Electronic Journal of Differential Equations, Vol. 2006(2006), No. 129, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

EXISTENCE OF POSITIVE SOLUTIONS FOR MULTI-TERM NON-AUTONOMOUS FRACTIONAL DIFFERENTIAL EQUATIONS WITH POLYNOMIAL COEFFICIENTS

AZIZOLLAH BABAKHANI, VARSHA DAFTARDAR-GEJJI

Abstract. In the present paper we discuss the existence of positive solutions in the case of multi-term non-autonomous fractional differential equations with polynomial coefficients; the constant coefficient case has been studied in [2]. We consider the equation

$$\Big(D^{\alpha_n} - \sum_{j=1}^{n-1} p_j(x) D^{\alpha_{n-j}}\Big) y = f(x, y).$$

We state various conditions on f and p_j's under which this equation has: positive solutions, a unique solution which is positive, and a unique solution which may not be positive.

1. Introduction

Let E be a real Banach space with a cone $K \subset E$. K introduces a partial order $\leq$ in $E : x \leq y$ if and only if $y - x \in E$. A cone K is said to be normal, if there exists a positive constant τ such that $\theta \leq f \leq g$ implies $\|f\| \leq \tau \|g\|$, where θ denotes the zero element of K. For $x, y \in E$ the order interval $\langle x, y\rangle$ is define to be [4]:

$$\langle x, y\rangle = \{z \in E : x \leq z \leq y\}$$

Theorem 1.1 ([4]). *Let K be a normal cone in a partially ordered Banach space E. Let F be an increasing operator which transforms $\langle x_0, y_0\rangle$ into itself; i. e., $Fx_0 \geq x_0$ and $Fy_0 \leq y_0$. Assume further that F is compact and continuous. Then F has at least one fixed point $x^* \in \langle x_0, y_0\rangle$.*

Theorem 1.2 (Banach fixed point theorem [4]). *Let K be a closed subspace of a Banach space E. Let F be a contraction mapping with Lipschitz constant $k < 1$ from K to itself. Then F has a unique fixed point x^* in K. Moreover if x_0 is an arbitrary point in K and $\{x_n\}$ is defined by $x_{n+1} = Fx_n$, ($n = 0, 1, 2, \dots$) then $\lim_{n\to\infty} x_n = x^* \in K$ and $d(x_n, x^*) \leq (k^n/(1-k))\, d(x_1, x_0)$.*

2000 *Mathematics Subject Classification.* 26A33, 34B18.
Key words and phrases. Riemann-Liouville fractional derivatives and integrals; normal cone; semi-ordered Banach space; completely continuous operator; equicontinuous set.

Submitted July 27, 2005. Published October 16, 2006.

Definition 1.3. The left sided Riemann-Liouville fractional integral [5, 6, 7] of order α of a real function f is defined as

$$I_{a+}^{\alpha} y(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{y(t)}{(x-t)^{1-\alpha}} dt, \quad \alpha > 0,\ x > a. \tag{1.1}$$

Definition 1.4. The left sided Riemann-Liouville fractional derivative [5, 6, 7] of order α of a function f is

$$D_{a+}^{\alpha} y(x) = \frac{d^n}{dx^n} \left[I_{a+}^{n-\alpha} y(x) \right], \quad n-1 \le \alpha < n, \quad n \in \mathbb{N}. \tag{1.2}$$

We denote D_{a+}^{α} by $D_a^{\alpha} y(x)$ and $I_{a+}^{\alpha} y(x)$ by $I_a^{\alpha} y(x)$. Also $D^{\alpha} y(x)$ and $I^{\alpha} y(x)$ refers to $D_{0+}^{\alpha} y(x)$ and $I_{0+}^{\alpha} y(x)$, respectively.

Proposition 1.5. *(i) If the fractional derivative $D_a^{\alpha} y(x)$ is integrable, then*

$$I_a^{\alpha}(D_a^{\beta} y(x)) = I_a^{\alpha-\beta} y(x) - \left[I_a^{1-\beta} y(x) \right]_{x=a} \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}, \quad 0 < \beta \le \alpha < 1\,. \tag{1.3}$$

(ii) If y is continuous on $[a,b]$, then $D_a^{\alpha} y(x)$ is integrable, $I^{1-\beta} y(x)|_{x=a} = 0$ and

$$I_a^{\alpha} \left(D_a^{\beta} y(x) \right) = I_a^{\alpha-\beta} y(x), \quad 0 < \beta \le \alpha < 1. \tag{1.4}$$

Proof. For (i), we refer the reader to [6]. For (ii), let $M = \max_{a \le x \le b} y(x)$ then, using (1.2) we get

$$\Big| \int_a^x D_a^{\alpha} y(t)\, dt \Big| \le \frac{M}{\Gamma(1-\alpha)} \int_a^x (x-t)^{-\alpha} dt = \frac{M(x-a)^{1-\alpha}}{\Gamma(2-\alpha)},$$

so $D_a^{\alpha} y(t)$ is integrable. On the other hand

$$|I_a^{1-\beta} y(x)|_{x=a} \le \frac{M}{\Gamma(1-\beta)} \Big[\int_a^x (x-t)^{-\beta} dt \Big]_{x=a} = \frac{M}{\Gamma(2-\beta)} \big[(x-a)^{1-\beta} \big]_{x=a} = 0,$$

and hence (1.3) reduces to $I_a^{\alpha}(D_a^{\beta} y(x)) = I_a^{\alpha-\beta} y(x)$. □

Proposition 1.6. *Let y be continuous on $[0,\lambda]$, $\lambda > 0$ and n be a non negative integer, then*

$$I^{\alpha}(x^n y(x)) = \sum_{k=0}^{n} \binom{-\alpha}{k} \left[D^k x^n \right] \left[I^{\alpha+k} y(x) \right] = \sum_{k=0}^{n} \binom{-\alpha}{k} \frac{n! x^{n-k}}{(n-k)!} I^{\alpha+k} y(x), \tag{1.5}$$

where

$$\binom{-\alpha}{k} = (-1)^k \frac{\Gamma(\alpha+1)}{n!\Gamma(\alpha)} = (-1)^k \binom{\alpha+k-1}{k} = \frac{\Gamma(1-\alpha)}{\Gamma(k+1)\Gamma(1-\alpha-k)}. \tag{1.6}$$

The proof of the above proposition can be found in [5, p. 53].

Corollary 1.7. *Let $y \in C[0,\lambda]$, $\lambda > 0$ and $p_j(x) = \sum_{k=0}^{N_j} a_{jk} x^k$, $N_j \in \mathbb{N} \cup \{0\}$, $j = 1, 2, \dots, n$. Then*

$$I^{\alpha}\Big(\sum_{j=1}^{n} p_j(x) y(x) \Big) = \sum_{j=1}^{n} \sum_{k=0}^{N_j} \sum_{r=0}^{k} a_{jk} \binom{-\alpha_n}{r} \frac{k! x^{k-r}}{(k-r)!} [I^{\alpha+r} y(x)]. \tag{1.7}$$

Proof. Using $I^\alpha(x^n y(x)) = \sum_{k=0}^n \binom{-\alpha}{k}[D^k x^n][I^{\alpha+k}y(x)]$ and $D^r(x^k) = \frac{k!x^{k-r}}{(k-r)!}$ we have

$$\begin{aligned} I^\alpha\Big(\sum_{j=1}^n p_j(x)y(x)\Big) &= \sum_{j=1}^n I^\alpha(p_j(x)y(x)) \\ &= \sum_{j=1}^n\sum_{k=0}^{N_j} a_{jk} I^\alpha(x^k y(x)) \\ &= \sum_{j=1}^n\sum_{k=0}^{N_j} a_{jk}\Big[\sum_{r=0}^k \binom{-\alpha_n}{r}(D^r x^k) I^{\alpha+r}y(x)\Big] \\ &= \sum_{j=1}^n\sum_{k=0}^{N_j}\sum_{r=0}^k a_{jk}\binom{-\alpha_n}{r}\frac{k!x^{k-r}}{(k-r)!}[I^{\alpha+r}y(x)]. \end{aligned} \tag{1.8}$$

□

2. Existence of Positive Solutions

In this section we discuss conditions under which the following fractional initial-value probelm has a positive solution.

$$\Big(D^{\alpha_n} - \sum_{j=1}^{n-1} p_j(x) D^{\alpha_{n-j}}\Big) y = f(x,y), \quad y(0) = 0, \quad 0 \le x \le \lambda, \lambda > 0, \tag{2.1}$$

where $0 < \alpha_1 < \alpha_2 < \dots < \alpha_n < 1$; $p_j(x) = \sum_{k=0}^{N_j} a_{jk}x^k, p_j^{(2m)}(x) \ge 0$, $p_j^{(2m+1)}(x) \le 0$, $m = 0, 1, \dots, [\frac{N_j}{2}]$, $j = 1, 2, \dots, n-1$, D^{α_j} is the standard Riemann-Liouville fractional derivative and $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ a given continuous function. Let us denote by $Y = C[0,\lambda]$, the Banach space of all continuous real functions on $[0,\lambda]$ endowed with the sup norm and K be the cone:

$$K = \{y \in Y : y(x) \ge 0, 0 \le x \le \lambda\}.$$

Definition 2.1. By a solution of (2.1), we mean a continuous function $y \in C[0,\lambda]$, that satisfies (2.1).

We remark that in [7] the initial-value problems

$$\begin{gathered} D_0^\alpha y(x) = f(x,y), \quad 0 < \alpha < 1, \\ I_0^{1-\alpha} y(x)|_{x=0} = b, \quad 0 < \alpha < 1, \end{gathered}$$

are studied where D_0^α denotes the Riemann-Liouville derivative and the underlying space of functions is $C(0,\lambda)$. However, in the present paper we are dealing with the space of functions $C[0,\lambda]$. For $y(x) \in C[0,\lambda]$, always $I_0^{1-\alpha}y(x)|_{x=0} = 0$, and is not free data.

Lemma 2.2. *The fractional initial-value problem* (2.1) *is equivalent to the Volterra integral equation*

$$y(x) = \sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^k a_{jk}\binom{-\alpha_n}{r}\frac{k!x^{k-r}}{(k-r)!} I^{\alpha_n-\alpha_{n-j}+r}y(x) + I^{\alpha_n}f(x,y(x)). \tag{2.2}$$

Proof. Suppose $y(x)$ satisfies (2.1), then

$$I^{\alpha_n}\big[\big(D^{\alpha_n}-\sum_{j=1}^{n-1}p_j(x)D^{\alpha_{n-j}}\big)y\big]=I^{\alpha_n}f(x,y).$$

Proposition 1.5 (ii) yields $I^{1-\alpha_n}y(x)\big|_{x=0}=0$, $I^{1-\alpha_n-\alpha_{n-j}}y(x)\big|_{x=0}=0$, hence $I^{\alpha_n}(D^{\alpha_n}y(x))=y(x)$ and using (1.7) we obtain the integral equation (2.2). Conversely, let $y(x)$ satisfy the integral equation (2.2). Then

$$\begin{aligned}
&\sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k}a_{jk}\binom{-\alpha_n}{r}\frac{k!x^{k-r}}{(k-r)!}I^{\alpha_n-\alpha_{n-j}+r}y(x)+I^{\alpha_n}f(x,y(x))\\
&=\sum_{j=1}^{n-1}\sum_{k=0}^{N_j}a_{jk}\Big[\sum_{r=0}^{k}\binom{-\alpha_n}{r}D^r x^k I^{\alpha_n-\alpha_{n-j}+r}y(x)\Big]+I^{\alpha_n}f(x,y(x))\\
&=\sum_{j=1}^{n-1}\sum_{k=0}^{N_j}a_{jk}I^{\alpha_n}(x^kD^{\alpha_{n-j}}y(x))+I^{\alpha_n}f(x,y(x))\\
&=\sum_{j=1}^{n-1}I^{\alpha_n}(p_j(x)D^{\alpha_{n-j}}y(x))+I^{\alpha_n}f(x,y(x))\\
&=I^{\alpha_n}(\sum_{j=1}^{n-1}[p_j(x)D^{\alpha_{n-j}}y(x))]+f(x,y(x)))=y(x)
\end{aligned}$$

But $y(x)=I^{\alpha_n}(D^{\alpha_n}y(x))$, hence $y(x)$ satisfies (2.1), and $y(0)=0$. □

Lemma 2.3. *If $p_j^{(2m)}(x)\geq 0$ and $p_j^{(2m+1)}(x)\leq 0$ for $m=0,1,\dots,[\frac{N_j}{2}]$ where $N_j=\deg(p_j)$, $j=1,2,\dots,n-1$ and p_j's are as in (2.1), then F defined as*

$$Fy(x)=\sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k}a_{jk}\binom{-\alpha_n}{r}\frac{k!x^{k-r}}{(k-r)!}I^{\alpha_n-\alpha_{n-j}+r}y(x)+I^{\alpha_n}f(x,y(x)),\quad(2.3)$$

is from K to itself.

Proof. The right-hand side of (2.3) can be expressed as:

$$\begin{aligned}
Fy(x)=&\sum_{j=1}^{n-1}\binom{-\alpha_n}{0}\Big(\sum_{k=0}^{N_j}a_{jk}x^k\Big)I^{\alpha_n-\alpha_{n-j}}y(x)\\
&+\sum_{j=1}^{n-1}\binom{-\alpha_n}{1}\Big(\sum_{k=1}^{N_j}ka_{jk}x^{k-1}\Big)I^{\alpha_n-\alpha_{n-j}+1}y(x)\\
&+\sum_{j=1}^{n-1}\binom{-\alpha_n}{2}\Big(\sum_{k=2}^{N_j}k(k-1)a_{jk}x^{k-2}\Big)I^{\alpha_n-\alpha_{n-j}+2}y(x)+\dots\\
&+\sum_{j=1}^{n-1}\binom{-\alpha_n}{N_j}(N_j!a_{jN_j})I^{\alpha_n-\alpha_{n-j}+N_j}y(x)\\
=&\sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\binom{-\alpha_n}{k}p_j^{(k)}(x)I^{\alpha_n-\alpha_{n-j}+k}y(x)
\end{aligned}$$

In view of (1.6), we have
$$\binom{-\alpha_n}{2m} > 0, \quad \binom{-\alpha_n}{2m+1} < 0, \quad m \in \mathbb{N}.$$
Then by assumptions on $p_j^{(k)}(x)$, $k = 0, 1, \dots, N_j$, $j = 1, 2, \dots, n-1$ we get $Fy(x) \in K$. $\square$

Furthermore, it is easy to show the following result.

Lemma 2.4. *The operator $F : K \to K$ defined in Lemma 2.3 is completely continuous.*

Lemma 2.5. *Let $M \subset K$ be bounded; i.e. there exists a positive constant l such that $\|y\| \le l$, for all $y \in M$. Then $\overline{F(M)}$ is compact.*

Proof. Let $L = \max\{1 + f(x, y) : 0 \le x \le 1, 0 \le y \le l\}$. For $y \in M$, we have
$$|F(y(x))| \le \sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k} \Big|a_{jk}\binom{-\alpha_n}{r}\Big| \frac{k! x^{\alpha_n-\alpha_{n-j}+k}}{(k-r)!\Gamma(\alpha_n-\alpha_{n-j}+r+1)} + \frac{Lx^{\alpha_n}}{\Gamma(\alpha_n+1)}.$$
Hence
$$\|Fu\| \le \Big[\sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k} \Big|a_{jk}\binom{-\alpha_n}{r}\Big| \frac{k!}{(k-r)!\Gamma(\alpha_n-\alpha_{n-j}+r+1)} + \frac{L}{\Gamma(\alpha_n+1)}\Big]\zeta,$$
where $\zeta = \max\{\lambda^{\alpha_n}, \lambda^{\alpha_n-\alpha_{n-1}}, \lambda^{\alpha_n-\alpha_{n-1}+b}\}$ and $b = \max\{N_1, N_2, \dots, N_{n-1}\}$. Hence $F(M)$ is bounded. Let $y \in M, x_1, x_2 \in [0, \lambda], x_1 < x_2$ then for given $\epsilon > 0$, choose
$$\delta = \min\Big\{\Big[\frac{\epsilon C(j,k,r)}{2}\Big]^{1/(\alpha_n-\alpha_{n-j}+r)} \quad \Big[\frac{\epsilon\Gamma(\alpha_n+1)}{4\|f\|_\infty}\Big]^{1/\alpha_n}\Big\}, \tag{2.4}$$
where $j = 1, 2, \dots, n-1$, $k = 0, 1, \dots, N_j$, $r = 0, 1, \dots, k$,
$$C(j,k,r) = \frac{(k-r)!}{\sum_{i=1}^{n-1}(N_i+1)(N_i+2)} \times \frac{\Gamma(\alpha_n-\alpha_{n-j}+r+1)}{\big|a_{jk}\binom{-\alpha_n}{r}\big| l\eta k!}$$
and $\eta = \max\{1, \lambda^{N_j}, j = 1, 2, \dots, n-1\}$. If $|x_1 - x_2| < \delta$,
$$\begin{aligned}
&|Fy(x_1) - Fy(x_2)| \\
&\le \Big|\sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k} \frac{|a_{jk}\binom{-\alpha_n}{r}| k! x_1^{k-r}}{(k-r)!\Gamma(\alpha_n-\alpha_{n-j}+r)} \\
&\quad \times \Big[\int_0^{x_1}\Big(\frac{y(t)}{(x_1-t)^{\alpha_{n-j}-\alpha_n-r+1}} - \frac{y(t)}{(x_2-t)^{\alpha_{n-j}-\alpha_n-r+1}}\Big)dt \\
&\quad - \int_{x_1}^{x_2}\frac{dt}{(x_2-t)^{\alpha_{n-j}-\alpha_n-r+1}}\Big] \\
&\quad + \frac{1}{\Gamma(\alpha_n)}\int_0^{x_1}\big((x_1-t)^{\alpha_n-1} - (x_2-t)^{\alpha_n-1}\big)f(t, y(t))dt \\
&\quad - \frac{1}{\Gamma(\alpha_n)}\int_{x_1}^{x_2}(x_2-t)^{\alpha_n-1}f(t, y(t))dt\Big| \\
&\le \sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k} \frac{|a_{jk}\binom{-\alpha_n}{r}| lk!\eta}{(k-r)!\Gamma(\alpha_n-\alpha_{n-j}+r)}(x_2-x_1)^{\alpha_n-\alpha_{n-j}+r} + \frac{2L(x_2-x_1)^{\alpha_n}}{\Gamma(\alpha_n+1)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k} \frac{|a_{jk}\binom{-\alpha_n}{r}|lk!\eta}{(k-r)!\Gamma(\alpha_n-\alpha_{n-j}+r)}\delta^{\alpha_n-\alpha_{n-j}+r} + \frac{2L\delta^{\alpha_n}}{\Gamma(\alpha_n+1)} \\
&\le \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.
\end{aligned}
$$

Hence $F(M)$ is equicontinuous and Arzela-Ascoli theorem implies that $\overline{F(M)}$ is compact. $\square$

Theorem 2.6. *Consider the fractional differential equation*

$$
\Big(D^{\alpha_n}-\sum_{j=1}^{n-1}p_j(x)D^{\alpha_{n-j}}\Big)y=g(y), \quad y(0)=0, \quad 0\le x\le\lambda,\ \lambda>0, \tag{2.5}
$$

where $0<\alpha_1<\alpha_2<\dots<\alpha_n<1$, $p_j(x)=\sum_{k=0}^{N_j}a_{jk}x^k$, $N_j\in\mathbb{N}\cup\{0\}$, $j=1,2,\dots,n-1$, $g:[0,1]\times[0,+\infty)\to[0,+\infty)$ *satisfies the Lipschitz condition with constant* L *and* $g(0)<\infty$.

If $p_j^{(2m)}(x)\ge 0$, $p_j^{(2m+1)}(x)\le 0$, $m=0,1,\dots,[\frac{N_j}{2}]$, *then* (2.5) *has a positive solution.*

Proof. In view Lemma 2.2, (2.5) is equivalent to the integral equation

$$
y(x)=\sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k} a_{jk}\binom{-\alpha_n}{r}\frac{k!x^{k-r}}{(k-r)!}I^{\alpha_n-\alpha_{n-j}+r}y(x)+I^{\alpha_n}g(y).
$$

In view of Lemma 2.4, $y(x)\in K$. Let $T:K\to K$ be defined as

$$
Ay(x)=\sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k} a_{jk}\binom{-\alpha_n}{r}\frac{k!x^{k-r}}{(k-r)!}I^{\alpha_n-\alpha_{n-j}+r}y(x)+I^{\alpha_n}g(y).
$$

A is completely continuous by Lemma 2.3.
Case (i) $g(0)\neq 0$. Let

$$
B(r)=\Big\{y(x)\in C[0,\delta]: y(x)\ge 0\ \|y-\frac{g(0)x^{\alpha_n}}{\Gamma(1+\alpha_n)}\|\le r\Big\},
$$

be a convex bounded and closed subset of the Banach space $C[0,\delta]$ where

$$
\delta<\min\Big\{\lambda,\ \Big(\frac{r\Gamma(\alpha_n+1)}{2Eg(0)}\Big)^{1/\alpha_n},\ \big(\frac{1}{2E}\big)^{1/\alpha_n}\Big\},
$$

where

$$
E=\xi\Big(\sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k}\frac{|a_{jk}\binom{-\alpha_n}{r}|k!}{(k-r)!\Gamma(\alpha_n-\alpha_{n-j}+r+1)}\Big)+\frac{L}{\Gamma(\alpha_n+1)},
$$

and

$$
\xi=\max\big\{x^{\alpha_n},x^{\alpha_n-\alpha_{n-1}+\rho}:0\le x\le\delta\big\}, \quad \rho=\max\{N_1,\dots,N_{n-1}\}.
$$

Note that, for all $y\in B(r)$,

$$
\begin{aligned}
&\big|Ay(x)-\frac{g(0)x^{\alpha_n}}{\Gamma(1+\alpha_n)}\big| \\
&\le \|u\|\Big[\sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k}\frac{|a_{jk}\binom{-\alpha_n}{r}|k!x^{\alpha_n-\alpha_{n-j}+k}}{(k-r)!\Gamma(\alpha_n-\alpha_{n-j}+r+1)}+\frac{Lx^{\alpha_n}}{\Gamma(\alpha_n+1)}\Big]
\end{aligned}
$$

$$\leq \|u\|\Big[\sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k}\frac{|a_{jk}\binom{-\alpha_n}{r}|k!}{(k-r)!\Gamma(\alpha_n-\alpha_{n-j})+r+1}+\frac{L}{\Gamma(\alpha_n+1)}\Big]\xi.$$

Since

$$\|u\| \leq \frac{g(0)}{\Gamma(1+\alpha_n)}x^{\alpha_n}+r \leq \frac{g(0)}{\Gamma(1+\alpha_n)}\delta^{\alpha_n}+r,$$

we have

$$\Big|Ay(x)-\frac{g(0)}{\Gamma(1+\alpha_n)}x^{\alpha_n}\Big| \leq E\Big(\frac{g(0)}{\Gamma(\alpha_n+1)}\delta^{\alpha_n+r}\Big) \leq \frac{r}{2}+\frac{r}{2}=r.$$

So we have $A(B(r)) \subseteq B(r)$. It can be seen that $A(B(r))$ is equicontinuous (the proof is similar to the proof of Lemma 2.5). Let $\{y_n\}$ be a bounded sequence in $B(r)$. Then $\{A(y_n)\} \subset T(B(r))$. Hence $\{A(y_n)\}$ is equicontinuous. Since $y_n \in C[a,b]$, Arzela-Ascoli theorem [1, 3] implies that $\{A(y_n)\}$ has a convergent subsequence. Therefore $A : B(r) \to B(r)$ is compact. Hence by Schauder fixed point theorem [4] it has a fixed point, which is a positive solution of(2.5). □

A similar proof can be given for the case $g(0) = 0$.

Example 2.7. Consider the equation

$$D^{\alpha_3}y(x)-(x^2-3x+2)D^{\alpha_2}y(x)-(1-x)D^{\alpha_1}y(x)=\frac{1+y}{1+y^2},$$

$y(0) = 0$, $0 \leq x \leq 1, 0 < \alpha_1 < \alpha_2 < \alpha_3 < 1$. Note that $p_1(x) = x^2 - 3x + 2$, $p_2(x) = 1 - x$ and $g(y) = \frac{1+y}{1+y^2}$ satisfy the conditions required in Theorem 2.6, hence this equation has a positive solution.

Theorem 2.8. *Let $f : [0,\lambda]\times[0,\infty) \to [0,\infty)$ be continuous and $f(x,.)$ be increasing for each $x \in [0,\lambda]$. Assume there exist v_0, w_0 satisfying $\mathcal{L}(D)v_0 \leq f(x,v_0)$, $\mathcal{L}(D)w_0 \geq f(x,w_0)$ and $0 \leq v_0(x) \leq w_0(x)$, $0 \leq x \leq 1$, where $\mathcal{L}(D) = D^{\alpha_n} - \sum_{j=1}^{n-1} p_j(x)D^{\alpha_{n-j}}$. Then (2.1) has a positive solution.*

Proof. We need to consider the fixed point of the operator F. Let $y_1, y_2 \in K$, $y_1 \leq y_2$, then

$$Fy_1=\sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k}a_{jk}\binom{-\alpha_n}{r}\frac{k!x^{k-r}}{(k-r)!}I^{\alpha_n-\alpha_{n-j}+r}y_1+I^{\alpha_n}f(x,y_1(x)) \leq Fy_2,$$

as f is nondecreasing. Hence F is an increasing operator. Assuming $Fv_0 \geq v_0, Fw_0 \leq w_0$, implies that $F : \langle v_0,w_0\rangle \to \langle v_0,w_0\rangle$ is compact operator in view of Lemma 2.4 and completely continuous in view of Lemma 2.3. Since K is a normal cone and F is compact continuous, by Theorem 1.1 F has a fixed point $u^* \in \langle v_0,w_0\rangle$, which is the required positive solution. □

Example 2.9. Consider the equation

$$D^{1/2}y(x)-(\Gamma(\frac{3}{2}))^{-1}\Gamma(\frac{7}{4})xD^{1/4}y(x)=(\Gamma(\frac{3}{2}))^{-1}f(x,y),$$

where $0 \leq x \leq 1$, $0 \leq y \leq +\infty$, $f(x,y) = \mu(x^{1/2} - x^{\frac{7}{4}})e^{2y-x}$ and $0 < \mu \leq 1$ which is equivalent to equation

$$\Gamma(\frac{3}{2})D^{1/2}y(x)-\Gamma(\frac{7}{4})xD^{1/4}y(x)=f(x,y).$$

If we let $v_0 = 0$, $w_0 = \frac{1}{2}x$, then $0 \le v_0 \le w_0$, $\mathcal{L}(D)v_0 = 0$, $\mathcal{L}(D)w_0 = x^{1/2} - x^{\frac{7}{4}}$, $\mathcal{L}(D)v_0 \le f(x,0)$ and $\mathcal{L}(D)w_0 \ge f(x, \frac{1}{2}x)$. Then this equation has a positive solution .

Theorem 2.10. *Let $f : [0,\lambda]\times[0,\infty) \to [0,\infty)$ be continuous and $f(x,.)$ increasing for each $x \in [0,\lambda]$. If $0 < \lim_{y\to+\infty} f(x,y) < +\infty$ for each $x \in [0,\lambda]$ then* (2.1) *has a positive solution.*

Proof. There exist positive constants N, R such that $f(x,y) \le N$, for all $x \in [0,\lambda]$, and all $y \ge R$. Let $C = \max\{f(x,y) | 0 \le y \le \lambda, 0 \le y \le R\}$. Then we have $f \le N + C$, for all $y \ge 0$. Now we consider the equation,

$$\Big(D^{\alpha_n} - \sum_{j=1}^{n-1} p_j(x) D^{\alpha_{n-j}}\Big) w(y) = N + C, \quad w(0) = 0, \quad 0 < x < \lambda.$$

Using Lemma 2.2, the above equation is equivalent to the integral equation

$$w(x) = \sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k} a_{jk}\binom{-\alpha_n}{r}\frac{k!x^{k-r}}{(k-r)!} I^{\alpha_n-\alpha_{n-j}+r} y(x) + I^{\alpha_n}(N+C).$$

This integral equation has a positive solution $w(x)$ in view of Theorem 2.8. Also

$$w(x) \ge \sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k} a_{jk}\binom{-\alpha_n}{r}\frac{k!x^{k-r}}{(k-r)!} I^{\alpha_n-\alpha_{n-j}+r} y(x) + I^{\alpha_n} f(x, w(x)) = Fw(x).$$

Now for $v(x) \equiv 0$, $F(v(x)) = I^{\alpha_n} f(x, v(x)) \ge v(x)$. Hence in view of Theorem 2.8, the result follows. □

It is easy to prove the following existence theorem using Theorems 2.8 and 2.10.

Theorem 2.11. *Let $f : [0,\lambda]\times[0,\infty) \to [0,\infty)$ be continuous and $f(x,.)$ increasing for each $x \in [0,\lambda]$. If*

$$0 \le \lim_{y\to\infty} \max_{0\le x\le\lambda} \frac{f(x,y)}{y} < +\infty.$$

Then(2.1) *has a positive solution.*

Example 2.12. *(1) $f(x,y) = x(1+e^{-y})^{-1}$, satisfies the condition required in Theorem 2.10.*
(2) $f(x,y) = x\ln(1+y)$ satisfies the conditions required in Theorem 2.11.

3. Uniqueness and existence of solutions

In this section we give conditions on f and p_j's, which render unique positive solution to (2.1).

Theorem 3.1. *Let $f : [0,\lambda]\times[0,\infty) \to [0,\infty)$ be continuous and Lipschitz with respect to the second variable with constant L. If (i) $p_j^{(2m)}(x) \ge 0$ and $p_j^{(2m+1)}(x) \le 0$, $m = 0, 1, \dots, [\frac{N_j}{2}]$, $N_j = \deg(p_j)$ $j = 1, 2, \dots, n-1$; and (ii)*

$$0 < \frac{L\lambda^{\alpha_n}}{\Gamma(\alpha_n+1)} + \sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k}\frac{|a_{jk}\binom{-\alpha_n}{r}| k!\lambda^{\alpha_n-\alpha_{n-j}+k}}{(k-r)!\Gamma(\alpha_n-\alpha_{n-j}+r+1)} < 1,$$

then (2.1) *has unique solution which is positive.*

Proof. As pointed out in the preceding section, (2.1) is equivalent to (2.2). For $y_1, y_2 \in K$ we have

$$\begin{aligned}
&\big|F(y_1(x)) - F(y_2(x))\big| \\
&\le \sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k} \frac{|a_{jk}\binom{-\alpha_n}{r}k!x^{k-r}}{(k-r)!} I^{\alpha_n-\alpha_{n-j}+r}|y_1(x)-y_2(x)| + LI^{\alpha_n}|y_1(x)-y_2(x)| \\
&\le \|y_1(x)-y_2(x)\|\Big[\frac{Lx^{\alpha_n}}{\Gamma(\alpha_n+1)} + \sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k}\frac{|a_{jk}\binom{-\alpha_n}{r}|k!x^{\alpha_n-\alpha_{n-j}+k}}{(k-r)!\Gamma(\alpha_n-\alpha_{n-j}+r+1)}\Big],
\end{aligned}$$

where F is given in (2.3). Hence

$$\begin{aligned}
&\|Fy_1 - Fy_2\| \\
&\le \Big[\frac{L\lambda^{\alpha_n}}{\Gamma(\alpha_n+1)} + \sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k}\frac{|a_{jk}\binom{-\alpha_n}{r}|k!\lambda^{\alpha_n-\alpha_{n-j}+k}}{(k-r)!\Gamma(\alpha_n-\alpha_{n-j}+r+1)}\Big]\|y_1(x)-y_2(x)\|.
\end{aligned}$$

In view of Theorem 1.2, F has unique fixed point in K, which is the unique positive solution of (2.1). □

In the following, we omit the condition on $p_j(x)'$s and study the equation

$$\Big(D_n^{\alpha} - \sum_{j=1}^{n-1} p_j(x)D^{\alpha_{n-j}}\Big)y = f(x,y), \quad y(0)=0, \quad 0\le x\le\lambda,\ \lambda>0, \tag{3.1}$$

where $0<\alpha_1<\alpha_2<\dots<\alpha_n<1$, $p_j(x)=\sum_{k=0}^{N_j} a_{jk}x^k, N_j\in\mathbb{N}\cup\{0\}$, $j=1,2,\dots,n-1$. Using Banach fixed point theorem for $F: C[0,\lambda]\to C[0,\lambda]$ we obtain the following result.

Example 3.2. Consider the equation

$$\Big(D^{1/2} - \frac{1}{60}(A-x)(B-x)D^{1/4} - \frac{1}{40}(C-x)D^{1/6} - MD^{1/8}\Big)y = Ly + e^x, \tag{3.2}$$

where $y(0)=0$, $0\le x\le 1$, $A\ge 1$ and $B\ge 1$. (3.2) is equivalent to the integral equation

$$y(x) = \sum_{j=1}^{3}\sum_{k=0}^{N_j}\sum_{r=0}^{k} a_{jk}\binom{-1/2}{r}\frac{k!}{(k-r)!}x^{k-r}I^{\frac12-\alpha_{n-j}+r}y(x) + I^{1/2}(Ly+e^x).$$

Here $p_1(x)=\sum_{k=0}^{2}a_{1k}x^k = \frac{1}{60}[x^2-(A+B)x+AB]$, hence $N_1=2$; $a_{10}=\frac{1}{60}AB$, $a_{11}=\frac{-1}{60}(A+B)$, $a_{12}=\frac{1}{60}$, $p_2(x)=\sum_{k=0}^{1}a_{2k}x^k=\frac{1}{40}(C-x)$, so $N_2=1$; $a_{20}=\frac{1}{40}C$, $a_{21}=\frac{-1}{40}$, $p_3(x)=\sum_{k=0}^{0}a_{3k}x^k=M$, so $N_3=0$, $a_{30}=M$. Hence

$$\begin{aligned}
y(x) &= a_{10}\binom{-1/2}{0}I^{\frac12-\frac14}y + a_{11}\Big[\binom{-1/2}{0}xI^{\frac12-\frac14}y + \binom{-1/2}{1}I^{\frac12-\frac14+1}y\Big] \\
&\quad + a_{12}\Big[\binom{-1/2}{0}x^2I^{\frac12-\frac14}y + 2\binom{-1/2}{1}xI^{\frac12-\frac14+1}y + 2\binom{-1/2}{2}I^{\frac12-\frac14+2}y\Big] \\
&\quad + a_{20}\binom{2\frac{-1}{2}}{0}I^{\frac12-\frac16}y + a_{21}\Big[\binom{-1/2}{0}xI^{\frac12-\frac16}y + \binom{-1/2}{1}I^{\frac12-\frac16+1}y\Big] \\
&\quad + a_{30}\binom{-1/2}{0}I^{\frac12-\frac18}y + LI^{\frac12}y + I^{1/2}e^x
\end{aligned}$$

$$
\begin{aligned}
&= \frac{AB}{60} I^{\frac{1}{2}-\frac{1}{4}} y - \frac{A+B}{60} \Big[x I^{\frac{1}{2}-\frac{1}{4}} y - \frac{1}{2} I^{\frac{1}{2}-\frac{1}{4}+1} y \Big] \\
&\quad + \frac{1}{60} \Big[x^2 I^{\frac{1}{2}-\frac{1}{4}} y - x I^{\frac{1}{2}-\frac{1}{4}+1} y + \frac{3}{4} I^{\frac{1}{2}-\frac{1}{4}+2} y + \Big] \\
&\quad + \frac{C}{40} I^{\frac{1}{2}-\frac{1}{6}} y - \frac{1}{40} \Big[x I^{\frac{1}{2}-\frac{1}{6}} y - \frac{1}{2} I^{\frac{1}{2}-\frac{1}{6}+1} y \Big] + M I^{\frac{1}{2}-\frac{1}{8}} y + L I^{1/2} y + I^{1/2} e^x.
\end{aligned}
$$

If $1 \le A \le 3$, $1 \le B \le 3$, $0 < M \le \frac{1}{40}$ and $0 < L \le \frac{1}{4}$ in the above equation satisfy the conditions required in Theorem 3.1. The iterated sequence is

$$y_1(x) = I^{1/2} e^x = x^{1/2} E_{1,\frac{3}{4}}(x)$$

$$
\begin{aligned}
y_2(x) = \Big[&\frac{AB}{60} I^{1/4} - \frac{A+B}{60} \big(x I^{\frac{1}{4}} - \frac{1}{2} I^{5/4} \big) + \frac{1}{60} \big(x^2 I^{1/4} - x I^{\frac{3}{4}} + \frac{3}{4} I^{9/4} \big) \\
&+ \frac{C}{40} I^{1/3} - \frac{1}{40} \big(x I^{1/3} - \frac{1}{2} I^{4/3} \big) + M I^{3/8} + L I^{1/2} \Big] y_1 + y_1,
\end{aligned}
$$

and

$$
\begin{aligned}
y_{n+1}(x) = \sum_{k=0}^{n} \Big[&\frac{AB}{60} I^{1/4} - \frac{A+B}{60} \big(x I^{1/4} - \frac{1}{2} I^{5/4} \big) + \frac{1}{60} \big(x^2 I^{1/4} - x I^{5/4} + \frac{3}{4} I^{9/4} \big) \\
&+ \frac{C}{40} I^{1/3} - \frac{1}{40} \big(x I^{\frac{1}{3}} - \frac{1}{2} I^{4/3} \big) + M I^{3/8} + L I^{1/2} \Big]^{n-k} y_1,
\end{aligned}
$$

$n = 1, 2, 3, \dots$, where $I^\alpha y_1 = x^{\alpha+\frac{1}{2}} E_{1,\alpha+\frac{3}{4}}(x)$, $\alpha > 0$. $y(x) = \lim_{n\to\infty} y_n(x)$ is the unique positive solution.

Theorem 3.3. *Let $f : [0,\lambda] \times [0,\infty) \to [0,\infty)$ be continuous and Lipschitz with respect to the second variable with constant L. Let a_{jk}'s satisfy*

$$0 < \frac{L\lambda^{\alpha_n}}{\Gamma(\alpha_n+1)} + \sum_{j=1}^{n-1} \sum_{k=0}^{N_j} \sum_{r=0}^{k} \frac{|a_{jk}\binom{-\alpha_n}{r}| k! \lambda^{\alpha_n - \alpha_{n-j} + k}}{(k-r)! \Gamma(\alpha_n - \alpha_{n-j} + r + 1)} < 1.$$

Then (3.1) *has unique solution, which may not necessarily be positive.*

Proof. Using Lemma 2.2, (3.1) is equivalent to the integral equation

$$y(x) = \sum_{j=1}^{n-1} \sum_{k=0}^{N_j} \sum_{r=0}^{k} a_{jk} \binom{-\alpha_n}{r} \frac{k! x^{k-r}}{(k-r)!} I^{\alpha_n - \alpha_{n-j} + r} y(x) + I^{\alpha_n} f(x, y(x)).$$

We define an operator $F : C[0,\lambda] \to C[0,\lambda]$ as

$$Fy(x) = \sum_{j=1}^{n-1} \sum_{k=0}^{N_j} \sum_{r=0}^{k} a_{jk} \binom{-\alpha_n}{r} \frac{k! x^{k-r}}{(k-r)!} I^{\alpha_n - \alpha_{n-j} + r} y(x) + I^{\alpha_n} f(x, y(x)),$$

For $y_1, y_2 \in C[0,\lambda]$,

$$
\begin{aligned}
&\|Fy_1 - Fy_2\| \\
&\le \Big[\frac{L\lambda^{\alpha_n}}{\Gamma(\alpha_n+1)} + \sum_{j=1}^{n-1} \sum_{k=0}^{N_j} \sum_{r=0}^{k} \frac{|a_{jk}\binom{-\alpha_n}{r}| k! \lambda^{\alpha_n - \alpha_{n-j} + k}}{(k-r)! \Gamma(\alpha_n - \alpha_{n-j} + r + 1)} \Big] \|y_1(x) - y_2(x)\|.
\end{aligned}
$$

Hence in view of Theorem 1.2, F will have unique fixed point in $C[0,\lambda]$, which is the unique solution of (3.1). This solution is not necessarily positive one. $\square$

Example 3.4. Consider the euation

$$\big(D^{1/2} - ax^2 D^{1/4} - bxD^{1/6} - cD^{\frac{1}{8}}\big)y = Ly + e^x, \quad y(0) = 0,\ 0 \le x \le 1. \tag{3.3}$$

This equation is equivalent to the integral equation

$$y(x) = \sum_{j=1}^{3}\sum_{k=0}^{N_j}\sum_{r=0}^{k} a_{jk}\binom{-\frac{1}{2}}{r}\frac{k!x^{k-r}}{(k-r)!} I^{\frac{1}{2}-\alpha_{n-j}+r} y(x) + I^{1/2}(Ly + e^x).$$

Here $p_1(x) = \sum_{k=0}^{2} a_{1k}x^k = ax^2$, then $N_1 = 2$, $a_{10} = a_{11} = 0$, $a_{12} = a$, $p_2(x) = \sum_{k=0}^{1} a_{2k}x^k = bx$, then $N_2 = 1$, $a_{20} = a_{21} = b$, and $p_3(x) = \sum_{k=0}^{0} a_{3k}x^k = c$, then $N_3 = 0$, $a_{30} = c$. Hence

$$\begin{aligned}
y(x) = a_{10}&\binom{-1/2}{0} I^{\frac{1}{2}-\frac{1}{4}} y + a_{11}\Big[\binom{-1/2}{0} x I^{\frac{1}{2}-\frac{1}{4}} y + \binom{-1/2}{1} I^{\frac{1}{2}-\frac{1}{4}+1} y\Big] \\
&+ a_{12}\Big[\binom{-1/2}{0} x^2 I^{\frac{1}{2}-\frac{1}{4}} y + 2\binom{-1/2}{1} x I^{\frac{1}{2}-\frac{1}{4}+1} y + 2\binom{-1/2}{2} I^{\frac{1}{2}-\frac{1}{4}+2} y + \Big] \\
&+ a_{20}\binom{2\frac{-1}{2}}{0} I^{\frac{1}{2}-\frac{1}{6}} y + a_{21}\Big[\binom{-1/2}{0} x I^{\frac{1}{2}-\frac{1}{6}} y + \binom{-1/2}{1} I^{\frac{1}{2}-\frac{1}{6}+1} y\Big] \\
&+ a_{30}\binom{-1/2}{0} I^{\frac{1}{2}-\frac{1}{8}} y + L I^{\frac{1}{2}} y + I^{1/2} e^x.
\end{aligned}$$

In view of (1.6) and that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(\frac{-1}{2}) = -2\sqrt{\pi}$ and $\Gamma(\frac{-3}{2}) = \frac{4\sqrt{\pi}}{3}$ we obtain

$$\begin{aligned}
y(x) = a&\Big[x^2 I^{1/4} y(x) - I^{5/4} y(x) + \frac{3}{4} I^{9/4} y(x)\Big] + b\Big[x I^{1/3} y(x) - \frac{1}{2} I^{4/3} y(x)\Big] \\
&+ c I^{3/8} y(x) + L I^{1/2} y(x) + I^{1/2} e^x.
\end{aligned}$$

If $|a| \le \frac{3}{5}$, $|b| \le \frac{2}{5}$, $|c| \le \frac{1}{5}$, $0 < L \le \frac{4}{5}$ in the above equation satisfy the conditions required in Theorem 3.3. The iterated sequence is

$$\begin{gathered}
y_1(x) = I^{1/2} e^x = x^{1/2} E_{1,\frac{3}{4}}(x), \\
y_2(x) = \Big[a(x^2 I^{1/4} - I^{5/4} + \frac{3}{4} I^{\frac{9}{4}}) + b(x I^{1/3} - \frac{1}{2} I^{4/3}) + c I^{3/8} + L I^{1/2}\Big] y_1 + y_1, \\
y_{n+1} = \sum_{k=0}^{n}\Big[a\big(x^2 I^{1/4} - I^{5/4} + \frac{3}{4} I^{9/4}\big) + b\big(x I^{1/3} - \frac{1}{2} I^{4/3}\big) + c I^{3/8} + L I^{1/2}\Big]^{n-k} y_1,
\end{gathered}$$

for $n = 1, 2, 3, \dots$, where $I^\alpha y_1 = x^{\alpha+\frac{1}{2}} E_{1,\alpha+\frac{3}{4}}(x), \alpha > 0$. $y(x) = \lim_{n\to\infty} y_n(x)$ is the unique solution, which may not be positive.

V. Daftardar-Gejji acknowledges University Grants Commission, N. Delhi, India for the support through Major Research Project.

References

[1] C. D. Aliprantis, O. Burkinshaw; *Principles of Real Analysis*, (Second Edition), Academic Press, New York, 1990.

[2] A. Babakhani, V. Daftardar-Gejji; *Existence of Positive Solutions of Nonlinear Fractional Differential Equations*, **278** J. Math. Anal. Appl. (2003) 434-442.

[3] R. R. Goldberg; *Methods of Real Analysis*, Oxford and IBH Publishing Company, New Delhi, 1970.

[4] M. C. Joshi, R. K. Bose; *Some Topics in Nonlinear Functional Analysis*, Wiley Eastern Limited, New Delhi, 1985.

[5] K. S. Miller, B. Ross; *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley Interscience, New York, 1993.
[6] I. Podlubny; *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[7] S. G. Samko, A. A. Kilbas, O. I. Marichev; *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Yverdon, 1993.

Azizollah Babakhani
Department of Mathematics, University of Mazanderan, Babol, Iran
E-mail address: babakhani@nit.ac.ir

Varsha Daftardar-Gejji
Department of Mathematics, University of Pune, Ganeshkhind, Pune - 411007, India
E-mail address: vsgejji@math.unipune.ernet.in