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POSITIVE SOLUTIONS FOR A CLASS OF SINGULAR BOUNDARY-VALUE PROBLEMS

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Abstract. This paper concerns the existence and multiplicity of positive solutions for Sturm-Liouville boundary-value problems. We use fixed point theorems and the sub-super solutions method to two solutions to the problem studied.

Introduction

Consider the boundary-value problem

$$\begin{gathered} Lu = \lambda f(t,u), \quad 0 < t < 1 \\ \alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0, \end{gathered} \tag{0.1}$$

where $Lu = -(ru')' + qu$, $r, q \in C[0,1]$ with $r > 0$, $q \geq 0$ on $[0,1]$, $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha\delta + \alpha\gamma + \beta\gamma > 0$, $f : (0,1) \times [0,\infty) \to [0,\infty)$, and λ is a positive parameter.

The existence and nonexistence of positive solutions of problem (0.1) with f possibly singular have been established by Choi [1], Dalmasso [2], Wong [7], and recently by Erbe and Mathsen [4]. In this paper, we shall obtain positive solutions to (0.1) under assumptions less stringent than in [4]. In particular, we do not need the condition that $f(t,u)$ be nondecreasing in u, which is essential in [1, 2, 4, 7]. Our approach depends on fixed point theorems and sub-super solutions method.

1. Main results

Let $G(t,s)$ be the Green's function for (0.1). Then u is a solution of (0.1) if and only if

$$u(t) = \lambda \int_0^1 G(t,s) f(s, u(s)) ds.$$

Recall that

$$G(t,s) = \begin{cases} c^{-1}\phi(t)\psi(s) & \text{if } t \leq s \\ c^{-1}\phi(s)\psi(t) & \text{if } t > s, \end{cases}$$

where ϕ and ψ satisfy

$$\begin{gathered} L\phi = 0, \quad \phi(0) = \beta, \quad \phi'(0) = \alpha \\ L\psi = 0, \quad \psi(1) = \delta, \quad \psi'(1) = -\gamma \end{gathered} \tag{1.1}$$

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and $c = -r(t)(\phi(t)\psi'(t) - \phi'(t)\psi(t)) > 0$. Note that $\phi' > 0$ on $(0,1]$, $\psi' < 0$ on $[0,1)$.

We shall make the following assumptions:

(H1) $f : (0,1) \times [0,\infty) \to [0,\infty)$ is continuous

(H2) For each $M > 0$, there exists a continuous function g_M on $(0,1)$ such that $f(t,u) \le g_M(t)$ for $t \in (0,1)$, $0 \le u \le M$, and

$$\int_0^1 G(s,s) g_M(s) ds < \infty.$$

(H3) There exist an interval $I \subset (0,1)$ and a function $m \in L^1(I)$ with $m \ge 0$, $m \not\equiv 0$ such that for every $a > 0$, there exists $r_a > 0$ such that

$$f(t,u) \ge a m(t) u \quad \text{for } t \in I,\ u \in (0, r_a)$$

(H4) There exist an interval $J \subset (0,1)$ and a positive number d such that

$$f(t,u) \ge du \quad \text{for } t \in J,\ u \ge 0.$$

(H5) There exist an interval $I_1 \subset (0,1)$ and a function $m_1 \in L^1(I_1)$ with $m_1 \ge 0, m_1 \not\equiv 0$ such that for every $b > 0$, there exists $R_b > 0$ such that

$$f(t,u) \ge b m_1(t) u \quad \text{for } t \in I_1,\ u \ge R_b.$$

Our main results are stated as follows.

Theorem 1.1. *Let (H1)–(H3) hold. Then there exists $\lambda_0 > 0$ such that* (0.1) *has a positive solution for $0 < \lambda < \lambda_0$. If, in addition, (H5) holds, then* (0.1) *has at least two positive solutions for $0 < \lambda < \lambda_0$.*

Theorem 1.2. *Let (H1)–(H4) hold. Then there exists $\lambda^* > 0$ such that* (0.1) *has a positive solution for $0 < \lambda < \lambda^*$ and no positive solution for $\lambda > \lambda^*$.*

Remark 1.3. Let $f(t,u) = m(t)g(u)$, where $g : [0,\infty) \to [0,\infty)$ be continuous with $\lim_{u\to 0^+} \frac{g(u)}{u} = \infty$, $\lim_{u\to\infty} \frac{g(u)}{u} = \infty$, and $m \in L^1(0,1)$ with $m \ge 0$, $m \not\equiv 0$. Then f satisfies (H1)–(H3) and (H5) and therefore Theorem 1.1 applies. If we take $m(t) = 1/\sqrt{t}$, $g(u) = u^p + u^q + h(u)$, where $p < 1 \le q$ and h is a nonnegative continuous function, then it is easily seen that $f(t,u)$ satisfies (H1)–(H5) and Theorem 1.2 applies. However, the results in [1, 2, 4, 7] may not apply since g may not be nondecreasing.

To prove our main results, we first establish the following results.

Lemma 1.4. *Let $h \in L^1(0,1)$ be such that $h \ge 0$ and let u satisfy*

$$Lu = h \quad \text{in } (0,1)$$
$$\alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0.$$

Then

$$u(t) \ge |u|_0 p(t),$$

where $p(t) = \min\big(\frac{\phi(t)}{|\phi|_0}, \frac{\psi(t)}{|\psi|_0}\big)$, and $\|\cdot\|_0$ denotes the supremum norm.

Proof. We proceed as in [3]. It is easy to see that

$$u(t) = \int_0^1 G(t,s) h(s) ds.$$

Let $|u|_0 = u(t_0)$ for some $t_0 \in (0,1)$. We verify that

$$\frac{G(t,s)}{G(t_0,s)} \geq p(t).$$

If $t, t_0 \leq s$ then

$$\frac{G(t,s)}{G(t_0,s)} = \frac{\phi(t)}{\phi(t_0)} \geq \frac{\phi(t)}{|\phi|_0},$$

and if $t_0 \leq s \leq t$ then

$$\frac{G(t,s)}{G(t_0,s)} = \frac{\phi(s)\psi(t)}{\phi(t_0)\psi(s)} \geq \frac{\psi(t)}{\psi(s)} \geq \frac{\psi(t)}{|\psi|_0}$$

since $\phi(s) \geq \phi(t_0)$. The other two cases are treated in a similar manner. Hence

$$u(t) \geq p(t)u(t_0) = |u|_0 p(t).$$

□

Lemma 1.5. *Let (H1)–(H3) hold. Then for each $\lambda > 0$, there exists $c_\lambda > 0$ such that if u is a nonzero solution of* (0.1) *then $|u|_0 \geq c_\lambda$. Furthermore, (c_λ) is nondecreasing in λ.*

Proof. Let $p_0 = \min_{t \in I} p(t)$, where p is defined in Lemma 1.4, and

$$K = \int_I G(\frac{1}{2}, s)m(s)ds.$$

By (H3), there exists $r_\lambda \in (0,1)$ such that

$$\frac{f(t,u)}{u} \geq \frac{2m(t)}{\lambda p_0 K} \quad \text{for } t \in I,\ 0 < u < r_\lambda$$

Define

$$c_\lambda = \sup\big\{r \in (0,1) : \frac{f(t,u)}{u} \geq \frac{2m(t)}{\lambda p_0 K} \text{ for } t \in I,\ 0 < u < r\big\}.$$

Then $0 < c_\lambda \leq 1$ and

$$\frac{f(t,u)}{u} \geq \frac{2m(t)}{\lambda p_0 K} \quad \text{for } t \in I,\ 0 < u \leq c_\lambda. \tag{1.2}$$

Clearly (c_λ) is nondecreasing in λ. Let u be a nonzero solution of (0.1) and suppose that $|u|_0 < c_\lambda$. Using Lemma 1.4 and (1.2), we obtain

$$\begin{aligned} u(t) &= \lambda \int_0^1 G(t,s)f(s,u(s))ds \\ &\geq \lambda \int_I \frac{2m(s)}{\lambda p_0 K} G(t,s)u(s)ds \\ &\geq 2K^{-1}|u|_0 \int_I G(t,s)m(s)ds, \end{aligned}$$

which implies

$$|u|_0 \geq u(\frac{1}{2}) \geq 2K^{-1}\Big(\int_I G(\frac{1}{2}, s)m(s)ds)\Big)|u|_0 = 2|u|_0,$$

a contradiction. This completes the proof. □

Lemma 1.6. *Let (H1), (H2), (H4) hold. Then* (0.1) *has no positive solution for λ large.*

Proof. Let u be a positive solution of (0.1). Using (H4) and Lemma 1.4, we obtain

$$u(\frac{1}{2}) = \lambda \int_0^1 G(\frac{1}{2}, s) f(s, u(s)) ds \geq \lambda d \int_J G(\frac{1}{2}, s) u(s) ds \geq \lambda d C |u|_0,$$

where $C = \big(\min_{s\in J} p(s)\big)\big(\int_J G(\frac{1}{2}, s) ds\big)$, which implies $\lambda \leq (dC)^{-1}$. □

The next Lemma establishes the existence of a solution once a pair of ordered sub- and supersolution are known, without assuming monotonicity of $f(t,u)$ in u.

Lemma 1.7. *Let (H1), (H2) hold. Suppose that* $\underline{u}$ *and* $\bar{u}$ *in* $C[0,1] \cap C^1(0,1)$ *are sub- and supersolutions of* (0.1) *respectively with* $0 \leq \underline{u} \leq \bar{u}$*, i.e.,*

$$\begin{gathered} L\underline{u}(t) \leq \lambda f(t, \underline{u}) \quad \text{in } (0,1) \\ \alpha \underline{u}(0) - \beta \underline{u}'(0) \leq 0, \quad \gamma \underline{u}(1) + \delta \underline{u}'(1) \leq 0 \end{gathered}$$

and

$$\begin{gathered} L\bar{u}(t) \geq \lambda f(t, \bar{u}(t)) \quad \text{in } (0,1) \\ \alpha \bar{u}(0) - \beta \bar{u}'(0) \geq 0, \quad \gamma \bar{u}(1) + \delta \bar{u}'(1) \geq 0. \end{gathered}$$

Then (0.1) *has a solution* u *with* $\underline{u} \leq u \leq \bar{u}$*.*

Proof. The proof is essentially given in [6], where nonsingular problems were considered. For convenience, we give a proof. Without loss of generality, we assume that $\lambda = 1$. Define

$$\bar{f}(t,v) = \begin{cases} f(t, \bar{u}(t)) + \frac{\bar{u}(t) - v}{1 + v^2} & \text{if } v > \bar{u}(t) \\ f(t,v) & \text{if } \underline{u}(t) \leq v \leq \bar{u}(t) \\ f(t, \underline{u}(t)) + \frac{\underline{u}(t) - v}{1 + v^2} & \text{if } v \leq \underline{u}(t). \end{cases}$$

For each $v \in C[0,1]$, let $u = Tv$ be the solution of

$$\begin{gathered} Lu = \bar{f}(t,v), \quad 0 < t < 1 \\ \alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0. \end{gathered}$$

Then $T : C[0,1] \to C[0,1]$ is completely continuous. Since T is bounded, T has a fixed point u by the Schauder fixed point Theorem. We verify that $\underline{u} \leq u \leq \bar{u}$. Suppose to the contrary that there exists $t_0 \in (0,1)$ such that $u(t_0) > \bar{u}(t_0)$. Let $w = u - \bar{u}$ and $t_1 \in [0,1]$ be such that $w(t_1) = \max_{0 \leq t \leq 1} w(t) > 0$. If $t_1 \in (0,1)$ then $w'(t_1) = 0$ and $(rw')'(t_1) \leq 0$, which implies that $Lw(t_1) \geq 0$. On the other hand,

$$Lw(t_1) = Lu(t_1) - L\bar{u}(t_1) \leq -\frac{w(t_1)}{1 + u^2(t_1)} < 0,$$

a contradiction. Suppose that $t_1 = 0$. Then $w'(0) \leq 0$, and since $\alpha w(0) - \beta w'(0) \leq 0$, we have a contradiction if $\alpha > 0$. If $\alpha = 0$ then $\beta > 0$ and therefore $w'(0) = 0$. Since $-(rw')'(t) + qw(t) \equiv Lw(t) < 0$ for small $t > 0$, it follows by integrating that $(rw')(t) > 0$ and so $w'(t) > 0$ for small $t > 0$, a contradiction. Similarly, we reach a contradiction if $t_1 = 1$. Hence $u \leq \bar{u}$ on $(0,1)$. The lower inequality can be derived in a similar manner. □

In view of Lemmas 1.4 and 1.5, we see that u is a positive solution of (0.1) if and only if u is a solution of

$$\begin{gathered} Lu = \lambda \tilde{f}(t,u), \quad 0 < t < 1 \\ \alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0, \end{gathered} \tag{1.3}$$

where $\tilde{f}(t,u(t)) = f(t,\max(u(t),c_\lambda p(t))$, or equivalently, u is a fixed point of A_λ, where

$$A_\lambda u(t) = \lambda \int_0^1 G(t,s)\tilde{f}(s,u(s))ds$$

Note that $A_\lambda : C[0,1] \to C[0,1]$ is completely continuous (see [3]).

We are now in a position to prove our main result.

Proof of Theorem 1.1. Let

$$\lambda_0 = \Big(\int_0^1 G(s,s)g_1(s)ds\Big)^{-1}$$

and suppose that $0 < \lambda < \lambda_0$, where g_1 is defined in (H2). Let u be a solution of

$$u = \theta A_\lambda u \quad \text{for some } \theta \in [0,1].$$

We claim that $|u|_0 \neq 1$. Indeed, if $|u|_0 = 1$ then since $c_\lambda |p|_0 \le c_\lambda \le 1$, it follows from (H2) that $\tilde{f}(s,u(s)) \le g_1(s)$, which implies

$$1 = |u|_0 \le \lambda \int_0^1 G(s,s)g_1(s)ds < 1$$

for $\lambda < \lambda_0$, a contradiction, and the claim is proved. Hence the Leray-Schauder fixed point Theorem gives the existence of a fixed point u of A_λ with $|u|_0 < 1$.

Next, suppose that (H5) holds. We shall employ fixed point theorems in a cone to show the existence of a second solution. Let $\mathbb{K}$ be the cone of nonnegative functions in $C[0,1]$. By the above arguments, we have

$$u \in \mathbb{K} \text{ and } u \le A_\lambda u \ \Rightarrow\ |u|_0 \neq 1.$$

Let

$$b = 2\Big(\lambda p_1 \int_{I_1} G\big(\frac{1}{2},s\big)m_1(s)ds\Big)^{-1},$$

where $p_1 = \min_{s\in I_1} p(s)$. By (H5), there exists $R_b > p_1$ such that

$$\tilde{f}(s,u) \ge b m_1(s)u \quad \text{for } s \in I_1,\ u \ge R_b.$$

We claim that

$$u \in \mathbb{K} \text{ and } u \ge A_\lambda u \ \Rightarrow\ |u|_0 \neq R_b p_1^{-1}$$

Suppose that $u \in \mathbb{K}$ and $u \ge A_\lambda u$. If $|u|_0 = R_b p_1^{-1}$ then it follows from Lemma 1.4 that

$$u(s) \ge R_b p_1^{-1} p(s) \ge R_b \quad \text{for } s \in I_1.$$

Hence

$$\begin{aligned} R_b p_1^{-1} &= |u|_0 \ge u\big(\frac{1}{2}\big) \\ &\ge \lambda \int_0^1 G\big(\frac{1}{2},s\big)\tilde{f}(s,u(s))ds \\ &\ge bR_b\lambda\Big(\int_{I_1} G\big(\frac{1}{2},s\big)\Big)m_1(s)ds\Big) = 2R_b p_1^{-1}, \end{aligned}$$

a contradiction, and the claim is proved. By Krasnoselskii's fixed point Theorem, [5], A_λ has a fixed point $\tilde{u}$ in $\mathbb{K}$ with $1 < |\tilde{u}|_0 < R_b p_1^{-1}$. This completes the proof □

Proof of Theorem 1.2. Let Λ be the set of all $\lambda > 0$ such that (0.1) has a positive solution and let $\lambda^* = \sup \Lambda$. By Theorem 1.1 and Lemma 1.6, $0 < \lambda^* < \infty$. Let $0 < \lambda < \lambda^*$. Then there exists $\lambda_0 > 0$ such that $\lambda < \lambda_0$ and $(0.1)_{\lambda_0}$ has a positive solution u_{λ_0}. Then u_{λ_0} satisfies

$$u_{\lambda_0}(t) \geq c_{\lambda_0} p(t) \geq c_\lambda p(t),$$

and therefore

$$\begin{aligned} L u_{\lambda_0}(t) &= \lambda_0 f(t, u_{\lambda_0}(t)) \\ &= \lambda_0 f(t, \max(u_{\lambda_0}(t), c_\lambda p(t)) \\ &\geq \lambda f(t, \max(u_{\lambda_0}(t), c_\lambda p(t)) \\ &= \lambda \tilde{f}(t, u_{\lambda_0}(t)), \end{aligned}$$

i.e., u_{λ_0} is a supersolution of (1.3). Since 0 is a subsolution of (1.3), it follows from Lemma 1.7 that (1.3) has a solution u_λ with $0 \leq u_\lambda \leq u_{\lambda_0}$. Thus u_λ is a positive solution of (0.1), completing the proof of Theorem 1.2. □

References

[1] Y. S. Choi, *A singular boundary value problem arising from near-ignition analysis of flame structures*, Diff. Integral Eqns. **4** (1991), 891-895.

[2] R. Dalmasso, *Positive solutions of singular boundary value problems*, Nonlinear Anal. **27** (1996), 645-652.

[3] H. Dang and K. Schmitt, *Existence of positive solutions for semilinear elliptic equations in annular domains*, Diff. Integral Eqns **7** (1994), 747-758.

[4] L. H. Erbe and R. M. Mathsen, *Positive solutions for singular nonlinear boundary value problems*, Nonlinear Anal.**46** (2001), 979-986.

[5] M. A. Krasnoselskii, *Positive solutions of operator equations*, Noordhoff, Groningen (1964).

[6] K. Schmitt, *Boundary value problems for quasilinear second order elliptic equations*, Nonlinear Anal. **3** (1978), 263-309.

[7] F. H. Wong, *Existence of positive solutions of singular boundary value problems*, Nonlinear Anal. **16** (1993), 397-406.

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