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THE DIRICHLET PROBLEM FOR THE MONGE-AMPÈRE EQUATION IN CONVEX (BUT NOT STRICTLY CONVEX) DOMAINS

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Abstract. It is well-known that the Dirichlet problem for the Monge-Ampère equation $\det D^2u = \mu$ in a bounded strictly convex domain Ω in $\mathbb{R}^n$ has a weak solution (in the sense of Aleksandrov) for any finite Borel measure μ on Ω and for any continuous boundary data. We consider the Dirichlet problem when Ω is only assumed to be convex, and give a necessary and sufficient condition on the boundary data for solvability.

1. Introduction

This note concerns the solvability of the Dirichlet problem for the Monge-Ampère equation:

$$\begin{aligned} \det D^2u &= \mu \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded convex domain in $\mathbb{R}^n$, μ is a finite Borel measure on Ω, and $g \in C(\partial\Omega)$. The solution to this problem is in the Aleksandrov sense and we look for a solution $u \in C(\bar\Omega)$ that is convex in Ω.

It is well-known (see Theorem 2.1 below) that this problem has a solution when Ω is strictly convex. An obvious necessary condition for the solvability of (1.1) is that there exists a convex function $\tilde g \in C(\bar\Omega)$ such that $\tilde g = g$ on $\partial\Omega$, or equivalently, that g can be extended to a $C(\bar\Omega)$ function, convex in Ω. We demonstrate that this condition is also sufficient, so that the following theorem holds.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain, and let μ be a finite Borel measure on Ω. The Dirichlet problem* (1.1) *has a unique Aleksandrov solution $u \in C(\bar\Omega)$ if and only if $g \in C(\partial\Omega)$ can be extended to a function $\tilde g \in C(\bar\Omega)$ that is convex in Ω.*

It is not hard to see that a necessary condition for the existence of such an extension is that g, restricted to any line segment in $\partial\Omega$, is convex. Theorem 5.2 and the subsequent remark show that this condition is also sufficient.

Uniqueness of solutions follows from a comparison principle (Theorem 4.2 below). The existence of a solution for the special case where $\mu \equiv 0$ on a neighborhood of $\partial\Omega$

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is demonstrated by finding both a subsolution and a supersolution that continuously attain the boundary data, and invoking an appropriate sub-/supersolution result. The general case of a finite measure on Ω is then handled by approximation.

We begin by reviewing the definition and some of the basic theory of the Aleksandrov solution (Section 2). In Section 3, the portion of the theory of viscosity solutions for the Monge-Ampère equation that is relevant for this problem is presented. Section 4 concerns the use of the sub-/supersolution approach and the Perron method. The proofs of Theorems 1.1 and 5.2 are in Section 5.

2. Aleksandrov solutions

In this section, the Aleksandrov solution and its main properties are reviewed. Most of this material can be found in Chapter 1 of [6], which includes notes on the original sources of the results below. See also [1].

The definition of the Aleksandrov solution depends on the normal mapping of a function. Let $\mathcal{U} \subset \mathbb{R}^n$, and let $u : \mathcal{U} \to \mathbb{R}$. The normal map (or subdifferential) of u at $x \in \mathcal{U}$, denoted by $\nabla u(x)$, is the set

$$\nabla u(x) = \{p \in \mathbb{R}^n : u(y) \geq u(x) + p \cdot (y - x)\ \forall y \in \mathcal{U}\}.$$

The normal map of u of a set $E \subset \mathcal{U}$, denoted by $\nabla u(E)$, is the union of the normal maps of u at the points of E:

$$\nabla u(E) = \bigcup_{x \in E} \nabla u(x).$$

When $\mathcal{U} \subset \mathbb{R}^n$ is open and $u \in C(\mathcal{U})$, the normal map defines a Borel measure, denoted Mu, on $\mathcal{U}$ by

$$Mu(E) = |\nabla u(E)|,$$

where $|\cdot|$ denotes Lebesgue measure. This measure is called the Monge-Ampère measure associated to u, and is finite on compact subsets of $\mathcal{U}$.

We can now define the Aleksandrov solution of the Monge-Ampère equation. Let μ be a Borel measure on Ω, a bounded convex domain in $\mathbb{R}^n$. A convex function $u \in C(\Omega)$ is called an Aleksandrov solution of $\det D^2 u = \mu$ in Ω if $Mu = \mu$ in Ω. We now state a basic existence and uniqueness theorem for the Dirichlet problem.

Theorem 2.1 ([6, Theorem 1.6.2]). *Let Ω be bounded and strictly convex and μ be a finite Borel measure on Ω. Then for any $g \in C(\partial\Omega)$, problem* (1.1) *has a unique solution (in the Aleksandrov sense) in $C(\bar{\Omega})$.*

The following estimate, due to Aleksandrov, allows for the estimation of a convex function, vanishing on the boundary of its domain, at a point x in terms of the distance from x to the boundary and the Monge-Ampère measure of the domain.

Theorem 2.2 ([6, Theorem 1.4.2]). *If $\Omega \subset \mathbb{R}^n$ is a bounded convex domain and $u \in C(\bar{\Omega})$ is convex with $u = 0$ on $\partial\Omega$, then*

$$|u(x)|^n \leq C \operatorname{dist}(x, \partial\Omega)(\operatorname{diam}\Omega)^{n-1} Mu(\Omega)$$

for all $x \in \Omega$, where C is a dimensional constant.

The following result will be needed to run the approximation argument in the proof of Theorem 1.1.

Lemma 2.3 ([6, Lemma 1.2.3]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. If u_j is a sequence of convex functions in Ω such that $u_j \to u$ uniformly on compact subsets of Ω, then $Mu_j \to Mu$ weakly in Ω, meaning that for any $f \in C(\Omega)$ with compact support, we have*

$$\int_\Omega f\, dMu_j \to \int_\Omega f\, dMu.$$

In the paper [9] the notions of the Monge-Ampère measure of a convex function and the related weak solution of the Monge-Ampère equation described above were extended to arbitrary domains in $\mathbb{R}^n$. Properties of these measures (and more generally the $k-$Hessian measures also defined in [9]), were also studied in [10] and [11]. For domains considered below that are not convex, we will use Mu to denote the measure defined in [9]. See also the remarks in the next section.

3. Viscosity solutions

Another weak solution that has been effectively used in the study of Monge-Ampère equations is the viscosity solution (see for example, [3], [4], [5], [12]), and in this section we review the definition and the relation between viscosity and Aleksandrov solutions.

Let $\mathcal{U} \subset \mathbb{R}^n$ be open and bounded. Let $f \in C(\mathcal{U})$ be nonnegative in $\mathcal{U}$. An upper semicontinuous function $v : \mathcal{U} \to \mathbb{R}$ is called a viscosity subsolution of $\det D^2u = f$ in $\mathcal{U}$ if whenever $x \in \mathcal{U}$ and $\phi \in C^2(\mathcal{U})$ are such that $v - \phi$ has a local maximum at $x_0 \in \mathcal{U}$, then $\det D^2\phi(x_0) \geq f(x_0)$. Similarly (but note the additional requirement of local convexity), a lower semicontinuous function $w : \mathcal{U} \to \mathbb{R}$ is called a viscosity supersolution of $\det D^2u = f$ in $\mathcal{U}$ if whenever $x \in \mathcal{U}$ and $\phi \in C^2(\mathcal{U})$, satisfying $D^2\phi \geq 0$ in $\mathcal{U}$, are such that $w - \phi$ has a local minimum at $x_0 \in \mathcal{U}$, then $\det D^2\phi(x_0) \leq f(x_0)$. A function u which is both a viscosity supersolution and a viscosity subsolution is called a viscosity solution of $\det D^2u = f$.

The nonnegativity of f in the previous definition is required for ellipticity. Without the additional requirement of local convexity for test functions in the definition of the supersolution, classical solutions may fail to be viscosity solutions (see [5] for a simple example). As shown in [12], an equivalent formulation of viscosity solution is obtained if local convexity is required of the test functions for a subsolution rather than supersolution.

Using viscosity solutions, we can define what it means for a continuous function to be convex in a domain that is not convex. A function u will be said to be convex in a domain $\mathcal{U}$ if it satisfies $\det D^2u \geq 0$ in the viscosity sense in $\mathcal{U}$. This is needed in Definition 4.1 and Theorem 4.2 below. See also the remarks on p. 226 of [9] and p. 580 of [10].

We remark that when $\mathcal{U}$ is convex and $\mu = f\,dx$, where $f \in C(\overline{\mathcal{U}})$ is positive in $\mathcal{U}$, a function $u \in C(\mathcal{U})$ is a viscosity solution of $\det D^2u = f$ if and only if it is an Aleksandrov solution of $Mu = \mu$ in $\mathcal{U}$. See [3] or [6, Chapter 1]. Furthermore, this equivalence of viscosity and measure-theoretic solutions (as in [9] and [10] and mentioned at the end of Section 2) extends to domains that are not convex; see [10, p. 586].

4. Subsolutions and supersolutions

In this section, we define appropriate sub- and superfunctions for this problem and show that the conditions for using a Perron argument (as described in [2] and

[8], and reviewed and modified in [7]) are met. In particular, we show that [7, Theorem 4.8] holds. Problem (1.1) then has a solution in the Aleksandrov sense that continuously attains its boundary values if there exist both a subsolution and a supersolution that equal g on $\partial\Omega$.

Let $\mathcal{U} \subset \mathbb{R}^n$ be open and connected, and let μ be a finite Borel measure on $\mathcal{U}$. We let $\mathcal{F}$ denote the family of local weak solutions of $Mu = \mu$. In other words, a function $u \in \mathcal{F}$ if u satisfies $Mu = \mu$ in some domain $D \subset \mathcal{U}$. We now show that the family $\mathcal{F}$ satisfies the hypotheses in [7, Theorem 4.8], which consist of several postulates.

Let B be a ball compactly contained in $\mathcal{U}$, and let $h \in C(\partial B)$. Then by Theorem 2.1, there is a unique $u \in C(\bar{B})$ that satisfies $Mu = \mu$ in B and $u = h$ on ∂B. Thus Postulate 4.1 in [7] is satisfied.

Definition 4.1. A convex function (in the viscosity sense, see Section 3) $v \in C(D)$ is called a *subfunction (superfunction)* in D ($D \subset \mathcal{U}$) if for any ball $B \subset D$ and any $w \in C(\partial B)$ for which $v \le (\ge)\, w$ on ∂B, we also have that $v \le (\ge)\, w$ in B, where w is that element of $\mathcal{F} \cap C(\bar{B})$ with boundary values w. If D is a bounded domain with $\bar{D} \subset \mathcal{U}$ and h is a bounded function on ∂D, $v \in C(\bar{D})$ is called a *subsolution* (with respect to h, $\mathcal{F}$ and D) if v is a subfunction in D and $v \le h$ on ∂D. $v \in C(\bar{D})$ is called a *supersolution* if v is a superfunction in D and $v \ge h$ on ∂D.

Postulate 4.6 in [7] and the weak version of Postulate 4.2 (Comparison principles) in [7] both hold by the following comparison principle, in which convexity of the functions is in the viscosity sense (see Section 3). ([9, Corollary 2.4] is also used to establish [7, Postulate 4.6].)

Theorem 4.2 ([9, Theorem 3.1])**.** *Let u, $v \in C(\bar{D})$ be convex and satisfy*

$$\begin{gathered} Mu \ge Mv \quad \text{in } D \\ u \le v \quad \text{on } \partial D. \end{gathered}$$

Then $u \le v$ in D.

We remark that by Theorem 4.2, if v is convex and $Mv \ge (\le)\, \mu$ in a domain D, then v is a subfunction (superfunction). This observation will be needed in the next section.

Let B be a ball with $\bar{B} \subset \mathcal{U}$, and let $\{h_j\} \subset C(\partial B)$ be uniformly bounded, say $|h_j(x)| \le M$ for all j and $x \in \partial B$. To establish [7, Postulate 4.3], we need to demonstrate that the sequence $\{u_j\} \subset C(\bar{B})$ of solutions to the problems $Mu = \mu$ in B, $u_j = h_j$ on ∂B is equicontinuous. By subtracting M from h_j (and hence from u_j), we may assume that $h_j \le 0$. Let $v \in C(\bar{B})$ solve

$$\begin{gathered} Mu = \mu \quad \text{in } B \\ u = -2M \quad \text{on } \partial B. \end{gathered}$$

Then, by Theorem 4.2 and the convexity of u_j, $v \le u_j \le 0$ in $\bar{B}$ for all j. Now suppose $p_j \in \nabla u_j(x)$ for some j and some $x \in B$. Then, by [6, Lemma 3.2.1],

$$|p_j| \le \frac{|u_j(x)|}{\operatorname{dist}(x, \partial B)} \le \frac{|v(x)|}{\operatorname{dist}(x, \partial B)}. \tag{4.1}$$

This inequality gives a uniform estimate on $|p_j|$ and implies that the u_j are uniformly Lipschitz in compact subsets of B and hence equicontinuous.

Let D be any bounded domain with $\bar{D} \subset \mathcal{U}$ and let h be a bounded function on ∂D. To verify [7, Postulate 4.4], we need to show that there exists a subsolution and a supersolution in D. Let $w(x) = \text{const} \geq \sup_{\partial D} h$. Then $Mw = 0$, so that w is a supersolution. Extend μ to a measure on $\mathbb{R}^n$ by setting $\mu(\mathbb{R}^n \setminus \mathcal{U}) = 0$. Let Ω_0 be a strictly convex domain that contains D, and let $h_0 \in C(\partial\Omega_0)$ be such that $\max_{\partial\Omega_0} h_0 \leq \inf_{\partial D} h$, and let $u \in C(\bar{\Omega}_0)$ solve (by Theorem 2.1)

$$\begin{gathered} Mu = \mu \quad \text{in } \Omega_0 \\ u = h_0 \quad \text{on } \partial\Omega_0. \end{gathered}$$

Then since u is convex, its maximum occurs on $\partial\Omega_0$, so $u \leq h$ on ∂D. By the comparison principle (Theorem 4.2), u is a subfunction in D and hence a subsolution.

Therefore all of the conditions in [7, Theorem 4.8] are met and the Perron process can be used to produce a generalized solution in any bounded domain with bounded boundary data. The continuous assumption of boundary data $h \in C(\partial D)$ is guaranteed when a subsolution and a supersolution, both of which equal h on ∂D, exist (as is also indicated on p. 233 of [9]). In the next section, we produce a subsolution and a supersolution for problem (1.1) both equal to g on $\partial\Omega$ when μ vanishes on a neighborhood of $\partial\Omega$.

5. Proof of Theorem 1.1

Because g is assumed to extend continuously to $\tilde{g}$, which is a convex function in Ω, the problem

$$\begin{gathered} \det D^2 u = 0 \quad \text{in } \Omega \\ u = g \quad \text{on } \partial\Omega \end{gathered} \tag{5.1}$$

has a solution $w \in C(\bar{\Omega})$, that is convex in Ω (see [10, Theorem 3.5]). By Theorem 4.2, w is a supersolution for problem (1.1) and $w = g$ on $\partial\Omega$. Since w is convex, it agrees with its convex envelope; in other words, for all $x \in \Omega$,

$$w(x) = \sup\{L(x) : L \leq w \text{ in } \Omega,\ L \text{ affine}\}.$$

For $x \in \bar{\Omega}$, define

$$U(x) = \sup\{L(x) : L \leq g \text{ on } \partial\Omega,\ L \text{ affine}\}. \tag{5.2}$$

We now show that $U = w$. Suppose the affine function L satisfies $L \leq w$ in Ω. Then by the continuity of L and w and since $w = g$ on $\partial\Omega$, $L \leq g$ on $\partial\Omega$, so $w \leq U$. Now suppose that $L \leq g$ on $\partial\Omega$ and L is affine. Then since $\det D^2 L = \det D^2 w = 0$ in Ω and $L \leq w$ on $\partial\Omega$, we get by Theorem 4.2 that $L \leq w$, so that $U \leq w$. Therefore w is given by the following formula (and this formula extends to $\partial\Omega$, see Theorem 5.2 below), which will be needed in the proof of Lemma 5.1 below:

$$w(x) = \sup\{L(x) : L \leq g \text{ on } \partial\Omega,\ L \text{ affine}\}. \tag{5.3}$$

Because Ω is convex the function $\operatorname{dist}(\cdot, \partial\Omega)$ is concave in Ω. Then for any constant $K \geq 0$,

$$w_K(x) = w(x) - K \operatorname{dist}(x, \partial\Omega) \tag{5.4}$$

is convex in Ω, continuous in $\bar{\Omega}$, and equal to g on $\partial\Omega$. By choosing K sufficiently large, w_K can be made as small as desired on any compact subset of Ω.

Now suppose that $\mu \equiv 0$ on a neighborhood of $\partial\Omega$, and let Ω_1 be a compact subset of Ω that contains the support of μ. Let Ω_0 be a strictly convex domain that contains Ω, and extend μ to Ω_0 by defining $\mu(\Omega_0 \setminus \Omega) \equiv 0$. Let $v \in C(\bar{\Omega}_0)$ be the unique Aleksandrov solution of

$$\begin{gathered}\det D^2 u = \mu \quad \text{in } \Omega_0 \\ u = 0 \quad \text{on } \partial\Omega_0,\end{gathered}$$

which exists by Theorem 2.1. Let $v_{K_1} = v - K_1$ where K_1 is chosen so that $v_{K_1} < g$ on $\partial\Omega$, and note that $\det D^2 v = \det D^2 v_{K_1}$. Let K in (5.4) be chosen so that $w_K < v_{K_1}$ in Ω_1. For $x \in \bar{\Omega}$, define

$$u(x) = \max\{w_K(x),\ v_{K_1}(x)\}.$$

Then $u \in C(\bar{\Omega})$, $u = g$ on $\partial\Omega$ and since it is the maximum of two convex functions, u is convex in Ω. We claim that u is a subsolution. Let $E \subset \Omega$ be a Borel set. Let $E_1 = E \cap \Omega_1$ and $E_2 = E \cap (\Omega \setminus \Omega_1)$. Let $p \in \nabla v(x)$ for some $x \in E_1$. Then

$$v_{K_1}(y) \geq v_{K_1}(x) + p \cdot (y - x)$$

for all $y \in \Omega_0$. We have that $u(x) = v_{K_1}(x)$ and that $u \geq v_{K_1}$ in Ω, so

$$u(y) \geq u(x) + p \cdot (y - x),$$

for all $y \in \Omega$, implying that $p \in \nabla u(x)$ and $\nabla v(E_1) \subset \nabla u(E_1)$, and hence that $Mv(E_1) \leq Mu(E_1)$. Therefore,

$$\begin{aligned} Mu(E) &= Mu(E_1) + Mu(E_2) \\ &\geq Mv(E_1) + Mu(E_2) \\ &= \mu(E_1) + Mu(E_2) \\ &= \mu(E) + Mu(E_2) \geq \mu(E). \end{aligned}$$

By the remark following Theorem 4.2, this proves that u is a subsolution. Note that Mu may be singular outside of the support of μ, but that this does not matter in the construction of the subsolution u.

Thus, by Section 4, the conclusion of Theorem 1.1 holds when μ is compactly supported in Ω. To handle the general case, we use the following approximation lemma, see [6, Lemma 1.6.1] for the corresponding result for strictly convex domains.

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain, and let μ be a finite Borel measure on Ω. Suppose $g \in C(\partial\Omega)$ and that there exists a sequence of convex functions $u_j \in C(\bar{\Omega})$ such that:*

(1) *$u_j = g$ on $\partial\Omega$,*
(2) *$\mu_j = Mu_j \to \mu$ weakly in Ω, and*
(3) *$\mu_j(\Omega) \leq A < \infty$ for all j.*

Then there exists a convex function $u \in C(\bar{\Omega})$ and a subsequence of the u_j that converges uniformly to u on compact subsets of Ω, such that $Mu = \mu$ in Ω and $u = g$ on $\partial\Omega$.

Proof. We use an argument similar to the one in [6]. By hypothesis, g must extend as a $C(\bar{\Omega})$ function that is convex in Ω. Therefore, the Dirichlet problem with zero right-hand side and boundary data g (problem (5.1)) has a solution $w \in C(\bar{\Omega})$. By the comparison principle (Theorem 4.2), $u_j \leq w$ in $\bar{\Omega}$ for all j, so the u_j are uniformly bounded above.

The next step is to show the u_j are bounded below. Let $\xi \in \partial\Omega$ and let $\epsilon > 0$. Then there exists an affine function L such that $L \le g$ on $\partial\Omega$ and $L(\xi) \ge g(\xi) - \epsilon$ (see (5.3) and Theorem 5.2 below). Let $v_j = u_j - L$. Then v_j is convex in Ω, $v_j \ge 0$ on $\partial\Omega$, and $Mv_j = Mu_j$. If $v_j \ge 0$ in Ω for all j, then $u_j \ge L$ for all j, and we have our lower bound. If $v_j(x) < 0$ for any $x \in \Omega$, we can then apply Theorem 2.2 to v_j in the convex set $\tilde\Omega \subset \Omega$ where $v_j < 0$:

$$\begin{aligned}(-v_j(x))^n &\le C \operatorname{dist}(x, \partial\tilde\Omega)(\operatorname{diam}(\tilde\Omega))^{n-1} Mv_j(\tilde\Omega)\\ &\le C \operatorname{dist}(x, \partial\Omega)(\operatorname{diam}(\Omega))^{n-1} Mv_j(\Omega)\\ &\le C \operatorname{dist}(x, \partial\Omega)(\operatorname{diam}(\Omega))^{n-1} A.\end{aligned}$$

Therefore,

$$v_j(x) \ge -C(\operatorname{dist}(x, \partial\Omega))^{1/n}(\operatorname{diam}(\Omega))^{(n-1)/n} A^{1/n} \equiv F(x).$$

Therefore $u_j \ge F + L$ in $\tilde\Omega$. Since $u_j \ge L$ in $\Omega \setminus \tilde\Omega$ and $F \le 0$, we have that $u_j \ge F + L$ in Ω, and the u_j are uniformly bounded below.

Using (4.1), one easily sees that the u_j are locally uniformly Lipschitz in Ω. Then by Arzela-Ascoli, there is a subsequence of the u_j that converges uniformly on compact sets to a function $u \in C(\Omega)$. Since the u_j are convex, so is u. By Lemma 2.3 the measures Mu_j converge weakly to Mu in Ω, but by hypothesis, $Mu_j \to \mu$ weakly in Ω, so $Mu = \mu$.

It remains to show that $u \in C(\bar\Omega)$ and that $u = g$ on $\partial\Omega$. For all j and any $x \in \Omega$,

$$w(x) \ge u_j(x) \ge F(x) + L(x),$$

so that

$$w(x) \ge u(x) \ge F(x) + L(x).$$

If $x \to \xi$, $w(x) \to w(\xi) = g(\xi)$, $F(x) \to 0$ and $L(x) \to L(\xi) \ge g(\xi) - \epsilon$. Since $\epsilon > 0$ is arbitrary, $u \in C(\bar\Omega)$ and $u = g$ on $\partial\Omega$. This completes the proof of Lemma 5.1.

We now use Lemma 5.1 to complete the proof of Theorem 1.1. Let Ω_j be a sequence of domains compactly contained in Ω that increase to Ω. Given a finite Borel measure μ on Ω, define $\mu_j = \chi_{\Omega_j}\mu$, where χ_{Ω_j} is the characteristic function of Ω_j. Then μ_j vanishes on a neighborhood of $\partial\Omega$, so by the previous argument, the problem

$$\begin{gathered}\det D^2 u = \mu_j \quad \text{in } \Omega\\ u = g \quad \text{on } \partial\Omega.\end{gathered}$$

has a solution $u_j \in C(\bar\Omega)$. Then all the conditions of Lemma 5.1 are met (with $A = \mu(\Omega)$), and we obtain a convex function $u \in C(\bar\Omega)$ that satisfies $\det D^2 u = \mu$ and is equal to g on $\partial\Omega$. □

We now give a necessary and sufficient condition for the extendability of g as a convex function, and justify the remark preceding (5.3).

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. $g \in C(\partial\Omega)$ can be extended to a function $\tilde g \in C(\bar\Omega)$, convex in Ω, if and only if the function U, defined by (5.2), is continuous in $\bar\Omega$ and $U = g$ on $\partial\Omega$.*

Proof. Note that U, being the supremum of bounded convex functions, is convex in Ω. Therefore, if U is continuous in $\bar\Omega$ with boundary values g, g extends as a convex function.

Now suppose that g extends to a convex function $\tilde{g} \in C(\bar{\Omega})$. Then if T is a flat portion of $\partial\Omega$, $g|_T$ must be convex. We saw earlier that U agrees with the solution v of problem (5.1) in Ω, so it only remains to check that U is continuous up to $\partial\Omega$ and that $U = g$ on $\partial\Omega$. Let $\xi \in \partial\Omega$ and suppose $\{x_n\} \subset \Omega$ is such that $x_n \to \xi$. Since $U = v$ in Ω, $v = g$ on $\partial\Omega$ and $v \in C(\bar{\Omega})$, we have $U(x_n) \to g(\xi)$. Therefore, if $U = g$ on $\partial\Omega$ we are done.

By definition, $U \le g$ on $\partial\Omega$. Suppose that there exists $\xi \in \partial\Omega$ for which $U(\xi) < g(\xi)$, and let $\epsilon = g(\xi) - U(\xi) > 0$. Either $\xi \in T$, where T is a flat portion of $\partial\Omega$, or ξ is a point of strict convexity, meaning that there exists an affine function $P : \mathbb{R}^n \to \mathbb{R}$ such that $P(\xi) = 0$ and $P > 0$ in $\Omega \setminus \xi$. The same argument as that on pp. 17-18 of [6] can be used to show that if η is a point of strict convexity, then $U(\eta) = g(\eta)$, so ξ must lie in a line segment contained in $\partial\Omega$.

There exists an affine function P such that $P(\xi) = 0$ and $P \ge 0$ in $\bar{\Omega}$. Let $T = \bar{\Omega} \cap \{P(x) = 0\}$. Let $0 < m < n$ be the dimension of T. Write a point $x \in \mathbb{R}^n$ as $(x_1, x_2, \dots, x_n)$. We may assume $\bar{\Omega} \subset \{x_n \ge 0\}$ and T is a subset of $\{x_{m+1} = x_{m+2} = \dots = x_n = 0\}$, which we identify with $\mathbb{R}^m$. Identifying points of T with their projection into $\mathbb{R}^m$, there exists $p \in \mathbb{R}^m$ such that

$$g(y) \ge g(\xi) + p \cdot (y - \xi) \tag{5.5}$$

for all $y \in T$. We now claim that there exists $\delta > 0$ such that

$$g(x) \ge g(\xi) + p \cdot (y - \xi) - \epsilon/2 \tag{5.6}$$

for all $x \in \partial\Omega \cap \{P(x) \le \delta\}$, where y is the projection of x into $\mathbb{R}^m$. If (5.6) does not hold for any $\delta > 0$, then there is a decreasing sequence $\delta_n \to 0$ and a corresponding sequence of points x_n such that

$$\begin{gathered} x_n \in \partial\Omega \cap \{P(x) \le \delta_n\}, \\ g(x_n) < g(\xi) + p \cdot (y_n - \xi) - \epsilon/2. \end{gathered} \tag{5.7}$$

Passing to a subsequence if necessary, we get $x_n \to x \in \partial\Omega$. Since $\delta_n \ge P(x_n) \ge 0$ and P is continuous, we get that $P(x) = 0$ and $x \in T$. On the other hand, from (5.7) we conclude that

$$g(x) \le g(\xi) + p \cdot (y - \xi) - \epsilon/2$$

contradicting (5.5). Therefore, we have (5.6) for some $\delta > 0$. Let

$$L(x) = g(\xi) + p \cdot (y - \xi) - \epsilon/2 - CP(x)$$

where $C \ge 0$ is a constant to be chosen and $x = (y, z) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$. Then, regardless of the choice of $C \ge 0$, L is an affine function for which $L(\xi) = g(\xi) - \epsilon/2 > U(\xi)$ and $L \le g$ in $\partial\Omega \cap \{P(x) \le \delta\}$. If C can be chosen such that $L \le g$ on the rest of $\partial\Omega$, this will contradict the definition of $U(\xi)$ and we will obtain $U = g$ on $\partial\Omega$ and $U \in C(\bar{\Omega})$. If $C \ge 0$, then for any $x \in \partial\Omega$ with $P(x) \ge \delta$, we have that

$$L(x) \le g(\xi) + p \cdot (y - \xi) - \epsilon/2 - C\delta.$$

Let $M = \max_{\partial\Omega}(p \cdot (y - \xi) - g(x))$. Since Ω is bounded and g is continuous, M is finite. Choose

$$C \ge \max\Big\{\frac{g(\xi) - \epsilon/2 + M}{\delta}, 0\Big\}.$$

Then for any $x \in \partial\Omega$, with $P(x) \ge \delta$, we have $L(x) \le g(x)$, and hence $L \le g$ on $\partial\Omega$, and the proof of Theorem 5.2 is complete. $\square$

We remark that the preceding argument shows that $g \in C(\partial\Omega)$ can be extended as a convex function, continuous in $\bar{\Omega}$ if and only if g, restricted to any portion of a hyperplane that lies in $\partial\Omega$, is convex. In particular, there is no restriction on g at any point of strict convexity.

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