } \partial\Omega$$

Then we conclude this section with a monotonicity result for the exterior (respectively interior) case. The last part is the proof of the main result for the exterior (respectively interior) case.

In Part II, we study the multi-layer case. Let D_0^* and D_1^* be $\mathcal{C}^2$-regular, compact sets in $\mathbb{R}^N$ and star shaped with respect to the origin such that D_1^* strictly contains D_0^*. We look for (D, v, u) where D is $\mathcal{C}^2$-regular domain such that $D_0^* \subset D \subset \bar{D} \subset D_1^*$ and v and u are solutions of the problems

$$\begin{aligned} -\Delta_p v = 0 \quad &\text{in } D_1^* \backslash D \qquad & -\Delta_p u = 0 \quad &\text{in } D \backslash D_0^* \\ v = 1 \quad &\text{on } \partial D & u = 0 \quad &\text{on } \partial D \\ v = 0 \quad &\text{on } \partial D_1^* & u = 1 \quad &\text{on } \partial D_0^* \end{aligned} \tag{1.5}$$

respectively, and satisfy the nonlinear joining condition

$$\|\nabla v\|^p - \|\nabla u\|^p = \lambda \quad \text{on } \partial D,\ \lambda \in \mathbb{R}. \tag{1.6}$$

This joining condition is justified by the method which we are using.

The described above problem appears in several physical situations and can be appropriately interpreted in many industrial applications. In the case $p = 2$, we refer the reader to [1, 2] and the references therein.

Inspired by the pioneering work of Beurling, where the notion of sub and super solutions in the geometrical case is used, Acker et al [7] studied this problem with a more general non linear joining junction. They assumed that D_0^* and D_1^* are bounded convex domains and D_1^* contains D_0^* strictly. By considering the p-laplacian operator, we show the existence of a solution to the exterior (resp interior) free boundary problem with non constant Bernoulli free boundary condition. In the second part of this article, we study the existence of solutions to the two-layer shape optimization problem. From a monotonicity result, we show the existence of classical solutions to the two-layer Bernoulli free-boundary problem with nonlinear joining conditions. Also we extend the existence result to the multi-layer case.. They proved there exists a convex $\mathcal{C}^1$ domain D, $D_0^* \subset\subset D \subset\subset D_1^*$ which is a classical solution of the two-layer free-boundary problem (1.5)-(1.6).

Using convex domains, Acker [1, 2] proved the existence for multi-layer free boundary problems by using the operator method in the case where $p = 2$. Laurence and Stredulinsky [18, 19], use convex domains, when proving an existence result in the case $p = 2, N = 2$.

Now, by combining a variational approach and a sequential method, we establish an existence result for non necessarily convex domains.

The structure of the Part II is as follows. In the first part, we present the main result. By considering the auxiliary results in the Part I, we study in the second part the shape optimization problem and the existence of Lagrange multiplier function λ. The existence result for the shape optimization problem consists of finding a domain D such that

$$J(D) = \min\{J(w), w \in \mathcal{O}_\epsilon\},$$

where $\mathcal{O}_\epsilon = \{w \subset \mathbb{R}^N, D_0^* \subset\subset D \subset\subset D_1^*, w \text{ verifying the } \epsilon\text{-cone property }, \operatorname{vol}(w) = m_0\}$, where vol denotes the volume, m_0 is a fixed value in $\mathbb{R}_+^*$. The functional J is defined on $\mathcal{O}_\epsilon$ by

$$J(D) = \frac{1}{p} \int_{D_1^* \backslash D} \|\nabla v\|^p + \frac{1}{p} \int_{D \backslash D_0^*} \|\nabla u\|^p, \quad 1 < p < \infty,$$

where v and u are solutions of

$$\begin{aligned} -\Delta_p v = 0 \quad \text{in } D_1^*\backslash D \qquad\qquad & -\Delta_p u = 0 \quad \text{in } D\backslash D_0^* \\ v = 1 \quad \text{on } \partial D \qquad\qquad & u = 0 \quad \text{on } \partial D \\ v = 0 \quad \text{on } \partial D_1^* \qquad\qquad & u = 1 \quad \text{on } \partial D_0^* \end{aligned} \tag{1.7}$$

respectively. Next, we obtain the joining condition as an optimality condition:

$$\|\nabla v\|^p - \|\nabla u\|^p = \frac{p}{p-1}\lambda_D \quad \text{on } \partial D,\ \lambda_D \in R. \tag{1.8}$$

Then we conclude this section with a monotonicity result. The third section is devoted to the proof of the main result. And the last part is devoted to the extension to the multi-layer case.

2. main results of Part I

For the exterior case, let K be a $\mathcal{C}^2$-regular, star-shaped with respect to the origin and bounded domain.

Theorem 2.1. *If Ω solution of the shape optimization problem* $\min\{J_1(w), w \in \mathcal{O}_\epsilon^1\}$ *is $\mathcal{C}^2$-regular domain, then the free boundary problem (1.1) admits a classical unique solution Ω.*

For the interior case, let K be a $\mathcal{C}^2$-regular, star-shaped with respect to the origin and bounded domain. Let

$$\alpha(R_K, p, N) := \begin{cases} e/R_K & \text{if } p = N \\ \dfrac{|\frac{p-N}{p-1}|}{\left|(\frac{p-1}{N-1})^{\frac{N-1}{N-p}} - (\frac{p-1}{N-1})^{\frac{p-1}{N-p}}\right|}\frac{1}{R_K} & \text{if } p \neq N. \end{cases}$$

where $R_K = \sup\{R > 0 : B(o, R) \subset K\}$, Here c_K is the minimal value for which the interior Bernoulli problem (1.2) admits a solution.

Theorem 2.2. *If the solution Ω of the shape optimization problem* $\min\{J(w), w \in \mathcal{O}_\epsilon^2\}$ *is $\mathcal{C}^2$-regular, then for all constant $c > 0$ satisfying $c \geq \alpha(R_K, p, N)$, Ω is the classical solution of the free-boundary problem* (1.2)*. Moreover The constant c_K satisfies $0 < c_K \leq \alpha(R_K, p, N)$.*

To prove these theorems we need the following results.

3. Auxiliary results

For the rest of this article, we consider a fixed, closed domain D which contains all the open subsets used.

Let ζ be an unitary vector of $\mathbb{R}^N$, ϵ be a real number strictly positive and y be in $\mathbb{R}^N$. We call a cone with vertex y, of direction ζ and angle to the vertex and height ϵ, the set defined by

$$\mathcal{C}(y, \zeta, \epsilon, \epsilon) = \{x \in \mathbb{R}^N : |x - y| \leq \epsilon \text{ and } |(x - y)\zeta| \geq |x - y| \cos \epsilon\}.$$

Let Ω be an open set of $\mathbb{R}^N$, Ω is said to have the ϵ-cone property if for all $x \in \partial\Omega$ then there exists a direction ζ and a strictly positive real number ϵ such that

$$\mathcal{C}(y, \zeta, \epsilon, \epsilon) \subset \Omega, \quad \text{for all } y \in B(x, \epsilon) \cap \bar{\Omega}.$$

Let K_1 and K_2 be two compact subsets of D. Let

$$d(x, K_1) = \inf_{y \in K_2} d(x, y), \quad d(x, K_2) = \inf_{y \in K_1} d(x, y).$$

Note that

$$\rho(K_1, K_2) = \sup_{x \in K_2} d(x, K_1), \quad \rho(K_2, K_1) = \sup_{x \in K_1} d(x, K_2).$$

Let

$$d_H(K_1, K_2) = \max[\rho(K_1, K_2), \rho(K_2, K_1)],$$

which is called the Hausdorff distance of K_1 and K_2.

Let (Ω_n) be a sequence of open subsets of D and Ω be an open subset of D. We say that the sequence (Ω_n) converges on Ω in the Hausdorff sense and we denote by $\Omega_n \xrightarrow{H} \Omega$ if $\lim_{n \to +\infty} d_H(\bar{D} \backslash \Omega_n, \bar{D} \backslash \Omega) = 0$.

Let (Ω_n) be a sequence of open sets of $\mathbb{R}^N$ and Ω be an open set of $\mathbb{R}^N$. We say that the sequence (Ω_n) converges on Ω in the sense of L^p, $1 \le p < \infty$ if χ_{Ω_n} converges on χ_Ω in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$, χ_Ω being the characteristic functions of Ω.

Let (Ω_n) be a sequence of open subsets of D and Ω be an open subset of D. We say that the sequence (Ω_n) converges on Ω in the compact sense if:

1. Every compact G subset of Ω, is included in Ω_n for n large enough,
2. every compact Q subset of $\bar{\Omega}^c$, is included in $\bar{\Omega}_n^c$ for n large enough.

Lemma 3.1. *Let Ω_1 and Ω_2 be two different domains star-shaped with respect to the origin and bounded. If $\bar{\Omega}_1 \subset \bar{\Omega}_2$ then there exists $0 < t_0 < 1$ such that $t_0 \Omega_2 \subset \Omega_1$ and $t_0 \partial\Omega_2 \cap \partial\Omega_1 \neq \emptyset$.*

The proof of the above lemma can be found in [22].

Lemma 3.2. *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions of $L^p(\Omega)$, $1 \le p < \infty$ and $f \in L^p(\Omega)$. We suppose f_n converges on f a.e. and $\lim_{n \to \infty} \|f_n\|_p = \|f\|_p$. Then we have $\lim_{n \to \infty} \|f_n - f\|_p = 0$.*

For the proof of the above lemma see for example [17].

Lemma 3.3 (Brezis-Lieb)**.** *Let $(f_n)_{n \in \mathbb{N}}$ be a bounded sequence in $L^p(\Omega)$, $1 \le p < \infty$. We suppose that f_n converges on f a.e., then $f \in L^p(\Omega)$ and*

$$\|f\|_p = \lim_{n \to \infty} (\|f_n - f\|_p + \|f_n\|_p).$$

For the proof of the above lemma, see for example [17].

Lemma 3.4. *Let $(\Omega_n)_{n \in \mathbb{N}}$ be a sequence of open sets in $\mathbb{R}^N$ having the ϵ-cone property, with $\bar{\Omega}_n \subset F \subset D$, F a compact set and D a ball, then, there exists an open set Ω, included in F, which satisfies the $\frac{\epsilon}{2}$-cone property and a subsequence $(\Omega_{n_k})_{k \in \mathbb{N}}$ such that*

$$\chi_{\Omega_{n_k}} \xrightarrow{L^1} \chi_\Omega, \quad \Omega_{n_k} \xrightarrow{H} \Omega$$
$$\partial\Omega_{n_k} \xrightarrow{H} \partial\Omega, \quad \bar{\Omega}_{n_k} \xrightarrow{H} \bar{\Omega}.$$

The above lemma is a well known result in functional analysis related to shape optimization; its proof can be found for example in [26].

4. Shape optimization result and monotonicity result

For the exterior case, we have the following result, whose proof can be found in [25].

Proposition 4.1. *The problem: Find $\Omega \in \mathcal{O}_\epsilon^1$ such that $J_1(\Omega) = \min\{J_1(w), w \in \mathcal{O}_\epsilon^1\}$ admits a solution.*

For the interior case, we have the following result, whose proof can be found in [26].

Proposition 4.2. *The problem: Find $\Omega \in \mathcal{O}_\epsilon^2$ such that $J_2(\Omega) = \min\{J_2(w), w \in \mathcal{O}_\epsilon^2\}$ admits a solution.*

Remark 4.3. Let us define another class of domains:

$$\mathcal{O}_0 = \{w \supset K : w \text{ is an open set of} \mathbb{R}^N.\mathcal{C}^k\text{-regular domain, } \int_w \frac{g^p}{c^p}(x)dx = V_0\}$$

where $k \geq 3$. It is possible to use the oriented distance, the results in [9, theorems 5.3 5.5,5.6], [10] and the Ascoli theorem to prove the existence of a domain at least of class $\mathcal{C}^{k-1}$ which is minimum for the shape optimization problem.

For the rest of this article, we assume that Ω is $\mathcal{C}^2$-regular in order to use the shape derivatives. The next theorems give a necessary condition optimality condition. We follow the approach of Sokolowski-Zolesio [29] to define the shape derivatives (see also [28]).

For the exterior case, we have the following result.

Proposition 4.4. *If Ω is the solution of the shape optimization problem*

$$\min\{J_1(w) : w \in \mathcal{O}_\epsilon^1\},$$

then there exists a Lagrange multiplier $\lambda_\Omega < 0$ such that $-\frac{\partial u}{\partial \nu_e} = (\frac{p}{1-p}\lambda_\Omega)^{\frac{1}{p}} \frac{g}{c}(x)$ on $\partial\Omega$.

Proof. Let J_1 be a functional defined on $\mathcal{O}_\epsilon^1$ by

$$J_1(w) := \frac{1}{p}\int_{w\setminus K} \|\nabla u_w\|^p dx,$$

where u_w is a solution to the Dirichlet problem

$$\begin{aligned} -\Delta_p u_w &= 0 \quad \text{in } \Omega\setminus K,\ 1 < p < \infty \\ u_w &= 0 \quad \text{on} \quad \partial w \\ u_w &= 1 \quad \text{on} \quad \partial K. \end{aligned} \tag{4.1}$$

We use classical Hadamard's formula to compute the Eulerian derivative of the functional J_1 at the point Ω in the direction V. A standard computation, see [22], shows

$$dJ_1(\Omega; V) = \int_{\partial\Omega} \|\nabla u\|^{p-2}\frac{\partial u}{\partial \nu_e}u' ds + \frac{1}{p}\int_{\partial\Omega} \|\nabla u\|^p V(0).\nu_e ds$$

where $u' = -\frac{\partial u}{\partial \nu_e}V(0).\nu_e$ on $\partial\Omega$. This implies

$$dJ(\Omega; V) = \frac{1-p}{p}\int_{\partial\Omega} \|\nabla u\|^p V(0).\nu_e ds.$$

Let us take $J(\Omega) = \int_\Omega \frac{g^p}{c^p}(x)dx$. Then

$$dJ(\Omega; V) = \int_\Omega \operatorname{div}(\frac{g^p}{c^p}(x)V(0))dx = \int_{\partial\Omega} \frac{g^p}{c^p}(x)V(0).\nu_e ds.$$

Ω is optimal then there exists a Lagrange multiplier $\lambda_\Omega \in R$ such that $dJ_1(\Omega, V) = \lambda_\Omega dJ(\Omega; V)$. We obtain

$$\int_{\partial\Omega} (\frac{1-p}{p}\|\nabla u\|^p - \lambda_\Omega \frac{g^p}{c^p}(x)V(0).\nu)ds = 0 \quad \text{for all} \quad V.$$

Then

$$\|\nabla u\|^p = \frac{p}{1-p}\lambda_\Omega \frac{g^p}{c^p}(x) \quad \text{on } \partial\Omega,$$
$$\|\nabla u\| = (\frac{p}{1-p}\lambda_\Omega)^{\frac{1}{p}} \frac{g}{c}(x) \quad \text{on } \partial\Omega$$

Since Ω is $\mathcal{C}^2$-regular and $u = 0$ on $\partial\Omega$, we get

$$-\frac{\partial u}{\partial \nu_e} = (\frac{p}{1-p}\lambda_\Omega)^{\frac{1}{p}} \frac{g}{c}(x) \quad \text{on } \partial\Omega.$$

□

For the interior case, we have the following result.

Proposition 4.5. *If Ω is the solution of the shape optimization problem*

$$\min\{J_2(w) : w \in \mathcal{O}^2_\epsilon\},$$

then there exists a Lagrange multiplier $\lambda_\Omega > 0$ such that $\frac{\partial u}{\partial \nu_i} = (\frac{p}{p-1}\lambda_\Omega)^{\frac{1}{p}} \frac{g}{c}(x)$ on $\partial\Omega$.

For the proof of the above proposition, we use the same technics as in proposition 4.4. To conclude this section, we state a monotonicity result. For the exterior case, we have the following result, whose proof can be found in [25].

Proposition 4.6. *Suppose that K is star-shaped with respect to the origin. Let Ω_1 and Ω_2 be two different solutions to the shape optimization problem $\min\{J_1(w), w \in \mathcal{O}^1_\epsilon\}$, star-shaped with respect to the origin such that $\bar{\Omega}_1 \subset \bar{\Omega}_2$. The mapping which associates to every Ω the corresponding Lagrange multiplier λ_Ω is strictly increasing i.e $\lambda_{\Omega_2} > \lambda_{\Omega_1}$.*

For the interior case, we have the following result, whose proof is found in [26].

Proposition 4.7. *Suppose that K is star-shaped with respect to the origin. Let Ω_1 and Ω_2 be two different solutions to the shape optimization problem $\min\{J_2(w), w \in \mathcal{O}^2_\epsilon\}$, star-shaped with respect to the origin such that $\Omega_1 \subset \Omega_2$ and $\partial\Omega_1 \cap \partial\Omega_2 \neq \emptyset$. The mapping which associates to every Ω the corresponding Lagrange multiplier λ_Ω is decreasing i.e $\lambda_{\Omega_1} \geq \lambda_{\Omega_2}$.*

5. Proof of the main results of Part I

We use the preceding properties to prove the main result. Exterior case:

Proof of the Theorem 2.1. We choose a ball $B(O,R)$ centered at the origin and radius R and a ball $B(O,r)$ such that $B(O,r) \subset K \subset B(O,R)$. First, we have to look for a solution u_0 to the problem

$$\begin{gathered} -\Delta_p u = 0 \quad \text{in } B_R \backslash B_r \\ u = 0 \quad \text{on } \partial B_R \\ u = 1 \quad \text{on } \partial B_r. \end{gathered} \tag{5.1}$$

The solution u_0 is explicitly determined by

$$u_0(x) = \begin{cases} \dfrac{\ln \|x\| - \ln R}{\ln r - \ln R} & \text{if } p = N \\ \dfrac{\|x\|^{\frac{p-N}{p-1}} - R^{\frac{p-N}{p-1}}}{r^{\frac{p-N}{p-1}} - R^{\frac{p-N}{p-1}}} & \text{if } p \neq N, \end{cases} \tag{5.2}$$

and

$$\|\nabla u_0(x)\| = \begin{cases} \dfrac{1}{\|x\|^2(\ln R - \ln r)} & \text{if } p = N \\ \dfrac{|\frac{p-N}{p-1}| \|x\|^{\frac{-N-p+2}{p-1}}}{|r^{\frac{p-N}{p-1}} - R^{\frac{p-N}{p-1}}|} & \text{if } p \neq N. \end{cases}$$

In particular $\|\nabla u_0\| < c$ on ∂B_R for R big enough.

Now consider the problem

$$\begin{gathered} -\Delta_p u = 0 \quad \text{in } B_R \backslash K \\ u = 0 \quad \text{on } \partial B_R \\ u = 1 \quad \text{on } \partial K. \end{gathered} \tag{5.3}$$

This problem admits a solution denoted by u_R. This solution is obtained by minimizing the functional J_1 defined on the Sobolev space

$$V' = \{v \in W_0^{1,p}(B_R), v = 1 \text{on } \partial K\}$$

and $J_1(v) = \frac{1}{p} \int_{B_R \backslash K} \|\nabla v\|^p dx$.

Consider the problem

$$\begin{gathered} -\Delta_p v = 0 \quad \text{in } B_R \backslash K \\ v = 0 \quad \text{on } \partial B_R \\ v = u_0 \quad \text{on } \partial K. \end{gathered} \tag{5.4}$$

It is easy to see that $v = u_r$ is a solution to problem (5.4). By the comparison principle [30], we obtain $0 \le u_0 \le 1$ and $0 \le u_R \le 1$. On $\partial(B_R \backslash K)$, we obtain $u_R \ge u_0$ and then, $u_R \ge u_0$ in $B_R \backslash K$. Finally, we have $\|\nabla u_R\| \ge \|\nabla u_0\|$ on ∂B_R.

Case $p = N$**.** If $R_1 < R_0$, we get $\|\nabla u_0\|_{|\partial B_{R_0}} \le \|\nabla u_0\|_{|\partial B_{R_1}}$ then the mapping for all R associates $\|\nabla u_0\|_{|\partial B_R}$ is decreasing.

Initially, we choose a radius R_0 big enough and we compute $\|\nabla u_0\|_{|\partial B_{R_0}}$ and if $\big|\|\nabla u_0\|_{|\partial B_{r_0}} - c\big| > \delta$, where $\delta > 0$ is a fixed and sufficiently small number. We continue the process by varying R in the increasing sense, we will achieve a step denoted N such that $\big|\|\nabla u_0\|_{|\partial B_{R_N}} - c\big| < \delta$.

Consider $\mathcal{O}_N$ the class of admissible domains defined as follows

$$\mathcal{O}_N = \left\{ w \in \mathcal{O}_\epsilon : w \subset B_{R_N}, \int_w \frac{g^p}{c^p} = V_0 \right\},$$

where V_0 denotes a fixed positive constant. We look for $\Omega \in \mathcal{O}_N$ and λ_Ω a real such that

$$\begin{gathered} -\Delta_p u = 0 \quad \text{in } \Omega \backslash K \\ u = 0 \quad \text{on } \partial\Omega \\ u = 1 \quad \text{on} \partial K \\ -\frac{\partial u}{\partial \nu} = c_\Omega \quad \text{on } \partial\Omega \end{gathered} \tag{5.5}$$

where $c_\Omega = (\frac{-p}{p-1}\lambda_\Omega)^{\frac{1}{p}} \frac{g}{c}(x)$. Applying proposition (4.1), the shape optimization problem $\min\{J_1(w), w \in \mathcal{O}_N\}$ admits a solution and by proposition 4.4, Ω satisfies the overdetermined boundary condition $-\frac{\partial u}{\partial \nu} = c_\Omega$.

We have $\Omega \in \mathcal{O}_N$, then $\Omega \subset B_{R_N}$, according to the lemma 3.1 there exists $t_0 < 1$ such that $t_0 B_{R_N} \subset \Omega$, and $t_0 \partial B_{R_N} \cap \partial\Omega \neq \emptyset$. Let us take $x_0 \in t_0 \partial B_{R_N} \cap \partial\Omega$ and set $u_{t_0}(x) = u_{R_N}(\frac{x}{t_0})$, $\frac{x}{t_0} \in B_{R_N} \backslash K$. u_{t_0} satisfies

$$\begin{gathered} -\Delta_p u_{t_0} = 0 \quad \text{in } t_0(B_{R_N} \backslash K) \\ u_{t_0} = 0 \quad \text{on } t_0 \partial B_{R_N} \\ u_{t_0} = 1 \quad \text{on } t_0 \partial K. \end{gathered} \tag{5.6}$$

On the other hand, we have $t_0 B_{R_N} \subset \Omega$, let us take $w_3 = u_{|t_0 B_{R_N}}$, then w_3 satisfies

$$\begin{gathered} -\Delta_p w_3 = 0 \quad \text{in } t_0 B_{R_N} \backslash K \\ w_3 = u_{|t_0 \partial B_{R_N}} \quad \text{on } t_0 \partial B_{R_N} \\ w_3 = 1 \quad \text{on } \partial K. \end{gathered} \tag{5.7}$$

Let us consider the problem

$$\begin{gathered} -\Delta_p z = 0 \quad \text{in } t_0 B_{R_N} \backslash K \\ z = 0 \quad \text{on } \partial t_0 \partial B_{R_N} \\ z = u_{t_0 |\partial K} \quad \text{on } \partial K. \end{gathered} \tag{5.8}$$

It is easy to see that $z = u_{t_0}$ is a solution to the problem (5.8). And we get $0 \leq u_{t_0} \leq 1$ and $0 \leq u \leq 1$. On $\partial(t_0 B_{R_N} \backslash K)$, we have $u_{t_0} \leq u$, by the comparison principle [30], we obtain $u_{t_0} \leq u \quad \text{in} (t_0 B_{R_N} \backslash K)$. We have

$$\lim_{t \to 0} \frac{u_{t_0}(x_0 - \nu_e t) - u_{t_0}(x_0)}{t} \leq \lim_{t \to 0} \frac{u(x_0 - \nu_e t) - u(x_0)}{t},$$

which is equivalent to

$$-\frac{\partial u_{t_0}}{\partial \nu_e}(x_0) \leq -\frac{\partial u}{\partial \nu_e}(x_0)\,.$$

This implies

$$\|\nabla u_{R_N}(x_0)\| \leq -\frac{\partial u}{\partial \nu_e(x_0)}$$

Let us consider $\Omega = \Omega_0$ as the first iteration and

$$\mathcal{O}_N^1 = \big\{w \in \mathcal{O}_\epsilon : w \subset \Omega_0 \subset B_{R_N},\ \int_w \frac{g^p}{c^p} = V_1\big\}, \quad (V_1 < V_0)$$

where V_1 denotes a fixed positive constant.

We iterate by looking for $\Omega_1 \in \mathcal{O}_N^1$ and λ_{Ω_1} such that such that

$$\begin{gathered}
-\Delta_p u_1 = 0 \quad \text{in } \Omega_1 \backslash K \\
u_1 = 0 \quad \text{on } \partial\Omega_1 \\
u_1 = 1 \quad \text{on} \partial K \\
-\frac{\partial u}{\partial \nu} = c_{\Omega_1} \quad \text{on } \partial\Omega_1
\end{gathered} \tag{5.9}$$

where $c_{\Omega_1} = (\frac{-p}{p-1}\lambda_{\Omega_1})^{\frac{1}{p}} \frac{g}{c}(x)$. Applying proposition (4.1), the shape optimization problem $\min\{J_2(w), w \in \mathcal{O}_N^1\}$ admits a solution and by proposition 4.4, Ω_1 satisfies the overdetermined boundary condition $-\frac{\partial u_1}{\partial \nu} = c_{\Omega_1}$. We have $\Omega \in \mathcal{O}_N^1$, then $\Omega_1 \subset B_{R_N}$, according the lemma 3.1 there exists $t_1 < 1$ such that $t_1 B_{R_N} \subset \Omega$, then $t_1 \partial B_{R_N} \cap \partial\Omega_1 \neq \emptyset$.

Let us take $x_1 \in t_1 \partial B_{R_N} \cap \partial\Omega_1$ and set $u_{t_1}(x) = u_{R_N}(\frac{x}{t_1})$, $\frac{x}{t_1} \in B_{R_N} \backslash K$. u_{t_1} satisfies

$$\begin{gathered}
-\Delta_p u_{t_1} = 0 \quad \text{in } t_1(B_{R_N} \backslash K) \\
u_{t_1} = 0 \quad \text{on } t_1 \partial B_{R_N} \\
u_{t_1} = 1 \quad \text{on } t_1 \partial K.
\end{gathered} \tag{5.10}$$

On the other hand, we have $t_1 B_{R_N} \subset \Omega_1$, let us take $w_4 = u_{|t_1 B_{R_N}}$, then w_4 satisfies

$$\begin{gathered}
-\Delta_p w_4 = 0 \quad \text{in } t_1 B_{R_N} \backslash K \\
w_4 = u_{1_{|t_1 \partial B_{R_N}}} \quad \text{on } t_1 \partial B_{R_N} \\
w_4 = 1 \quad \text{on } \partial K.
\end{gathered} \tag{5.11}$$

Let us consider the problem

$$\begin{gathered}
-\Delta_p z = 0 \quad \text{in } t_1 B_{R_N} \backslash K \\
z = 0 \quad \text{on } t_1 \partial B_{R_N} \\
z = u_{t_1 |\partial K} \quad \text{on } \partial K.
\end{gathered} \tag{5.12}$$

It is easy to see that $z = u_{t_1}$ is a solution to (5.12). And we get $0 \leq u_{t_1} \leq 1$ and $0 \leq u_1 \leq 1$. On $\partial(t_1 B_{R_N} \backslash K)$, we have $u_{t_1} \leq u_1$, by the comparison principle [30], we obtain $u_{t_1} \leq u_1 \quad \text{in}(t_1 B_{R_N} \backslash K)$. We have

$$\lim_{t \to 0} \frac{u_{t_1}(x_1 - \nu_e t) - u_{t_1}(x_1)}{t} \leq \lim_{t \to 0} \frac{u_1(x_1 - \nu_e t) - u_1(x_1)}{t},$$

which is equivalent to

$$-\frac{\partial u_{t_1}}{\partial \nu_e}(x_1) \leq -\frac{\partial u_1}{\partial \nu_e}(x_1).$$

This implies

$$\|\nabla u_{R_N}(x_1)\| \leq -\frac{\partial u_1}{\partial \nu_e}(x_1).$$

We can continue the process until a step denoted by k such that

$$-\frac{\partial u_k}{\partial \nu_e}(x_k) = c_{\Omega_k}$$

For all $s \in \partial B_{R_N}$, we get $\|\nabla u_0(s)\| \leq \|\nabla u_{R_N}(s)\|$ then there exists $s_0 \in \partial B_{R_N}$, such that $\|\nabla u_0(s_0)\| > c_{\Omega_k}$.

The sequence $(c_{\Omega_j})_{(0 \leq j \leq k)}$ is strictly decreasing and positive, then $(\frac{-p}{p-1}\lambda_{\Omega_j})^{\frac{1}{p}}$ converges on c. Then there exists Ω solution to problem (1.1), the sequence

$(\Omega_j)_{(0\le j\le k)}$ gives a good approximation to Ω. The uniqueness of the solution Ω is given by the monotonicity result.

Case $p \neq N$**.** If $R_1 < R_0$, we get $\|\nabla u_0\|_{|\partial B_{R_1}} \geq \|\nabla u_0\|_{|\partial B_{R_0}}$ then the mapping for all R associates $\|\nabla u_0\|_{|\partial B_R}$ is decreasing. Initially, we choose a radius R_0 big enough and we compute $\|\nabla u_0\|_{|\partial B_{R_0}}$ and if $\big|\|\nabla u_0\|_{|\partial B_{R_0}} - c\big| > \delta, \delta > 0$ fixed and sufficiently small number. We continue the process by varying R in the increasing sense, we will achieve a step denoted N such that $\big|\|\nabla u_0\|_{|\partial B_{R_N}} - c\big| < \delta$. Here the reasoning is identical to the case $p = N$. □

Interior case.

Proof of the Theorem 2.2. Let $R_K = \sup\{R > 0 : B(o,R) \subset K\}$. Let $r > 0$ such that $B(o,r) \subset B(o,R_K)$. First, we have to look for a solution u_0 to the problem

$$\begin{gathered} -\Delta_p u = 0 \quad \text{in } B_{R_K}\backslash B_r \\ u = 0 \quad \text{on } \partial B_{R_K} \\ u = 1 \quad \text{on } \partial B_r. \end{gathered} \tag{5.13}$$

The solution u_0 is explicitly determined by

$$u_0(x) = \begin{cases} \dfrac{\ln\|x\| - \ln R_K}{\ln r - \ln R_K} & \text{if } p = N \\ \dfrac{-\|x\|^{\frac{p-N}{p-1}} + R_K^{\frac{p-N}{p-1}}}{R_K^{\frac{p-N}{p-1}} - r^{\frac{p-N}{p-1}}} & \text{if } p \neq N, \end{cases} \tag{5.14}$$

and

$$\|\nabla u_0(x)\| = \begin{cases} \dfrac{1}{r(\ln R_K - \ln r)} & \text{if } p = N \\ \dfrac{|\frac{p-N}{p-1}|\|x\|^{\frac{-N+1}{p-1}}}{|r^{\frac{p-N}{p-1}} - R^{\frac{p-N}{p-1}}|} & \text{if } p \neq N. \end{cases}$$

In particular $\|\nabla u_0\| > c$ on ∂B_r for r small enough. Now let us consider the problem

$$\begin{gathered} -\Delta_p u = 0 \quad \text{in } K\backslash B_r \\ u = 1 \quad \text{on } \partial B_r \\ u = 0 \quad \text{on } \partial K. \end{gathered} \tag{5.15}$$

Then problem (5.15) admits a solution denoted by u_r. This solution is obtained by minimizing the functional J defined on the Sobolev space

$$V' = \{v \in W^{1,p}(K\backslash B_r), v = 1\text{on } \partial B_r \text{ and } v = 0 \text{ on } \partial K\}$$

and $J(v) = \frac{1}{p}\int_{K\backslash B_r} \|\nabla v\|^p dx$. Consider the problem

$$\begin{gathered} -\Delta_p v = 0 \quad \text{in } B_{R_K}\backslash B_r \\ v = 1 \quad \text{on } \partial B_r \\ v = u_r \quad \text{on } \partial B_{R_K}. \end{gathered} \tag{5.16}$$

It is easy to see that $v = u_r$ is a solution to (5.16). By the comparison principle [30], we obtain $0 \le u_0 \le 1$ and $0 \le u_r \le 1$. On $\partial(B_{R_K}\backslash B_r)$, we obtain $u_r \ge u_0$ and then, $u_r \ge u_0$ in $B_{R_K}\backslash B_r$. Finally, we have $\|\nabla u_r\| \le \|\nabla u_0\|$ on ∂B_r.

Case $p = N$.

$$\|\nabla u_0\|_{|\partial B_r} = \frac{1}{r(\ln R_K - \ln r)} = h(r), \quad \forall r \in]0, R_K[.$$

It is easy to see that $h(r)$ is a strictly decreasing function on $]0, \frac{R_K}{e}[$ and a strictly increasing function on $]\frac{R_K}{e}, R_K[$. Then for all $r \in]0, R_K[$, $\|\nabla u_0\|_{|\partial B_r} \geq h(\frac{R_K}{e}) = \frac{e}{R_K}$.

(1) For $g(x) = e/R_K$, let $\delta > 0$ be a fixed and sufficiently small number. To initialize we choose $r_0 \in]0, \frac{R_K}{e}[\cup]\frac{R_K}{e}, R_K[$ such that $\big|\|\nabla u_0\|_{|\partial B_{r_0}} - c\big| > \delta$. To fix ideas let us consider $r_0 \in]0, \frac{R_K}{e}[$. The process will be identical if $r_0 \in]\frac{R_K}{e}, R_K[$.

By varying r in the increasing sense, we will achieve a step denoted n such that

$$r_n \in]0, \frac{R_K}{e}[\text{and}\big|\|\nabla u_0\|_{|\partial B_{r_n}} - c\big| < \delta.$$

Consider $\mathcal{O}_n$ the class of admissible domains defined as follows

$$\mathcal{O}_n = \big\{w \in \mathcal{O}_\epsilon, B_{r_n} \subset w, \partial B_{r_n} \cap \partial w \neq \emptyset, \text{ and } \int_w \frac{g^p}{c^p} = V_0\big\},$$

where V_0 denotes a fixed positive constant. We look for $\Omega \in \mathcal{O}_n$ and λ_Ω such that

$$\begin{gathered} -\Delta_p u = 0 \quad \text{in } K\backslash\bar{\Omega} \\ u = 1 \quad \text{on } \partial\Omega \\ u = 0 \quad \text{on} \partial K \\ \frac{\partial u}{\partial \nu} = c_\Omega \quad \text{on } \partial\Omega \end{gathered} \tag{5.17}$$

where $c_\Omega = (\frac{p}{p-1}\lambda_\Omega)^{\frac{1}{p}} \frac{g}{c}(x)$. Applying the proposition (4.2), the shape optimization problem $\min\{J_2(w), w \in \mathcal{O}_n\}$ admits a solution and by proposition 4.5, Ω satisfies the overdetermined boundary condition $\frac{\partial u}{\partial \nu} = c_\Omega$. Then problem (5.5) admits a solution.

Since $\Omega \in \mathcal{O}_n$, we have $B_{r_n} \subset \Omega$, $\partial B_{r_n} \cap \partial\Omega \neq \emptyset$ and u_{r_n} satisfies

$$\begin{gathered} -\Delta_p u_{r_n} = 0 \quad \text{in } K\backslash B_{r_n} \\ u_{r_n} = 1 \quad \text{on } \partial B_{r_n} \\ u_{r_n} = 0 \quad \text{on } \partial K. \end{gathered} \tag{5.18}$$

Let us consider the problem

$$\begin{gathered} -\Delta_p z = 0 \quad \text{in } K\backslash\bar{\Omega} \\ z = u_{r_n} \quad \text{on } \partial\Omega \\ z = 0 \quad \text{on } \partial K. \end{gathered} \tag{5.19}$$

It is easy to see that $z = u_{r_n}$ is a solution to (5.19), and we get $0 \leq u_{r_n} \leq 1$ and $0 \leq u \leq 1$. On $\partial(K\backslash\bar{\Omega})$, we have $u_{r_n} \leq u$. Since $\partial\Omega \cap \partial B_{r_n} \neq \emptyset$, let $x_0 \in \partial\Omega \cap \partial B_{r_n}$, we have

$$\lim_{t \to 0} \frac{u_{r_n}(x_0 - \nu t) - u_{r_n}(x_0)}{t} \leq \lim_{t \to 0} \frac{u(x_0 - \nu t) - u(x_0)}{t},$$

This is equivalent to

$$\frac{\partial u_{r_n}}{\partial \nu}(x_0) \geq \frac{\partial u}{\partial \nu}(x_0) = c_\Omega.$$

Let $\Omega = \Omega_0$ as the first iteration. We iterate by looking for $\Omega_1 \in \mathcal{O}_n^1$ such that

$$\begin{gathered} -\Delta_p u_1 = 0 \quad \text{in } K\backslash\bar{\Omega}_1 \\ u_1 = 1 \quad \text{on } \partial\Omega_1 \\ u_1 = 0 \quad \text{on } \partial K \\ \frac{\partial u_1}{\partial\nu} = c_{\Omega_1} \quad \text{on } \partial\Omega_1. \end{gathered} \tag{5.20}$$

where $c_{\Omega_1} = (\frac{p}{p-1}\lambda_{\Omega_1})^{\frac{1}{p}}\frac{g}{c}(x)$, and

$$\mathcal{O}_n^1 = \big\{w \in \mathcal{O}_\epsilon : \Omega_0 \subset w,\ \partial w \cap \partial B_{r_n} \neq \emptyset \int_w \frac{g^p}{c^p} = V_1\big\},$$

where V_1 is a strictly positive constant and $V_0 < V_1$. By the same reasoning as above, we conclude that

$$\frac{\partial u_{r_n}}{\partial\nu}(x_1) \geq \frac{\partial u_1}{\partial\nu}(x_1) = c_{\Omega_1}$$

where $x_1 \in \partial\Omega_1 \cap \partial B_{r_n}$. We can continue the process until a step denoted by k which we will determine and we have

$$\frac{\partial u_{r_n}}{\partial\nu}(x_k) \geq \frac{\partial u_k}{\partial\nu}(x_k) = c_{\Omega_k} \quad \text{and} \quad x_k \in \partial\Omega_k \cap \partial B_{r_n}.$$

Finally, we have constructed an increasing sequence of domain solutions: $\Omega_0 \subset \Omega_1 \subset \Omega_2 \dots \subset \Omega_k$. By the monotonicity result, we have $c_{\Omega_0} \geq c_{\Omega_1} \geq c_{\Omega_2} \dots \geq c_{\Omega_k}$.

Since $\|\nabla u_{r_n}\| \leq \|\nabla u_0\|$ on ∂B_{r_n}, k is chosen as follows: At each point $s_0 \in \partial B_{r_n}$, we have

$$c_{\Omega_k} \leq \frac{\partial u_0}{\partial\nu}(s_0) \leq c_{\Omega_{k-1}}.$$

Then we obtain the inequality

$$c_{\Omega_k} - \frac{e}{R_K} \leq \frac{\partial u_0}{\partial\nu}(s_0) - \frac{e}{R_K} \leq c_{\Omega_{k-1}} - \frac{e}{R_K}. \tag{5.21}$$

The sequence $(c_{\Omega_j})_{(0\leq j\leq k)}$ is decreasing and strictly positive, then it converges on l. Passing to the limit in (5.21), we obtain that $l = \frac{e}{R_K}$ and there exists Ω solution to problem (1.2). The sequence $(\Omega_j)_{(0\leq j\leq k)}$ gives a good approximation to Ω. The uniqueness of the solution Ω is given by the monotonicity result.

(2) For $g(x) > \frac{e}{R_K}$ and $r \in]0, \frac{R_K}{e}[\cup]\frac{R_K}{e}, R_K[$. We have the same reasoning and we show that the problem (1.2) admits a solution.

Case $p \neq N$. Here the reasoning is identical to the case $p = N$. We note that

$$\|\nabla u_0\|_{|\partial B_{r_n}} = |\frac{p-N}{p-1}| \frac{1}{1 - (\frac{r}{R_K})^{\frac{N-p}{p-1}}} \frac{1}{r} = h(r)$$

and h is strictly increasing on $](\frac{p-1}{N-1})^{\frac{p-1}{N-p}} R_K, R_K[$ and a strictly decreasing on $]0, (\frac{p-1}{N-1})^{\frac{p-1}{N-p}} R_K[$. For all

$$g(x) \geq |\frac{p-N}{p-1}| \frac{1}{|(\frac{p-1}{N-1})^{\frac{N-1}{N-p}} - (\frac{p-1}{N-1})^{\frac{p-1}{N-p}}|} \frac{1}{R_K} = h((\frac{p-1}{N-1})^{\frac{p-1}{N-p}} R_K),$$

problem (1.1) admits a solution.

It is easy to have, $0 < c_K \leq \alpha(R_K, p, N)$. If K is a ball of radius R, an explicit computation gives $c_K = \alpha(R, p, N)$ and for all $0 < c < c_K$ problem (1.2) has no solution. $\square$

6. Main result of Part II

Let D_0^* and D_1^* be $\mathcal{C}^2$-regular, compact sets in $\mathbb{R}^N$ and starshaped with respect to the origin such that D_1^* strictly contains D_0^*. We want to find (v, u) solutions of

$$\begin{aligned} -\Delta_p v = 0 \quad &\text{in } D_1^* \backslash D \qquad & -\Delta_p u = 0 \quad &\text{in } D \backslash D_0^* \\ v = 1 \quad &\text{on } \partial D & u = 0 \quad &\text{on } \partial D \\ v = 0 \quad &\text{on } \partial D_1^* & u = 1 \quad &\text{on } \partial D_0^* \end{aligned} \tag{6.1}$$

respectively, and satisfy the non linear joining condition

$$\|\nabla v\|^p - \|\nabla u\|^p = \lambda \quad \text{on } \partial D, \tag{6.2}$$

where λ is a given real $\in \mathbb{R}$.

Theorem 6.1. *Let D_0^* and D_1^* be $\mathcal{C}^2$-regular, compact sets in $\mathbb{R}^N$ and starshaped with respect to the origin such that D_1^* strictly contains D_0^*. One supposes in add that there is $R_0 = \sup\{R > 0 : B(O, R) \in D_1^*\}$and $D_0^* \in B(O, R_0)$ If D, $\mathcal{C}^2$-regular domain solution to the shape optimization problem $\min\{J(w), w \in \mathcal{O}_\epsilon\}$ such that $D_0^* \subset\subset D \subset\subset D_1^*$, then D is a solution of the two-layer free boundary problem (6.1)-(6.2).*

To prove the main result of the Part II, we need to establish some results such as shape optimization and monotonicity results.

7. Shape optimization and monotonicity result

Theorem 7.1. *The problem: Find $D \in \mathcal{O}_\epsilon$ such that $J(D) = \min\{J(w), w \in \mathcal{O}_\epsilon\}$ admits a solution*

Proof. Let E be a functional defined on $W^{1,p}(D_1^*) \times W^{1,p}(D_1^*)$ by

$$E(\tilde{v}, \tilde{u}) = \frac{1}{p} \int_{D_1^*} \|\nabla \tilde{v}\|^p + \frac{1}{p} \int_{D_1^*} \|\nabla \tilde{u}\|^p, \quad 1 < p < \infty,$$

where $\tilde{v}$ is the extension of v in D_0 and $\tilde{u}$ is the extension by 0 in $D_1^* \backslash D$ of u. And v and u are solutions of

$$\begin{aligned} -\Delta_p v = 0 \quad &\text{in } D_1^* \backslash D \qquad & -\Delta_p u = 0 \quad &\text{in } D \backslash D_0^* \\ v = 1 \quad &\text{on } \partial D & u = 0 \quad &\text{on } \partial D \\ v = 0 \quad &\text{on } \partial D_1^* & u = 1 \quad &\text{on } \partial D_0^* \end{aligned} \tag{7.1}$$

Let $J(D) := E(\tilde{v}, \tilde{u})$. It is easy to see that $J(D) \geq 0$, this implies $\inf\{J(w), w \in \mathcal{O}_\epsilon\} > -\infty$. Let $\alpha = \inf\{J(w), w \in \mathcal{O}_\epsilon\}$. Then there exists a minimizing sequence $(D_n)_{(n \in \mathbb{N})} \subset \mathcal{O}_\epsilon$ such that $J(D_n)$ converges to α. Since the sequence is bounded, there exists a compact set F such that $D_0^* \subset\subset \bar{D}_n \subset F \subset\subset D_1^*$. By the lemma 3.4, there exists a subsequence $(D_{n_k})_{(n_k \in \mathbb{N})}$ and D verifying the ϵ-cone property such that

$$\chi_{D_{n_k}} \xrightarrow{L^1} \chi_D \quad \text{and} D_{n_k} \xrightarrow{H} D.$$

It is easy to see the sequence (v_n, u_n) is bounded in $W^{1,p}(D_1^*)$ see [25, 27]. Since $W^{1,p}(D_1^*)$ is a reflexive space, there exists a subsequence (v_{n_k}, u_{n_k}) and (v^*, u^*) such that v_{n_k} converges weakly on v^* in $W^{1,p}(D_1^*)$ and u_{n_k} converges weakly on u^* in $W^{1,p}(D_1^*)$. The norm is lower semi continuous for the weak topology in $W^{1,p}(D_1^*)$, then we have

$$\begin{aligned}&\frac{1}{p}\int_{D_1^*\backslash D}\|\nabla v^*\|^p+\frac{1}{p}\int_{D\backslash D_0^*}\|\nabla u^*\|^p\\ &\geq \liminf(\frac{1}{p}\int_{D_1^*\backslash D_{n_k}}\|\nabla v_{n_k}\|^p+\frac{1}{p}\int_{D_{n_k}\backslash D_0^*}\|\nabla u_{n_k}\|^p).\end{aligned}$$

From the above we get $J(D)\geq\alpha$, then $J(D)=\min\{J(w),w\in\mathcal{O}_\epsilon\}$. $\square$

Remark 7.2. On the one hand, see [25] [26], it is easy to verify that $v=v^*,u=u^*$ and v^*,u^* satisfy

$$\begin{aligned}-\Delta_p v^*=0 \quad &\text{in } \mathcal{D}'(D_1^*\backslash D) & -\Delta_p u^*=0 \quad &\text{in } \mathcal{D}'(D\backslash D_0^*)\\ v^*=1 \quad &\text{on } \partial D & u^*=0 \quad &\text{on } \partial D\\ v^*=0 \quad &\text{on } \partial D_1^* & u^*=1 \quad &\text{on } \partial D_0^*\end{aligned}$$

respectively. On the other hand, we have regularity for v,u as solutions to (7.1); see [11, 21, 31].

Remark 7.3. The remark 4.1 can be stated for the multilayer case. The theorem 4.3 and the lemma 4.4 proved in [26] are valid too for the multilayer case.

For the rest of this article, we assume that D is $\mathcal{C}^2$-regular domain in order to use the shape derivatives. We follow the approach of Sokolowski-Zolesio to define the shape derivatives [29] (see also [28]).

Theorem 7.4. *If D is a solution to the shape optimization problem* $\min\{J(w),w\in\mathcal{O}_\epsilon\}$*, then there exists a Lagrange multiplier function $\lambda_D\in\mathbb{R}$ such that*

$$\|\nabla v\|^p-\|\nabla u\|^p=\frac{p}{p-1}\lambda_D \quad \text{on} \quad \partial D. \tag{7.2}$$

Proof of the theorem 7.4.

$$J(D)=\frac{1}{p}\int_{D_1^*\backslash D}\|\nabla v\|^p+\frac{1}{p}\int_{D\backslash D_0^*}\|\nabla u\|^p,\quad 1<p<\infty,$$

where v and u are solutions of

$$\begin{aligned}-\Delta_p v=0 \quad &\text{in } D_1^*\backslash D & -\Delta_p u=0 \quad &\text{in } D\backslash D_0^*\\ v=1 \quad &\text{on } \partial D & u=0 \quad &\text{on } \partial D\\ v=0 \quad &\text{on } \partial D_1^* & u=1 \quad &\text{on } \partial D_0^*\end{aligned} \tag{7.3}$$

A standard computation , see [22], shows the Euleurian derivative of the functional J at the point D in the direction V is $dJ(D,V)=A+B$, where

$$\begin{aligned}A&=\int_{D_1^*\backslash D}\|\nabla v\|^{p-2}\nabla v'\nabla v dx+\frac{1}{p}\int_{D_1^*\backslash D}\operatorname{div}(\|\nabla v\|^p)V(0))dx\\ B&=\int_{D\backslash D_0^*}\|\nabla u\|^{p-2}\nabla u'\nabla u dx+\frac{1}{p}\int_{D\backslash D_0^*}\operatorname{div}(\|\nabla u\|^p)V(0))dx\end{aligned}$$

By the Green formula, we have

$$A = -\int_{D_1^*\backslash D} \operatorname{div}(\|\nabla v\|^{p-2}\nabla v)v' dx + \frac{1}{p}\int_{\partial(D_1^*\backslash D)} \|\nabla v\|^{p-2}\frac{\partial v}{\partial \nu_1} v' ds$$
$$+ \frac{1}{p}\int_{\partial(D_1^*\backslash D)} \|\nabla v\|^p V(0).\nu_1 ds.$$

In $D_1^*\backslash D$, we have $\operatorname{div}(\|\nabla v\|^{p-2}\nabla v) = 0$, then

$$A = \frac{1}{p}\int_{\partial(D_1^*\backslash D)} \|\nabla v\|^{p-2}\frac{\partial v}{\partial \nu_1} v' ds + \frac{1}{p}\int_{\partial(D_1^*\backslash D)} \|\nabla v\|^p V(0).\nu_1 ds.$$

By the same reasoning, we obtain

$$B = \frac{1}{p}\int_{\partial(D\backslash D_0^*)} \|\nabla u\|^{p-2}\frac{\partial u}{\partial \nu_2} u' ds + \frac{1}{p}\int_{\partial(D\backslash D_0^*)} \|\nabla u\|^p V(0).\nu_2 ds.$$

Let us take $\nu_1 = -\nu_2$ where ν_2 is the exterior normal unit to D. By the computations, see [22], we obtain

$$u' = -\frac{\partial u}{\partial \nu_2} V(0).\nu_2 \quad \text{on } \partial D,$$
$$v' = -\frac{\partial v}{\partial \nu_1} V(0).\nu_1 \quad \text{on } \partial D.$$

This implies

$$A = -\int_{\partial D} \|\nabla v\|^p V(0).\nu_1 ds + \frac{1}{p}\int_{\partial D} \|\nabla v\|^p V(0)\nu_1 ds,$$
$$B = -\int_{\partial D} \|\nabla u\|^p V(0).\nu_2 ds + \frac{1}{p}\int_{\partial D} \|\nabla u\|^p V(0).\nu_2 dx\,.$$

Then we have

$$dJ(D, V) = \frac{1-p}{p}\int_{\partial D} (-\|\nabla v\|^p + \|\nabla u\|^p) V(0).\nu_2 ds.$$

Let us take $J_2(D) = \int_D dx = V_0$, then

$$dJ_2(D, V) = \int_D \operatorname{div}(V(0))dx = \int_{\partial D} V(0).\nu_2\, ds.$$

There exists a Lagrange multiplier $\lambda_D \in R$ such that $dJ(D, V) = \lambda_D dJ_2(D, V)$. We obtain

$$\int_{\partial D} [\frac{1-p}{p}(-\|\nabla v\|^p + \|\nabla u\|^p) - \lambda_D)] V(0).\nu_2 ds = 0 \quad \text{for all} \quad V,$$

then $\|\nabla v\|^p - \|\nabla u\|^p = \frac{p}{p-1}\lambda_D$ on ∂D. $\square$

Remark 7.5. The consequence (D, v, u) in theorems (7.1) and (7.4) satisfies

$$\begin{aligned} -\Delta_p v &= 0 \quad \text{in } D_1^*\backslash D & -\Delta_p u &= 0 \quad \text{in } D\backslash D_0^* \\ v &= 1 \quad \text{on } \partial D & u &= 0 \quad \text{on } \partial D \\ v &= 0 \quad \text{on } \partial D_1^* & u &= 1 \quad \text{on } \partial D_0^* \end{aligned}$$

and satisfy the nonlinear joining condition

$$\|\nabla v\|^p - \|\nabla u\|^p = \frac{p}{p-1}\lambda_D \quad \text{on } \partial D.$$

To conclude this section, we state a monotonicity result, in the following sense.

Theorem 7.6. *Let D_0^* and D_1^* be $\mathcal{C}^2$-regular, compact sets in $\mathbb{R}^N$ and starshaped with respect to the origin such that D_1^* strictly contains D_0^*. Let D_1 and D_2 be two different solutions to the shape optimization problem $\min\{J(w), w \in \mathcal{O}_\epsilon\}$ starshaped with respect to the origin such that $D_1 \subset D_2$ and $\partial D_1 \cap \partial D_2 \neq \emptyset$ then $\lambda_{D_1} \geq \lambda_{D_2}$.*

Proof. For any $i \in \{1,2\}$, if D_i is the solution to the shape optimization problem, we have (v_i, u_i) satisfy the problem

$$
\begin{aligned}
-\Delta_p v_i = 0 \quad &\text{in } D_1^* \backslash D_i \qquad & -\Delta_p u_i = 0 \quad &\text{in } D_i \backslash D_0^* \\
v_i = 1 \quad &\text{on } \partial D_i & u_i = 0 \quad &\text{on } \partial D_i \\
v_i = 0 \quad &\text{on } \partial D_1^* & u_i = 1 \quad &\text{on } \partial D_0^*
\end{aligned}
$$

and the nonlinear joining condition

$$
\|\nabla v_i\|^p - \|\nabla u_i\|^p = \frac{p}{p-1}\lambda_{D_i} \quad \text{on } \partial D_i, \lambda_{D_i} \in R.
$$

Consider the problem

$$
\begin{aligned}
-\Delta_p v_3 = 0 \quad &\text{in } D_1^* \backslash D_2 \\
v_3 = v_1 \quad &\text{on } \partial D_2 \\
v_3 = 0 \quad &\text{on } \partial D_1^*
\end{aligned}
\tag{7.4}
$$

It is easy to see that $v_3 = v_1$ is a solution to (7.4). We get $0 \leq v_2 \leq 1$ and $0 \leq v_1 \leq 1$. On $\partial(D_1^* \backslash D_2)$, we have $v_2 \geq v_1$. By the comparison principle [30], we obtain $v_2 \geq v_1$ in $D_1^* \backslash D_2$.

Let $x_0 \in \partial D_1 \cap \partial D_2$ and ν be the exterior unit normal in x_0, then we get

$$
\frac{v_2(x_0+\nu h) - v_2(x_0)}{h} \geq \frac{v_1(x_0+\nu h) - v_1(x_0)}{h},
$$

By passing to the limit,

$$
\lim_{h\to 0} \frac{v_2(x_0+\nu h) - v_2(x_0)}{h} \geq \lim_{h\to 0} \frac{v_1(x_0+\nu h) - v_1(x_0)}{h},
$$

which implies

$$
\frac{\partial v_2}{\partial \nu}(x_0) \geq \frac{\partial v_1}{\partial \nu}(x_0).
$$

It suffices to remark that $\frac{\partial v_i}{\partial \nu}(x_0) < 0$ $(i = 1,2)$ to conclude that

$$
\|\nabla v_1(x_0)\|^p \geq \|\nabla v_2(x_0)\|^p. \tag{7.5}
$$

Consider the problem

$$
\begin{aligned}
-\Delta_p u_3 = 0 \quad &\text{in } D_1 \backslash D_0^* \\
u_3 = u_2 \quad &\text{on } \partial D_1 \\
u_3 = 1 \quad &\text{on } \partial D_0^*
\end{aligned}
\tag{7.6}
$$

It is easy to see that $u_3 = u_2$ is a solution to (7.6). We get $0 \leq u_1 \leq 1$ and $0 \leq u_2 \leq 1$. On $\partial(D_1 \backslash D_0^*)$, we have $u_2 \geq u_1$. By the comparison principle [30], we obtain $u_2 \geq u_1$ in $D_1 \backslash D_0^*$.

Let $x_0 \in \partial D_1 \cap \partial D_2$, then

$$
\frac{u_2(x_0-\nu h) - u_2(x_0)}{h} \geq \frac{u_1(x_0-\nu h) - u_1(x_0)}{h}.
$$

By passing to the limit,

$$
\lim_{h\to 0} \frac{u_2(x_0-\nu h) - u_2(x_0)}{h} \geq \lim_{h\to 0} \frac{u_1(x_0-\nu h) - u_1(x_0)}{h}
$$

which implies

$$-\frac{\partial u_2}{\partial \nu}(x_0) \geq -\frac{\partial u_1}{\partial \nu}(x_0).$$

That is, $\|\nabla u_2(x_0)\| \geq \|\nabla u_1(x_0)\|$, Then we have $\|\nabla u_2(x_0)\|^p \geq \|\nabla u_1(x_0)\|^p$. This implies

$$-\|\nabla u_1(x_0)\|^p \geq -\|\nabla u_2(x_0)\|^p. \tag{7.7}$$

By combining (7.5) and (7.7),

$$\|\nabla v_1(x_0)\|^p - \|\nabla u_1(x_0)\|^p \geq \|\nabla v_2\|^p - \|\nabla u_2\|^p.$$

Then $\lambda_{D_1} \geq \lambda_{D_2}$. □

8. Proof of the main result of Part II

In this section, we use the preceding theorems to prove the main result.

Proof of the theorem 6.1. Let $R_0 = sup\{R > 0, B(O,R) \subset D_1^*\}$. Let $r_0 > 0, r > 0$ such that $B(O, r_0) \subset D_0^* \subset B(O, r)$. First, we look for v_0 solution of the problem

$$\begin{gathered} -\Delta_p v_0 = 0 \quad \text{in } B_{R_0} \backslash B_r \\ v_0 = 1 \quad \text{on } \partial B_r \\ v_0 = 0 \quad \text{on } \partial B_{R_0} \end{gathered} \tag{8.1}$$

and second u_0 solution of the problem

$$\begin{gathered} -\Delta_p u_0 = 0 \quad \text{in } B_r \backslash B_{r_0} \\ u_0 = 0 \quad \text{on } \partial B_r \\ u_0 = 1 \quad \text{on } \partial B_{r_0} \end{gathered} \tag{8.2}$$

The problem (8.1) admits a solution v_0 which is explicitly determined by

$$v_0(x) = \begin{cases} \dfrac{\ln \|x\| - \ln R_0}{\ln r - \ln R_0} & \text{if } p = N \\ \dfrac{-\|x\|^{\frac{p-N}{p-1}} + R_0^{\frac{p-N}{p-1}}}{R_0^{\frac{p-N}{p-1}} - r^{\frac{p-N}{p-1}}} & \text{if } p \neq N, \end{cases}$$

and

$$\|\nabla v_0(x)\| = \begin{cases} \dfrac{1}{\|x\|(\ln R_0 - \ln r)} & \text{if } p = N \\ \dfrac{|\frac{p-N}{p-1}|\|x\|^{\frac{-N+1}{p-1}}}{|r^{\frac{p-N}{p-1}} - R_0^{\frac{p-N}{p-1}}|} & \text{if } p \neq N. \end{cases}$$

Also the problem (8.2) admits a solution u_0 which is explicitly determined by

$$u_0(x) = \begin{cases} \dfrac{\ln \|x\| - \ln r}{\ln r_0 - \ln r} & \text{if } p = N \\ \dfrac{-\|x\|^{\frac{p-N}{p-1}} - r^{\frac{p-N}{p-1}}}{r_0^{\frac{p-N}{p-1}} - r^{\frac{p-N}{p-1}}} & \text{if } p \neq N, \end{cases}$$

and

$$\|\nabla u_0(x)\| = \begin{cases} \dfrac{-1}{\|x\|(\ln r_0 - \ln r)} & \text{if } p = N \\ \dfrac{|\frac{p-N}{p-1}|\|x\|^{\frac{-N-p+2}{p-1}}}{|r_0^{\frac{p-N}{p-1}} - r^{\frac{p-N}{p-1}}|} & \text{if } p \neq N. \end{cases}$$

On ∂B_r, let us take $h(r) = \|\nabla v_0\|^p - \|\nabla u_0\|^p$.

Now consider the problem

$$\begin{gathered} -\Delta_p v = 0 \quad \text{in } D_1^*\backslash B_r \\ v = 1 \quad \text{on } \partial B_r \\ v = 0 \quad \text{on } \partial D_1^* \end{gathered} \tag{8.3}$$

Problem (8.3) admits a solution denoted by v_r. This solution is obtained by minimizing the functional J_1 on the Sobolev space

$$\mathcal{V}_1 = \{v \in W_0^{1,p}(D_1^*\backslash B_r), v = 1 \text{ on } \partial B_r\} \quad \text{and} \quad J_1(v) = \frac{1}{p}\int_{D_1^*\backslash B_r} \|\nabla v\|^p dx.$$

$$\begin{gathered} -\Delta_p u = 0 \quad \text{in } B_r\backslash D_0^* \\ u = 1 \quad \text{on } \partial D_0^* \\ u = 0 \quad \text{on } \partial B_r \end{gathered} \tag{8.4}$$

Then problem (8.4) admits a solution denoted by u_r. This solution is obtained by minimizing the functional J_2 on the Sobolev space

$$\mathcal{V}_2 = \{u \in W_0^{1,p}(B_r\backslash D_0^*), u = 1 \text{ on } \partial D_0^*\} \quad \text{and} \quad J_2(u) = \frac{1}{p}\int_{B_r\backslash D_0^*} \|\nabla u\|^p dx.$$

Consider the problem

$$\begin{aligned} -\Delta_p v = 0 \quad &\text{in } B_{R_0}\backslash B_r & -\Delta_p u = 0 \quad &\text{in } B_r\backslash D_0^* \\ v = 1 \quad &\text{on } \partial B_r & u = u_0 \quad &\text{on } \partial D_0^* \\ v = v_r \quad &\text{on } \partial B_{R_0} & u = 0 \quad &\text{on } \partial B_r. \end{aligned} \tag{8.5}$$

It is easy to see that $v = v_r$ and $u = u_0$ are respectively solutions to the problem (8.5). We have $0 \leq v_0 \leq 1$ and $0 \leq v_r \leq 1$. We obtain on $\partial(B_{R_0}\backslash B_r)$, $v_r \geq v_0$. By the comparison principle [30], we have $v_r \geq v_0$ in $B_{R_0}\backslash B_r$. Finally, we have $\|\nabla v_r\| \leq \|\nabla v_0\|$ then

$$\|\nabla v_0\|^p \geq \|\nabla v_r\|^p \quad \text{on } \partial B_r. \tag{8.6}$$

Also, we have $0 \leq u_0 \leq 1$ and $0 \leq u_r \leq 1$. We obtain on $\partial(B_r\backslash D_0^*)$, $u_r \geq u_0$. By the comparison principle [30], we have $u_r \geq u_0$ in $B_r\backslash D_0^*$. We get $\|\nabla u_r\| \geq \|\nabla u_0\|$ then

$$-\|\nabla u_0\|^p \geq -\|\nabla u_r\|^p \quad \text{on } \partial B_r. \tag{8.7}$$

By combining (8.6) and (8.7), we obtain

$$\|\nabla v_0\|^p - \|\nabla u_0\|^p \geq \|\nabla v_r\|^p - \|\nabla u_r\|^p \quad \text{on } \partial B_r.$$

Case $p = N$**.** Note that

$$h(r) = \frac{1}{r^p}\left(\frac{1}{(\ln R_0 - \ln r)^p} - \frac{1}{(-\ln r_0 + \ln r)^p}\right), \quad \text{for all } r \in]r_0, R_0[.$$

Let $\delta > 0$ be a fixed and sufficiently small number. To initialize, we choose $r_1 \in]r_0, R_0[$, such that $|h(r_1) - \lambda| > \delta, \lambda \in \mathbb{R}$. By varying r in the increasing sense, we will achieve a step denoted n such that $r_n \in]r_0, R_0[$ and $|h(r_n) - \lambda| < \delta$.

Consider $\mathcal{O}_n$ the class of admissible domains defined as follows

$$\mathcal{O}_n = \{w \in \mathcal{O}_\epsilon, B_{r_n} \subset w, \partial B_{r_n} \cap \partial w \neq \emptyset \quad \text{and} \quad \mathrm{vol}(w) = V_1\},$$

where V_1 denotes a fixed positive constant. We look for $D \in \mathcal{O}_n$ such that

$$\begin{aligned} -\Delta_p v &= 0 \quad \text{in } D_1^* \backslash D & \qquad -\Delta_p u &= 0 \quad \text{in } D \backslash D_0^* \\ v &= 1 \quad \text{on } \partial D & u &= 0 \quad \text{on } \partial D \\ v &= 0 \quad \text{on } \partial D_1^* & u &= 1 \quad \text{on } \partial D_0^* \end{aligned} \tag{8.8}$$

and satisfies the nonlinear joining condition

$$\|\nabla v\|^p - \|\nabla u\|^p = \frac{p}{p-1}\lambda_D \quad \text{on } \partial D. \tag{8.9}$$

Applying the theorem 7.1, the shape optimization problem $\min\{J(w), w \in \mathcal{O}_n\}$ admits a solution D and by the theorem 7.4, D satisfies the joining condition (8.9). Since $D \in \mathcal{O}_n$, we have $B_{r_n} \subset D$ and $\partial B_{r_n} \cap \partial D \neq \emptyset$ and v_{r_n} respectively u_{r_n} satisfy

$$\begin{aligned} -\Delta_p v_{r_n} &= 0 \quad \text{in } D_1^* \backslash B_{r_n} & \qquad -\Delta_p u_{r_n} &= 0 \quad \text{in } B_{r_n} \backslash D_0^* \\ v_{r_n} &= 1 \quad \text{on } \partial B_{r_n} & u_{r_n} &= 0 \quad \text{on } \partial B_{r_n} \\ v_{r_n} &= 0 \quad \text{on } \partial D_1^* & u_{r_n} &= 1 \quad \text{on } \partial D_0^*. \end{aligned} \tag{8.10}$$

Consider the problem

$$\begin{aligned} -\Delta_p z &= 0 \quad \text{in } D_1^* \backslash D & \qquad -\Delta_p g &= 0 \quad \text{in } D \backslash D_0^* \\ z &= v_{r_n} \quad \text{on } \partial D & g &= 0 \quad \text{on } \partial D \\ z &= 0 \quad \text{on } \partial D_1^* & g &= u_{r_n} \quad \text{on } \partial D_0^*. \end{aligned} \tag{8.11}$$

It is easy to see that $z = v_{r_n}$ and $g = u_{r_n}$ are respectively solutions to problem (8.11). We get $0 \leq v_{r_n} \leq 1$ and $0 \leq v \leq 1$. We have on $\partial(D_1^* \backslash D)$, $v_{r_n} \leq v$. By the comparison principle [30], we obtain $v_{r_n} \leq v$ in $(D_1^* \backslash D)$. Since $\partial B_{r_n} \cap \partial D \neq \emptyset$, let's take $x_1 \in \partial B_{r_n} \cap \partial D$, we have by passing to the limit

$$\lim_{h \to 0} \frac{v_{r_n}(x_1 + \nu h) - v_{r_n}(x_1)}{h} \leq \lim_{h \to 0} \frac{v(x_1 + \nu h) - v(x_1)}{h},$$

this is equivalent to (where ν is the exterior normal to D)

$$\|\nabla v_{r_n}(x_1)\|^p \geq \|\nabla v(x_1)\|^p \tag{8.12}$$

We get $0 \leq u \leq 1$ and$0 \leq u_{r_n} \leq 1$. We have on $\partial(D \backslash D_0^*)$, $u_{r_n} \leq u$. By the comparison principle [30], we obtain $u_{r_n} \leq u$ in $(D \backslash D_0^*)$. Since $\partial B_{r_n} \cap \partial D \neq \emptyset$, let us take $x_1 \in \partial B_{r_n} \cap \partial D$, we have by passing to the limit

$$\lim_{h \to 0} \frac{u_{r_n}(x_1 - \nu h) - u_{r_n}(x_1)}{h} \leq \lim_{h \to 0} \frac{u(x_1 - \nu h) - u(x_1)}{h},$$

that is

$$-\|\nabla u_{r_n}(x_1)\|^p \geq -\|\nabla u(x_1)\|^p \tag{8.13}$$

By combining the relations (8.12) and (8.13), we obtain

$$\|\nabla v_{r_n}(x_1)\|^p - \|\nabla u_{r_n}(x_1)\|^p \geq \|\nabla v(x_1)\|^p - \|\nabla u(x_1)\|^p.$$

Let us take $D = D_1$ as the first iteration. We iterate by looking $D_2 \in \mathcal{O}_n^2$ such that

$$\begin{aligned} -\Delta_p v_2 &= 0 \quad \text{in } D_1^* \backslash D_2 & \qquad -\Delta_p u_2 &= 0 \quad \text{in } D_2 \backslash D_0^* \\ v_2 &= 1 \quad \text{on } \partial D_2 & u_2 &= 0 \quad \text{on } \partial D_2 \\ v_2 &= 0 \quad \text{on } \partial D_1^* & u_2 &= 1 \quad \text{on } \partial D_0^*, \end{aligned} \tag{8.14}$$

and satisfies the nonlinear joining condition

$$\|\nabla v_2\|^p - \|\nabla u_2\|^p = \frac{p}{p-1}\lambda_{D_2} \quad \text{on } \partial D_2. \tag{8.15}$$

Also

$$\mathcal{O}_n^2 = \{w \in \mathcal{O}_\epsilon, D_1 \subset w, \partial B_{r_n} \cap \partial w \neq \emptyset \quad \text{and } \operatorname{vol}(w) = V_2\},$$

where V_2 is a strictly positive constant and $V_1 < V_2$. By the same reasoning as above, we obtain

$$\|\nabla v_{r_n}(x_2)\|^p - \|\nabla u_{r_n}(x_2)\|^p \geq \|\nabla v(x_2)\|^p - \|\nabla u(x_2)\|^p \quad \text{on } \partial B_{r_n}.$$

We can continue the process until a step denoted k, which we will be determined, and we have

$$\|\nabla v_{r_n}(x_k)\|^p - \|\nabla u_{r_n}(x_k)\|^p < \|\nabla v(x_k)\|^p - \|\nabla u(x_k)\|^p \quad \text{and} x_k \in \partial D_k \cap \partial B_{r_n}.$$

Finally, we constructed an increasing sequence of domain solutions

$$D_1 \subset D_2 \subset \cdots \subset D_{k-1} \subset D_k.$$

By the monotonicity result, in theorem 7.6, we have

$$\lambda_{D_1} \geq \lambda_{D_2} \geq \cdots \geq \lambda_{D_{k-1}} \geq \lambda_{D_k}.$$

Since $\|\nabla v_{r_n}(x_k)\|^p - \|\nabla u_{r_n}(x_k)\|^p \leq \|\nabla v_0\|^p - \|\nabla u_0\|^p$ on ∂B_{r_n}, k is chosen as follows in each point $s_0 \in \partial B_{r_n}$,

$$\lambda_{D_k} \leq h(s_0) \leq \lambda_{D_{k-1}}.$$

Then we obtain the inequality

$$\lambda_{D_k} - \lambda \leq h(s_0) - \lambda \leq \lambda_{D_{k-1}} - \lambda. \tag{8.16}$$

The sequence $(\lambda_{D_j})_{(0 \leq j \leq k)}$ is decreasing and underestimated because we cannot indefinitely generate a sequence domains if not we will leave D_1^*. We have $\lambda_{D_k} \geq \lambda_{D'_*}$ where D'_* is the greatest domain contained in $D_1^*, \partial D'_* \cap \partial B_{r_n} \neq \emptyset . D'_*$ is solution to the shape optimization problem $\min\{J(w), w \in \mathcal{O}_n\}$ and for all k, we have $D_k \subset D'_*$.

The sequence $(\lambda_{D_j})_{(0 \leq j \leq k)}$ converges to l. By passing to the limit in (8.16), we obtain $l = \lambda$ and there exists D solution to problem (1.5)-(1.6). The sequence $(D_j)_{(0 \leq j \leq k)}$ gives a good approximation to D.

Case $p \neq N$**.** Here the reasoning is identical to the case $p = N$. We note that

$$h(r) = (|\frac{p-N}{N-1}|)^p (r^{-|\frac{N-1}{p-1}|})^p \Big(\frac{1}{|r^{\frac{p-N}{p-1}} - R_0^{\frac{p-N}{p-1}}|^p} - \frac{1}{(r|r_0^{\frac{p-N}{p-1}} - r^{\frac{p-N}{p-1}}|)^p} \Big),$$

for all $r \in]r_0, R_0[$. $\square$

9. The multi-layer case

Let D_0 and D_{k+1} be $\mathcal{C}^2$-regular , compact sets in $\mathbb{R}^N$ and starshaped with respect to the origin such that D_{k+1} strictly contains D_0. One supposes that there is R_0 such that $D_0 \subset B(0, R_0) \subset D_{k+1}$ where $R_0 = \sup\{R > 0 : B(0, R) \subset D_{k+1}\}$. We find a sequence of domains, $\mathcal{C}^2$-regular, starshaped with respect to the origin and solution to the shape optimization problem $\min\{J(w), w \in \mathcal{O}_\epsilon\}$, $D_0 \subset D_1 \subset D_2 \subset \cdots \subset D_k \subset D_{k+1}$ such that (D_i, v_i, u_i) is solution of

$$\begin{aligned} -\Delta_p v_i &= 0 \quad \text{in } D_{i+1}\backslash D_i & -\Delta_p u_i &= 0 \quad \text{in } D_i\backslash D_{i-1} \\ v_i &= 1 \quad \text{on } \partial D_i & u_i &= 0 \quad \text{on } \partial D_i \\ v_i &= 0 \quad \text{on } \partial D_{i+1} & u_i &= 1 \quad \text{on } \partial D_{i-1} \end{aligned} \tag{9.1}$$

and satisfy the non linear joining condition

$$\|\nabla v_i\|^p - \|\nabla u_i\|^p = \frac{p}{p-1}\lambda_i \quad \text{on } \partial D_i, \lambda_i \in \mathbb{R}, 1 \le i \le k. \tag{9.2}$$

Theorem 9.1. *Let D_0 and D_{k+1} be $\mathcal{C}^2$-regular , compact sets in $\mathbb{R}^N$ and starshaped with respect to the origin such that D_{k+1} strictly contains D_0. Then there exists a sequence domains $(D_i)_{(1\le i\le k)}$, $\mathcal{C}^2$-regular domain solution to the shape optimization problem* $\min\{J(w), w \in \mathcal{O}_\epsilon\}$ *such that $D_0 \subset D_1 \subset D_2 \cdots \subset D_k \subset D_{k+1}$ solution of the multi-layer free boundary problem (9.1)-(9.2).*

To prove this theorem, we use the method presented in the proof of the two layer case. In fact we consider at first the domains D_0 and D_{k+1}. And according to the two layer case there is D_1 ($D_0 \subset D_1 \subset D_{k+1}$) which is solution to the problem. And sequentially, we seek D_i ($D_{i-1} \subset D_i \subset D_{k+1}, i = 2, \cdots k$).It is always possible to invoke the two layer case in order to solve these types of problems.

References

[1] Acker , A., *On the multi-layer fluid problem: regularity, uniqueness, convexity and successive approximation of solutions*, Comm. Part. Diff.Eq, **16**(1991), 674-666.

[2] Acker , A., *On the existence of convex solutions for multi-layer free boundary problems with general nonlinear joining condition*, Trans.Amer.Math.Soc, **350**(1998), no. 8, 2981-3020.

[3] Acker , A., *On the non convexity of solutions in free boundary problems arising in plasma physics and fluid dynamics,* Comm. Pure. Appl.Math., **42**(1989), 1165-1174.(Addendum: Comm. Pure. Appl.Math., **44**(1991), no. 7, 869-872.

[4] Acker , A., *Qualitative properties of the boundary derivative of the capacity potential for special classes of annular domains,* Math.Meth.in the Appl.Sci., **7**(1985), 251-259.

[5] Acker , A., *On 2-layer free boundary problems with generalized joining conditions. In: Comparison Methods and Stability Theory* (edited by Xinzhi Liu and David Siegel). Lecture notes in pure and applied mathematics, Vol. 162. Marcel dekker, Inc .1994.

[6] Acker, A, Meyer, R. A.; *A free boundary problem for the p-Laplacian: uniqueness,convexity, and sucessive approximation of solutions,* Electronic Journal of Differential Equations, **8**(1995), 1-20.

[7] Acker, A.,Henrot, A.,Poghosyan,M. and Shahgholian, H.; *The multi-layer free boundary problem for the $p-$Laplacian in convex domains,* Interfaces and Free boundaries, **6**(2004), 81-103.

[8] H. W. Alt and L. A. Caffarelli, *Existence and regularity for a minimum problem with free boundary,* J. Reine Angew. Math., **325**(1981), 105-144.

[9] Delfour, M. Zolesio, J.P.; *Shape analysis via oriented distance functions* , Journal of Functional Analysis **123**, 129-201,(1994)

[10] Delfour M., Zolesio J.P.; *Shapes and Geometries, Analysis, Differential Calculus, and Optimization* Advances in Design and Control, SIAM 2001.

[11] DiBenedetto, E.; $\mathcal{C}^{1+\alpha}$ *local regularity of weak solutions of degenerate elliptic equations,* Nonlinear Analysis, **7**(1983), 827-850.

[12] Dobrowolski, M., *On quasilinear elliptic equations in domains with conical boundary points,* J. Reine Angew. Math., **394**(1989), 186-195.
[13] Flucher, M. and Rumpf, M.; *Bernoulli's free boundary problem, qualitative theory and numerical approximation,* J. Reine Angew. Math, **486**(1997), 165-204.
[14] Henrot, A, Shahgholian, H.; *Existence of classical solutions to a free boundary problem for the p-Laplace operator (I): the exterior convex case,* J. Reine Angew. Math. **521**(2000), 85-97.
[15] Henrot, A, Shahgholian, H.; *Existence of classical solutions to a free boundary problem for the p-Laplace operator (II): the interior convex case,* Indiana Univ. Math. Journal, **49** (2000), No.1, 301-323.
[16] Henrot, A, Shahgholian, H.; *The one phase free boundary problem for the p-Laplacian with non-constant Bernoulli boundary condition,* Trans. Amer. Math. Soc. **354**(2002), no. 6, 2399-2416.
[17] Kavian, O.; *Introduction à la théorie des points critiques et applications aux problèmes elliptiques,* Springer-Verlag, France, Paris, 1993.
[18] Laurence, P., and Stredulinsky, E., *A New Approach to Queer Differential Equations,* Comm.in Pure Appl.Math., **38**(1985), 333-355.
[19] Laurence, P., and Stredulinsky, E., *Existence of regular solutions with convex levels sets for semilinear elliptic equations with nonmonotone L^1 nonlinearities. Part (I),* Indiana Univ. Math. J. vol. 39 (4)(1990), pp. 1081-1114.
[20] Lavrent'ev, M. A.; *Variational methods for boundary value problems for systems of elliptic equations,* Noordhoff, 1963.
[21] Lewis, J. L.; *Regularity of the derivatives of solutions to certain degenerate elliptic equations,* Indiana Univ . Math. J., **32**(1983), 849-858.
[22] Ly, I.; *Thèse de doctorat de troisième cycle: Résultat d'existence en optimisation de forme et étude d'un problème extérieur à frontière libre: cas du p-laplacien,* Soutenue le 20 juin 2002 , UGB, Sénégal.
[23] Ly, I., Seck, D.; *Etude d'un problème à frontière libre pour le p-laplacien,* CRAS, Paris **332**(2001), Série I, 899-902.
[24] Ly, I., Seck, D.; *Estimation de la constante de Bernoulli dans le problème intérieur cas du p-laplacien,* CRAS, Paris **337**(2003), Série I, 393-396.
[25] Ly, I., Seck, D.; *Optimisation de forme et problème à frontière libre pour le p-laplacien,* Ann. Fac. Sci. Toulouse Math. (6) 12 (2003), no.1, 103-126.
[26] Ly, I. and Seck ,D., *Isoperimetric inequality for an interior free boundary problem with p-laplacian operator,* Electronic Journal of Differential Equations, **2004**(2004), n 109, pp 1-12.
[27] Ly, I., Seck ,D.; *About free boundary problem with quasi linear operator and non constant Bernoulli boundary condition,* to appear.
[28] Simon, J.; *Differential with respect to the domain in boundary value problems,* Numer. Funct. Anal. and Optimiz. **2** (7 - 8), 649-687(1980).
[29] Sokolowski, J., Zolesio, J. P.; *Introduction to Shape Optimization: shape sensitivity analysis,* Springer Series in Computational Mathematics,Vol. 10,Springer,Berlin, 1992.
[30] Tolksdorf, P.; *On the Dirichlet problem for quasilinear equations in domains with conical boundary points,* Comm. Partial Differential Equations, **8**(1983), 773-817.
[31] Tolksdorf, P.; *Regularity for a more general class of quasilinear elliptic equations,* J. Differential Equations, **52**(1984), 126-150.

Faculté des Sciences Economiques et de Gestion, Université Cheikh Anta Diop, B.P 5683, Dakar, Sénégal

E-mail address: `ndirkaly@ugb.sn`

Faculté des Sciences Economiques et de Gestion, Université Cheikh Anta Diop, B.P 5683, Dakar, Sénégal

E-mail address: `dseck@ucad.sn`