

## NONCLASSICAL SHOCK WAVES OF CONSERVATION LAWS: FLUX FUNCTION HAVING TWO INFLECTION POINTS

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ABSTRACT. We consider the Riemann problem for non-genuinely nonlinear conservation laws where the flux function admits two inflection points. This is a simplification of van der Waals fluid pressure, which can be seen as a function of the specific volume for a specific entropy at which the system lacks the non-genuine nonlinearity. Corresponding to each inflection point, a nonclassical Riemann solver can be uniquely constructed. Furthermore, two kinetic relations can be used to construct nonclassical Riemann solutions.

### 1. INTRODUCTION

The theory of nonclassical solutions of hyperbolic systems of conservation laws has been introduced by LeFloch and has been developed for many years. Nonclassical shocks may appear when the system fails to be genuinely nonlinear. Briefly, nonclassical shock waves violate the standard the Oleinik criterion [14] in the scalar case and the Lax shock inequalities [6] and the Liu entropy conditions [13] for the case of hyperbolic systems of conservation laws. To select nonclassical shock waves, by a standard way, one follows the strategy proposed by Abeyaratne-Knowles [1, 2], and by Hayes and LeFloch [3, 4, 8] to describe the whole family of *nonclassical Riemann solutions* and then to use a *kinetic relation* to determine the relevant physical solution. For material on this subject, see the text book [8]. Related works can be found in [7, 3, 4, 8, 9, 10, 11, 12, 5, 16, 18, 15, 20, 19, 17] and the references therein.

In this paper, we consider the Riemann problem for conservation laws, where the flux-function has two inflection points

$$\begin{aligned} \partial_t u + \partial_x f(u) &= 0, \\ u(x, 0) &= \begin{cases} u_l & \text{for } x < 0, \\ u_r & \text{for } x > 0. \end{cases} \end{aligned} \tag{1.1}$$

Here,  $u_l$  and  $u_r$  are constants.

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The flux function  $f$  is a twice differentiable function of  $u \in \mathbb{R}$  and is assumed to satisfy the following hypotheses

$$\begin{aligned} f''(u) &> 0 \quad \text{for } u \in (-\infty, 0) \cup (1, +\infty), \\ f''(u) &< 0 \quad \text{for } u \in (0, 1), \\ \lim_{u \rightarrow \pm\infty} f'(u) &= +\infty, \quad \lim_{u \rightarrow \pm\infty} f(u) = +\infty. \end{aligned} \tag{1.2}$$

Thus the flux  $f$  has two inflection points at  $u = 0$  and  $u = 1$ . The specification of these two values does not restrict the scope of consideration of this paper. By assumption, the function  $f$  is clearly convex in each interval  $(-\infty, 0)$  and  $(1, +\infty)$ , and is concave in the interval  $(0, 1)$ . To specify these intervals, we denote

$$\begin{aligned} E_I &:= (-\infty, 0), \\ E_{II} &:= [0, 1], \\ E_{III} &:= (1, +\infty), \end{aligned} \tag{1.3}$$

and call each of them a *phase*.

In studying nonclassical shocks, one is concerned at the break of the genuine nonlinearity of the system on a manifold. In many situations, this manifold may be reduced to be simply an inflection point of a flux function in appropriate coordinates. In several models such as the Van der Waals fluids, non-genuine nonlinearity may occur not only on one, but on two manifolds of phase domains.

In the work of LeFloch-Thanh [9], the presence of two inflection points in the flux function was studied. The nonclassical Riemann solver was constructed by restricting only on the first kinetic function, though we may have several kinetic functions on a Hugoniot curve. More clearly, following the strategy proposed by Abeyaratne-Knowles [1, 2], and by Hayes-LeFloch [3, 4, 8], the authors define the entropy dissipation to describe the whole set of nonclassical waves. It appears that the entropy dissipation may vanish three times. And this would lead to the definition of two kinetic functions. The domain as well as the range of each of these two kinetic function contains one inflection point and its values are symmetric to the variable values with respect to the inflection point. The difficulty to use the second kinetic function is that the shock speed involving the second kinetic function may be less than that of the shock speed using the first kinetic function. Consequently, the Riemann solution may not be well-defined when two kinetic functions are to be involved. In LeFloch-Thanh [12], phase transitions were observed. All nonclassical shock waves satisfying a single entropy condition that entropy should be nondecreasing in time were also characterized.

This paper will deal with the case of two apparent kinetic functions, continuing works in [9, 12]. For simplicity, we restrict our attention to the scalar case where we have a single conservation law. The flux function will have the shape of the pressure of van der Waals fluids in the region where it admits two inflection points. Accordingly, we may have two kinetic functions, and we will consider when we can use each of them, or both. Moreover, as the entropy dissipation selects nonclassical waves like the rule of equal areas, we will define the kinetic functions relying on the rule of equal areas to set up their domains. To select a unique solution, however, we have to restrict the range of kinetic functions such that the chord connecting two states of a nonclassical shock cuts the graph of the flux function at only one

point. The construction may be more visual in some sense. We note that a similar way was constructed for classical shock waves by Oleinik [14].

We note that the existence of nonclassical shock waves is related to the existence of travelling waves of a regularized problem for diffusive-dispersive models, when the diffusive and dispersive coefficients tend to zero, see [8].

This paper is organized as follows. In Section 2 we will investigate the properties of tangents to the graph of  $f$ , and then we review the Oleinik construction of the entropy solution. Section 3 will be devoted to selecting non-classical Riemann solutions relying on one kinetic relation corresponding to each inflection point. In Section 4 we will give a Riemann solver which involves two kinetic relations.

## 2. BASIC PROPERTIES AND OLEINIK CONSTRUCTION

In this section, first we describe several essential properties of the flux function  $f$ . Tangents to the graph of  $f$  will be used to select nonclassical shocks instead of an entropy dissipation. Second, we review the Oleinik construction for *classical entropy solutions* of the problem (1.1).

Recall that a discontinuity of (1.1) the form

$$u(x, t) = \begin{cases} u_l & \text{for } x < st, \\ u_r & \text{for } x > st, \end{cases} \quad (2.1)$$

connecting the left-hand state  $u_l$  and the right-hand state  $u_r$  with shock speed  $s$ , is called a classical shock of (1.1) if it satisfies the Rankine-Hugoniot relations

$$-s(u_r - u_l) + (f(u_r) - f(u_l)) = 0, \quad (2.2)$$

and the Oleinik entropy criterion

$$\frac{f(u) - f(u_l)}{u - u_l} \geq \frac{f(u_r) - f(u_l)}{u_r - u_l}, \quad \text{for } u \text{ between } u_r \text{ and } u_l. \quad (2.3)$$

The condition (2.3) means that the graph of  $f$  is lying below (above) the line connecting  $u_l$  to  $u_r$  when  $u_r < u_l$  (respectively  $u_r > u_l$ ).

Under the hypotheses (1.2), the tangents at 1 and 0 cut the graph of the flux function  $f$  at points  $a$  and  $b$ , respectively, with  $a < 0 < 1 < b$ ; see Figure 2.1. From each  $u \in (a, b)$ , one can draw two distinct tangents to the graph of  $f$ . Denote these tangent points by  $\phi^\sharp(u)$  and  $\psi^\sharp(u)$  with

$$\phi^\sharp(u) < \psi^\sharp(u).$$

In other words,

$$\begin{aligned} f'(\phi^\sharp(u)) &= \frac{f(u) - f(\phi^\sharp(u))}{u - \phi^\sharp(u)}, \\ f'(\psi^\sharp(u)) &= \frac{f(u) - f(\psi^\sharp(u))}{u - \psi^\sharp(u)}. \end{aligned} \quad (2.4)$$

To the end points of the interval under consideration  $a, b$  we set

$$\phi^\sharp(a) = \psi^\sharp(a) = 1 \quad \text{and} \quad \phi^\sharp(b) = \psi^\sharp(b) = 0.$$

There are no tangents to the graph of  $f$  from any point outside the interval  $[a, b]$ . Besides, the values  $u$  and  $\psi^\sharp(u)$  always lie on different sides with respect to  $u = 1$ ,

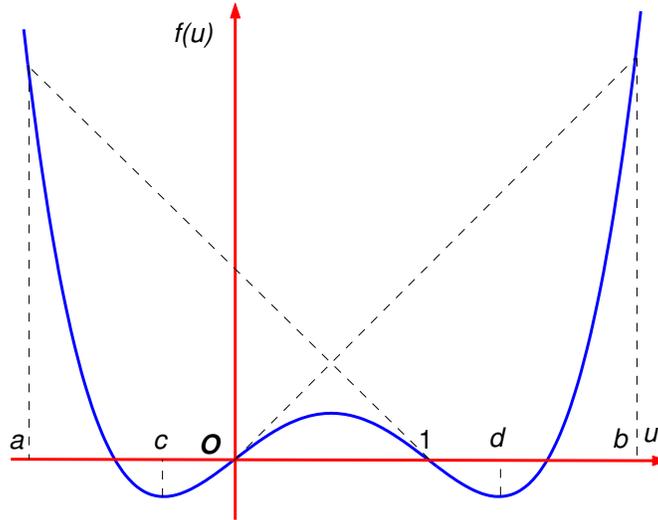


FIGURE 1. Flux function having two inflection points

and the values  $u$  and  $\phi^{\natural}(u)$  always lie on different sides with respect to  $u = 0$ , i.e.

$$\begin{aligned} \phi^{\natural}(u)u < 0 \quad \text{for } u \neq 0, \quad \phi^{\natural}(0) = 0, \\ (\psi^{\natural}(u) - 1)(u - 1) < 0 \quad \text{for } u \neq 1, \quad \psi^{\natural}(1) = 1. \end{aligned} \quad (2.5)$$

There are two points  $c < d$  such that the epigraph of the function  $\tilde{f}$  defined by

$$\tilde{f}(u) = \begin{cases} f(u), & \text{if } u \in (-\infty, c] \cup [d, +\infty), \\ \text{affine} & \text{on } [c, d], \end{cases} \quad (2.6)$$

coincides with the convex hull of that of the function  $f$ . Geometrically, the tangents to the graph of  $f$  from  $c$  and  $d$  coincide. Such points  $c$  and  $d$  are unique. More clearly,

$$f'(c) = \frac{f(d) - f(c)}{d - c} = f'(d).$$

It is not difficult to check that

**Proposition 2.1.** *The function  $\psi^{\natural}$  is increasing for  $u \in [a, c]$  and decreasing for  $u \in [c, b]$ . The function  $\phi^{\natural}$  is decreasing for  $u \in [a, d]$  and increasing for  $u \in [d, b]$ . Moreover  $\phi^{\natural}$  maps  $[a, b]$  onto  $[c, 1]$ , while  $\psi^{\natural}$  maps  $[a, b]$  onto  $[0, d]$ .*

Inversely, the tangent from a point  $u \in (c, d)$  cuts the graph of  $f$  at exactly two distinct points, denoted by  $\phi^{-\natural}(v)$  and  $\psi^{-\natural}(v)$ , with the convention

$$\phi^{-\natural}(v) < \psi^{-\natural}(v).$$

The definition can be extended to the end values  $c$  and  $d$  as

$$\phi^{-\natural}(c) = d, \quad \text{and} \quad \psi^{-\natural}(d) = c.$$

The functions  $\phi^{-\natural}$  and  $\psi^{-\natural}$  in the interval  $[c, d]$  are not monotone and therefore not one-to-one. However, they are monotone for  $u \in [0, 1]$ . Restricting consideration to the interval  $[0, 1]$ , they are the inverses of the functions  $\phi^{\natural}$  and  $\psi^{\natural}$  defined

above, respectively:

$$\phi^{\sharp} \circ \phi^{-\sharp} = \psi^{\sharp} \circ \psi^{-\sharp} = id \quad \text{on the interval } [0, 1]. \quad (2.7)$$

Since we want to discuss the tangent functions in the whole interval  $[a, b]$ , we can assume for the global purpose

$$\begin{aligned} \phi^{-\sharp}(u) &= \psi^{-\sharp}(u) = +\infty, & u \in [a, c), \\ \phi^{-\sharp}(u) &= \psi^{-\sharp}(u) = -\infty, & u \in (d, b]. \end{aligned} \quad (2.8)$$

We were dealing with tangent points and points from which we can draw tangents to the graph. Between these two kinds of points, there is another kind of points that will be concerned to the dynamics of phase transition.

The following proposition can easily be verified.

**Proposition 2.2.** *Given a point  $u \in (a, b)$ , any line between  $u$  and another point  $v \in (\phi^{\sharp}(u), \psi^{\sharp}(u))$  cuts the graph of  $f$  at exactly four points of which  $u$  and  $v$  are the two. Denote such the remaining two points by  $\phi^{\sharp}(u, v)$  and  $\psi^{\sharp}(u, v)$ , with convention*

$$\phi^{\sharp}(u, v) < \psi^{\sharp}(u, v).$$

For the limit cases, we set

$$\begin{aligned} \phi^{\sharp}(u, v = \phi^{\sharp}(u)) &:= \phi^{\sharp}(u) = v, \\ \psi^{\sharp}(u, v = \psi^{\sharp}(u)) &:= \psi^{\sharp}(u) = v. \end{aligned}$$

By definition, the values  $\phi^{\sharp}(u, v)$  and  $\psi^{\sharp}(u, v)$  satisfy

$$\frac{f(\phi^{\sharp}(u, v)) - f(u)}{\phi^{\sharp}(u, v) - u} = \frac{f(\psi^{\sharp}(u, v)) - f(u)}{\psi^{\sharp}(u, v) - u} = \frac{f(v) - f(u)}{v - u}. \quad (2.9)$$

Next, we turn to the Oleinik construction [14] of the entropy solution of the problem (1.1). The following lemma characterizes shock waves that are admissible by the Oleinik criterion.

**Lemma 2.3** (Classical shocks). *Given a left-hand state  $u_0$ , the set of right-hand states  $u_1$  attainable by a classical shock is given by*

- (i) *If  $u_0 \in (-\infty, c) \cup (b, +\infty)$ , then  $u_1 \in (-\infty, u_0]$ .*
- (ii) *If  $u_0 \in [c, 0]$ , then  $u_1 \in (-\infty, u_0] \cup [\phi^{-\sharp}(u_0), \psi^{\sharp}(u_0)]$ .*
- (iii) *If  $u_0 \in (0, 1)$ , then  $u_1 \in (-\infty, \phi^{-\sharp}(u_0)] \cup [u_0, \psi^{\sharp}(u_0)]$ .*
- (iv) *If  $u_0 \in [1, b]$ , then  $u_1 \in (-\infty, \phi^{-\sharp}(\psi^{\sharp}(u_0))] \cup [\psi^{\sharp}(u_0), u_0]$ .*

So we are at the position to construct the classical Riemann solutions. First, for  $u_l \in (-\infty, c)$ , Lemma 2.2 asserts that all the states  $u_r \in (-\infty, u_l)$  can be reached by a single shock. States  $u_r \in (u_l, 0]$  can be arrived at by a single rarefaction wave, since the characteristic speed is increasing when we move from  $u_l$  to  $u_r$ . If  $u_r \in [0, d]$ , we have  $\phi^{\sharp}(u_r) \in [c, 0]$ . So the solution is a composite of a rarefaction wave from  $u_l$  to  $\phi^{\sharp}(u_r)$  followed by a shock from  $\phi^{\sharp}(u_r)$  to  $u_r$ . If  $u_r > d$ , the solution is combined from three elementary waves: a rarefaction wave from  $u_l$  to  $c$ , followed by a shock from  $c$  to  $d$ , and then followed by a rarefaction wave from  $d$  to  $u_r$ .

Second, we deal with  $u_l \in [c, 0]$ . If  $u_r \in (-\infty, u_l)$ , the Riemann solution is a single shock. A single rarefaction wave can connect  $u_l$  with the states  $u_r \in (u_l, 0]$ . If  $u_r \in [0, \phi^{-\sharp}(u_l)]$ , then  $\phi^{\sharp}(u_r) \in [u_l, 0]$  and the Riemann solution is composed by a rarefaction wave from  $u_l$  to  $\phi^{\sharp}(u_r)$  followed by a shock from  $\phi^{\sharp}(u_r)$  to  $u_r$ . A

single shock from  $u_l$  can reach  $u_r \in (\phi^{-\natural}(u_l), \psi^{\natural}(u_l)]$ . Finally, if  $u_r > \psi^{\natural}(u_l)$ , the solution is a composite of a shock from  $u_l$  to  $\psi^{\natural}(u_l)$  followed with a rarefaction wave connecting  $\psi^{\natural}(u_l)$  to  $u_r$ .

Third,  $u_l \in (0, 1)$ . A single shock from  $u_l$  can reach  $u_r \in (-\infty, \phi^{-\natural}(u_l)] \cup [u_l, \psi^{\natural}(u_l)]$ . A single rarefaction wave from  $u_l$  can connect to  $u_r \in [0, u_l]$ . If  $u_r \in (\phi^{-\natural}(u_l), 0)$ , then there exists a unique value  $u^* \in (0, u_l)$  such that  $\phi^{-\natural}(u^*) = u_r$ . That is  $u^* = \phi^{\natural}(u_r)$ . In that case the Riemann solution is a rarefaction wave connecting  $u_l$  to  $u^*$  followed by a shock connecting  $u^*$  to  $u_r$ . Finally, if  $u_r > \psi^{\natural}(u_l)$ , the Riemann solution is a shock connecting  $u_l$  to  $\psi^{\natural}(u_l)$  followed with a rarefaction wave from  $\psi^{\natural}(u_l)$  to  $u_r$ .

Fourth, assume that  $u_l \in [1, b]$ . A single shock from  $u_l$  can reach

$$u_r \in (-\infty, \phi^{-\natural}(\psi^{\natural}(u_l))] \cup [\psi^{\natural}(u_l), u_l].$$

A single rarefaction wave from  $u_l$  can connect to  $u_r \in [u_l, +\infty)$ . If  $u_r \in [0, \psi^{\natural}(u_l))$ , the Riemann solution is combined by a shock from  $u_l$  to  $\psi^{\natural}(u_l)$  followed by a rarefaction from  $\psi^{\natural}(u_l)$  to  $u_r$ . If  $u_r \in (\phi^{-\natural}(\psi^{\natural}(u_l)), a)$ , the solution contained three waves: a shock from  $u_l$  to  $\psi^{\natural}(u_l)$ , followed by a rarefaction from  $\psi^{\natural}(u_l)$  to  $\phi^{\natural}(u_r)$ , and followed by a shock connecting  $\phi^{\natural}(u_r)$  to  $u_r$ .

Finally, if  $u_l \in (b, +\infty)$ , then the Riemann solution is simply a shock if  $u_r < u_l$  and a rarefaction wave otherwise. We arrive at the following conclusion.

**Theorem 2.4** (Classical Riemann solver). *Under the assumption (1.2), the Riemann problem (1.1) admits a unique classical solution in the class of piecewise smooth self-similar functions made of rarefaction fans and shock waves satisfying the Oleinik entropy criterion. This solution depends continuously on the Riemann data.*

### 3. NON-CLASSICAL RIEMANN SOLVERS USING ONE KINETIC RELATION

In this section, we will present two *non-classical* Riemann solvers. The first one relying on *non-classical* jumps (see definition below) crossing the first inflection point  $u = 0$ . This solver can be proved to depend continuously on Riemann data. The second one using non-classical jumps crossing the second inflection point  $u = 1$ . The Riemann solver, however, does not depend continuously on Riemann data.

#### 3.1. Riemann solver relying on jumps crossing the inflection point $u = 0$ .

A function  $\phi : [a, b] \rightarrow [a, b]$  is called a *kinetic relation corresponding to the inflection point  $u = 0$*  if there exists a point  $p \in (a, c)$  such that in  $\Omega_0 := [p, 1] \ni 0$ , the following conditions are satisfied (see Figure 2):

- (A1) The function  $\phi$  is monotone decreasing in  $\Omega_0$ ,  $\phi(u) \leq \psi^{\natural}(u)$ ,  $\phi(u)$  lies between  $\phi^{\natural}(u)$  and  $\phi^{-\natural}(u)$ , for  $u \in \Omega_0$  in the sense that

$$\begin{aligned} \phi^{-\natural}(u) &> \phi(u) > \phi^{\natural}(u), & \forall u < 0, \\ \phi^{-\natural}(u) &< \phi(u) < \phi^{\natural}(u), & \forall u > 0, \\ \phi(0) &= \phi^{\natural}(0) = \phi^{-\natural}(0) = 0; \end{aligned} \tag{3.1}$$

- (A2) The *contraction property*

$$|\phi \circ \phi(u)| < |u|, \quad \forall u \in \Omega_0, \quad u \neq 0. \tag{3.2}$$

(A3) Conditions at limits:

$$\phi(p) = \psi^\sharp(p), \quad \phi(1) = a. \quad (3.3)$$

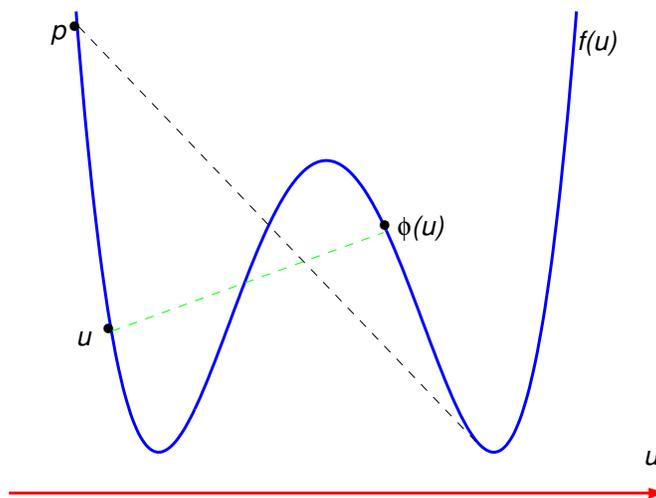


FIGURE 2. Kinetic Function  $\phi$

For an arbitrary non-classical shock between a given left-hand state  $u_0$  and a given right-hand state  $u_1$ , *kinetic relation* is the requirement that

$$u_1 = \phi(u_0). \quad (3.4)$$

To select non-classical shock rather than classical ones, we postulate that

(C) Non-classical shocks are preferred whenever available.

We now solve the Riemann problem relying on this condition. The construction in this section is similar to the non-classical one for the 1-wave family in [9], but we want to recall it here for completeness.

Suppose first that  $u_l \in (-\infty, p)$ . Any point  $u_r \in (-\infty, u_l)$  can be achieved by a single classical shock. Any point  $u_r \in (u_l, 0]$  is attainable by a single rarefaction wave. If  $u_r \in (0, \phi(p)]$ , there exists a unique point  $u_* \in [p, 0)$  such that  $u_r = \phi(u_*)$ . The solution is then the composite of a rarefaction wave from  $u_l$  to  $u_*$  followed by a nonclassical shock from  $u_*$  to  $u_r$ . If  $u_r \in (\phi(p), +\infty)$ , the solution consists of three parts: A rarefaction wave from  $u_l$  to  $p$  followed by a nonclassical shock from  $p$  to  $\phi(p)$ , followed by a rarefaction wave from  $\phi(p)$  to  $u_r$ .

Second, suppose that  $u_l \in [p, 0)$ . A point  $u_r \in (-\infty, u_l)$  can be attained by a single classical shock. A point  $u_r \in (u_l, 0]$  is attainable by a single rarefaction wave. If  $u_r \in (0, \phi(u_l)]$ , there exists a unique point  $u_* \in [u_l, a)$  such that  $u_r = \phi(u_*)$ . The solution is then the composite of the rarefaction wave from  $u_l$  to  $u_*$  followed by a nonclassical shock from  $u_*$  to  $u_r$ . If  $u_r \in (\phi(u_l), \phi(p)]$ , there exists a unique point  $u^* \in [p, u_l)$  such that  $u_r = \phi(u^*)$ . For this construction to make sense, one must here check whether the classical shock from  $u_l$  to  $u^*$  is slower than the nonclassical

shock from  $u^*$  to  $u_r$ . So, consider the function

$$\tilde{f}(v) := \begin{cases} f(v), & \text{if } v \in (-\infty, u_l], \\ f(u_l) + f'(u_l)(v - u_l), & \text{if } v \in (u_l, +\infty). \end{cases} \quad (3.5)$$

If  $u_r \in (\phi(u_l), \eta)$ , where

$$\eta := \min\{\phi(p), \phi^{-\natural}(u_l)\},$$

the function  $\tilde{f}$  is convex on  $(-\infty, +\infty)$  and the points  $u^*$  and  $u_r$  belong to its epigraph. Therefore, the line segment connecting  $u^*$  and  $u_r$  should lie above the graph of  $\tilde{f}$  in the interval  $(u^*, u_r) \ni u_l$ . That is to say

$$\frac{\tilde{f}(u_l) - \tilde{f}(u^*)}{u_l - u^*} < \frac{f(u_r) - f(u^*)}{u_r - u^*},$$

i.e.,

$$s(u_l, u^*) < s(u^*, u_r). \quad (3.6)$$

The latter inequality means precisely that the classical shock from  $u_l$  to  $u^*$  can be followed by the nonclassical shock from  $u^*$  to  $u_r$ .

In the latter construction, if  $u_l \in [p, \phi^{\natural}(\phi(p))]$ , then

$$\eta = \phi(p),$$

and we have completed the argument when  $u_r \in (\phi(u_l), \phi(p))$ . For  $u_r \in (\phi(p), +\infty)$ , the Riemann solution consists of three parts: A classical shock from  $u_l$  to  $p$  followed by a nonclassical shock from  $p$  to  $\phi(p)$ , followed by a rarefaction wave from  $\phi(p)$  to  $u_r$ .

Suppose next that  $u_l \in [\phi^{\natural}(\phi(p)), 0)$ , then

$$\eta = \phi^{-\natural}(u_l).$$

If  $u_r \in [\phi^{-\natural}(u_l), \phi(p)]$ , the solution can be a classical shock connecting  $u_l$  to  $u^*$  followed by a nonclassical shock from  $u^*$  to  $u_r$  provided (3.6) holds, or else a single classical shock. For  $u_r \in (\phi(p), +\infty)$ , if

$$s(u_l, p) < s(p, \phi(p)), \quad (3.7)$$

then the solution consists of a classical shock from  $u_l$  to  $p$ , followed by a nonclassical shock from  $p$  to  $\phi(p)$ , then followed by a rarefaction wave. If else, (3.7) fails, then the solution is either a classical shock from  $u_l$  to  $u_r$  if  $u_r \leq \psi^{\natural}(u_l)$  or a classical shock from  $u_l$  to  $\psi^{\natural}(u_l)$  followed by a rarefaction wave from  $\psi^{\natural}(u_l)$  to  $u_r$  if else.

Third, suppose that  $u_l \in [0, 1)$ . The points  $u_r \in [0, +\infty)$  are reached by the classical construction described in Section 2. If  $u_r \in [\phi(u_l), 0]$ , there exists a unique point  $u^* \in [0, u_l]$  such that  $u_r = \phi(u^*)$ . The solution then consists of a rarefaction wave connecting  $u_l$  to  $u^*$  followed by a nonclassical shock from  $u^*$  to  $u_r$ . If  $u_r \in [\phi^{-\natural}(u_l), \phi(u_l)]$ , then there exists a unique point  $u_* \in [u_l, 1)$  such that  $u_r = \phi(u_*)$ . Since both  $u_l$  and  $u_*$  belong to  $[0, 1]$  and the function  $f$  is concave in this interval, we have

$$\frac{f(u_l) - f(u_*)}{u_l - u_*} < \frac{f(\phi(u_l)) - f(u_*)}{\phi(u_l) - u_*} < \frac{f(u_r) - f(u_*)}{u_r - u_*}.$$

This means the shock speed  $s(u_l, u_*)$  is less than the shock speed  $s(u_*, u_r)$ . Therefore the Riemann solution can be a classical shock from  $u_l$  to  $u_*$  followed by a nonclassical shock from  $u_*$  to  $u_r$ . If  $u_r \in (a, \phi^{-\natural}(u_l)]$ , there exists a unique point

$u^* \in [u_l, 1)$  such that  $u_r = \phi(u^*)$ . The solution then consists of a classical shock from  $u_l$  to  $u^*$  followed by a nonclassical shock from  $u^*$  to  $u_r$  provided

$$s(u_l, u^*) < s(u^*, u_r),$$

or else a single classical shock. The states  $u_r \in (-\infty, a]$  are reached by single classical shocks.

Finally, when  $u_l \in [1, +\infty)$ , we also use the classical construction described in Section 2.

Denote by  $\phi^{-1} : [a, \phi(p)] \rightarrow [p, 1]$ , the inverse of the kinetic function  $\phi$ , which is also a monotone decreasing mapping.

The arguments presented above are summarized as follows:

**Theorem 3.1** (Construction of the Riemann solver). *Given the left-hand and the right-hand states  $u_l, u_r$ . Under the hypotheses (1.2), we have the following description of the Riemann solver that can be involved in a combination of rarefaction fans and shock waves, satisfying the kinetic relation (3.4) (for nonclassical shocks), and the condition (C):*

*Case 1:  $u_l \in (-\infty, p)$ .*

- *If  $u_r \in (-\infty, u_l)$ , the solution is a single classical shock.*
- *If  $u_r \in (u_l, 0]$ , the solution is a single rarefaction wave.*
- *If  $u_r \in (0, \phi(p))$ , the solution is the composite of a rarefaction wave connecting  $u_l$  to  $u_* := \phi^{-1}(u_r)$  followed by a nonclassical shock from  $u_*$  to  $u_r$ .*
- *If  $u_r \in (\phi(p), +\infty)$ , the solution consists of three parts: A rarefaction wave from  $u_l$  to  $e$  followed by a nonclassical shock from  $e$  to  $\phi(p)$ , followed by a rarefaction wave from  $\phi(p)$  to  $u_r$ .*

*Case 2:  $u_l \in [p, 0)$ .*

- *If  $u_r \in (-\infty, u_l)$ , the solution is a single classical shock.*
- *If  $u_r \in (u_l, 0]$ , the solution is a single rarefaction wave.*
- *If  $u_r \in (0, \phi(u_l))$ , the solution is the composite of a rarefaction wave from  $u_l$  to  $u_* := \phi^{-1}(u_r)$  followed by a nonclassical shock from  $u_*$  to  $u_r$ .*
- *If  $u_l \in [p, \phi^{\natural}(\phi(p)))$  and  $u_r \in (\phi(u_l), \phi(p))$ , then the solution consists of a classical shock from  $u_l$  to  $u^* := \phi^{-1}(u_r)$  followed by a nonclassical shock from  $u^*$  to  $u_r$ .*
- *If  $u_l \in [p, \phi^{\natural}(\phi(p)))$  and  $u_r \in (\phi(p), +\infty)$ , the solution consists of three waves: A classical shock from  $u_l$  to  $e$  followed by a nonclassical shock from  $e$  to  $\phi(p)$ , followed by a rarefaction wave from  $\phi(p)$  to  $u_r$ .*
- *If  $u_l \in [\phi^{\natural}(\phi(p)), 0)$  and  $u_r \in (\phi(u_l), \phi^{-\natural}(u_l))$ , the solution consists of the classical shock from  $u_l$  to  $u^* := \phi^{-1}(u_r)$  followed by a nonclassical shock from  $u^*$  to  $u_r$ .*
- *If  $u_l \in [\phi^{\natural}(\phi(p)), 0)$  and  $u_r \in [\phi^{-\natural}(u_l), \psi^{\natural}(u_l)]$ , the solution is a classical shock from  $u_l$  to  $u^*$  followed by a nonclassical shock from  $u^*$  to  $u_r$  if (3.5) holds, or else a single classical shock.*
- *If  $u_l \in [\phi^{\natural}(\phi(p)), 0)$  and  $u_r \in (\psi^{\natural}(u_l), +\infty)$ , the solution consists of a classical shock from  $u_l$  to  $\psi^{\natural}(u_l)$  followed by a rarefaction wave from  $\psi^{\natural}(u_l)$  to  $u_r$ .*

*Case 3:  $u_l \in [0, 1)$ .*

- *If  $u_r \in [0, +\infty)$ , the solution is classical (Section 2).*

- If  $u_r \in [\phi(u_l), 0]$ , the solution consists of the rarefaction wave from  $u_l$  to  $u^* := \phi^{-1}(u_r)$  followed by a nonclassical shock from  $u^*$  to  $u_r$ .
- If  $u_r \in [\phi^{-\natural}(u_l), \phi(u_l))$ , the solution consists of a classical shock from  $u_l$  to  $u_* := \phi^{-1}(u_r)$  followed by a nonclassical shock from  $u_*$  to  $u_r$ .
- If  $u_r \in [\phi^{-\natural}(u_l), a)$ , the solution consists of the classical shock wave from  $u_l$  to  $u^* := \phi(u_r)$  followed by a nonclassical shock from  $u^*$  to  $u_r$  provided (4.3) holds, or else a single classical shock.
- The states  $u_r \in (-\infty, a]$  are reached by a single classical shock.

Case 4:  $u_l \in [1, +\infty)$ . The construction is classical (Section 2).

### 3.2. Riemann solver relying on jumps crossing the inflection point $u = 1$ .

In this subsection, we will provide a Riemann solver using only non-classical shocks crossing through the inflection point  $u = 1$ . As the behavior of the graph of  $f$  changes across this point from concavity to convexity, another condition will be placed instead of the convex-concave condition (3.2).

A function  $\psi : [a, b] \rightarrow [a, b]$  is called a *kinetic function corresponding to the inflection point  $u = 1$*  if there are points  $\theta \in (c, 0), q \in (d, b)$  such that in  $\Omega_1 := [\theta, q] \ni 1$ , the following conditions are satisfied (see Figure 3):

- (B1) The function  $\psi$  is monotone decreasing in  $\Omega_1$ ,  $\psi(u) \geq \phi^{\natural}(u)$ ,  $\psi(u)$  lies between  $\psi^{\natural}(u)$  and  $\psi^{-\natural}(u)$ ,  $\forall u \in \Omega_1$  in the sense that

$$\begin{aligned} \psi^{-\natural}(u) &> \psi(u) > \psi^{\natural}(u), & \forall u < 1, \\ \psi^{-\natural}(u) &< \psi(u) < \psi^{\natural}(u), & \forall u > 1, \\ \psi(1) &= \psi^{\natural}(1) = \psi^{-\natural}(1) = 1; \end{aligned} \quad (3.8)$$

- (B2) The *contraction property*

$$|\psi \circ \psi(u) - 1| < |u - 1|, \quad \forall u \in \Omega_1, \quad u \neq 1. \quad (3.9)$$

- (B3) Conditions at limits:

$$\psi(q) = \phi^{\natural}(q), \quad \phi^{\natural}(\psi(\theta)) = \theta. \quad (3.10)$$

For any non-classical shock between a given left-hand state  $u_0$  and a given right-hand state  $u_1$ , kinetic relation for the coming construction is the requirement that

$$u_1 = \psi(u_0). \quad (3.11)$$

So, we begin to construct the Riemann solver, postulating the condition (C) in the previous subsection.

Assume first that  $u_l \in (-\infty, \theta)$ . A single classical shock can jump to any  $u_r \in (-\infty, u_l)$ . A single rarefaction wave can connect  $u_l$  from the left to any  $u_r \in [u_l, 0]$  from the right. If  $u_r \in (0, \phi^{-\natural}(\theta))$ , then  $\phi^{\natural}(u_r) \in [\theta, 0)$ , the solution thus is a rarefaction wave from  $u_l$  to  $\phi^{\natural}(u_r)$  followed by a classical shock from  $\phi^{\natural}(u_r)$  to  $u_r$ . If now  $u_r \in (\phi^{-\natural}(\theta), \psi(\theta))$ , the solution consists of a rarefaction wave from  $u_l$  to  $\theta$ , followed by a non-classical shock from  $\theta$  to  $\psi(\theta)$ , then followed by a classical shock from  $\psi(\theta)$  to  $u_r$ . If  $u_r \in [\psi(\theta), +\infty)$ , the solution is a composite of a rarefaction wave from  $u_l$  to  $\theta$ , followed by a non-classical shock from  $\theta$  to  $\psi(\theta)$ , then followed by a rarefaction wave from  $\psi(\theta)$  to  $u_r$ .

Second, let  $u_l \in [\theta, 0]$ . A single classical shock can jump to any  $u_r \in (-\infty, u_l)$ . A single rarefaction wave can connect  $u_l$  to any  $u_r \in [u_l, 0]$ . If  $u_r \in (0, \phi^{-\natural}(u_l))$ , then  $\phi^{\natural}(u_r) \in [u_l, 0)$ , and therefore the solution is a rarefaction wave from  $u_l$  to  $\phi^{\natural}(u_r)$

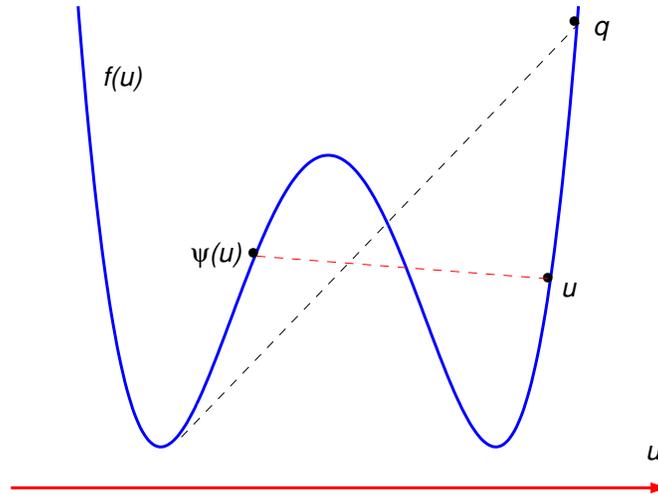


FIGURE 3. Kinetic Function  $\psi$

followed by a classical shock from  $\phi^{\sharp}(u_r)$  to  $u_r$ . If now  $u_r \in (\phi^{-\sharp}(u_l), \psi^{\sharp}(u_l, \psi(u_l)))$ , the solution is a single classical shock. If  $u_r \in (\psi^{\sharp}(u_l, \psi(u_l)), \psi(u_l))$ , then the solution is a non-classical shock from  $u_l$  to  $\psi(u_l)$  followed by a classical shock from  $\psi(u_l)$  to  $u_r$ . If  $u_r \in [\psi(u_l), +\infty)$ , then the solution is composed from a non-classical shock from  $u_l$  to  $\psi(u_l)$  followed by a rarefaction wave from  $\psi(u_l)$  to  $u_r$ .

Third, let  $u_l \in (0, 1)$ . A single classical shock can arrive at any  $u_r \in (-\infty, \phi^{-\sharp}(u_l)]$ . If  $u_r \in (\phi^{-\sharp}(u_l), 0]$ , then  $\phi^{\sharp}(u_r) \in [0, u_l]$ . The solution is thus a rarefaction wave from  $u_l$  to  $\phi^{\sharp}(u_r)$  attached by a classical shock from  $\phi^{\sharp}(u_r)$  to  $u_r$ . If  $u_r \in (0, u_l]$ , the solution is a single rarefaction wave. A single classical shock can arrive at any  $u_r \in (u_l, \psi^{\sharp}(u_l, \psi(u_l))]$ . If  $u_r \in (\psi^{\sharp}(u_l, \psi(u_l)), \psi(u_l))$ , then the solution is a composite of a non-classical shock from  $u_l$  to  $\psi(u_l)$  followed by a classical shock from  $\psi(u_l)$  to  $u_r$ . If  $u_r \in [\psi(u_l), +\infty)$ , then the solution is combined from a non-classical shock from  $u_l$  to  $\psi(u_l)$  followed by a rarefaction wave from  $\psi(u_l)$  to  $u_r$ .

Fourth, assume  $u_l \in (1, \psi^{-1}(0))$ . By the monotony, we have

$$\psi(u_l) > 0.$$

If  $u_r \in [u_l, +\infty)$ , then the solution is a rarefaction wave. A single classical shock can jump from  $u_l$  to any  $u_r \in [\psi^{\sharp}(u_l, \psi(u_l)), u_l]$ .

If  $u_r \in (\psi(u_l), \psi^{\sharp}(u_l, \psi(u_l)))$ , then the solution is combined from two shocks: a non-classical shock from  $u_l$  to  $\psi(u_l)$  followed by a classical shock from  $\psi(u_l)$  to  $u_r$ . If  $u_r \in [0, \psi(u_l)]$ , then the solution is a non-classical shock from  $u_l$  to  $\psi(u_l)$  followed by a rarefaction wave from  $\psi(u_l)$  to  $u_r$ . If now  $u_r \in (\phi^{-\sharp}(\psi(u_l)), 0)$ , then  $\phi^{\sharp}(u_r) \in (0, \psi(u_l))$ . The solution is thus a non-classical shock from  $u_l$  to  $\psi(u_l)$  followed by a rarefaction wave from  $\psi(u_l)$  to  $\phi^{\sharp}(u_r)$  attached by a classical shock from  $\phi^{\sharp}(u_r)$  to  $u_r$ . If  $u_r \in [\psi^{\sharp}(u_l, \psi(u_l)), \phi^{-\sharp}(\psi(u_l))]$ , then the solution is a non-classical shock from  $u_l$  to  $\psi(u_l)$  followed by a classical shock from  $\psi(u_l)$  to  $u_r$ . If  $u_r \in (-\infty, \psi^{\sharp}(u_l, \psi(u_l)))$ , then no non-classical shocks can be involved in the construction. We thus use the classical construction in Section 2 in this interval. The

discontinuity in this regime is

$$u_l \in (1, \psi^{-1}(0)), \quad u_r = \psi^\sharp(u_l, \psi(u_l)). \quad (3.12)$$

Fifth, let  $u_l \in [\psi^{-1}(0), q]$ . The monotony of  $\psi$  yields

$$\psi(u_l) \leq 0.$$

A single rarefaction wave can connect  $u_l$  to any  $u_r \in [u_l, +\infty)$ . A single classical shock can jump from  $u_l$  to any  $u_r \in [\psi^\sharp(u_l, \psi(u_l)), u_l)$ .

If  $u_r \in [\phi^{-\sharp}(\psi(u_l)), \psi^\sharp(u_l, \psi(u_l))]$ , then the solution is combined from two shocks: a non-classical shock from  $u_l$  to  $\psi(u_l)$  followed by a classical one from  $\psi(u_l)$  to  $u_r$ . If  $u_r \in (0, \phi^{-\sharp}(\psi(u_l)))$ , then  $\phi^\sharp(u_r) \in (\psi(u_l), 0)$ . The solution is therefore a composite of a non-classical shock from  $u_l$  to  $\psi(u_l)$  followed by a rarefaction wave from  $\psi(u_l)$  to  $\phi^\sharp(u_r)$ , then attached by a classical shock from  $\phi^\sharp(u_r)$  to  $u_r$ . If  $u_r \in [\psi(u_l), 0]$ , then the solution is a non-classical shock from  $u_l$  to  $\psi(u_l)$  followed by a rarefaction wave from  $\psi(u_l)$  to  $u_r$ . If  $u_r \in [\psi^\sharp(u_l, \psi(u_l)), \psi(u_l)]$ , then the solution is a non-classical shock from  $u_l$  to  $\psi(u_l)$  followed by a classical one from  $\psi(u_l)$  to  $u_r$ . If  $u_r \in (-\infty, \psi^\sharp(u_l, \psi(u_l)))$ , then there are no non-classical shocks and we have a situation similar to the previous one. In this construction, we have a discontinuity sharing the same formula for  $u_r$  but  $u_l \in [\psi^{-1}(0), q]$  instead. Combining this argument and (3.16), we obtain the curve of discontinuity of the construction

$$\{u_l \in (1, q], \quad u_r = \psi^\sharp(u_l, \psi(u_l))\}. \quad (3.13)$$

Finally, let  $u_l \in (q, +\infty)$ . In this case we have no non-classical shocks and we use the classical construction as well.

The above arguments can be summarized in the following theorem

**Theorem 3.2.** *Given the initial Riemann data  $(u_l, u_r)$ . Under the hypotheses (1.2) and the condition (C), the Riemann problem (1.1) admits a unique self-similar solution made of rarefaction waves, classical shocks and non-classical shocks satisfying the kinetic relation (3.15). The Riemann solver is described by*

*Case 1:  $u_l \in (-\infty, \theta)$ .*

- *If  $u_r \in (-\infty, u_l)$ , the solution is a single classical shock.*
- *If  $u_r \in [u_l, 0]$ , the solution is a single rarefaction wave.*
- *If  $u_r \in (0, \phi^{-\sharp}(\theta)]$ , the solution is a rarefaction wave from  $u_l$  to  $\phi^\sharp(u_r)$  followed by a classical shock from  $\phi^\sharp(u_r)$  to  $u_r$ .*
- *If  $u_r \in (\phi^{-\sharp}(\theta), \psi(\theta))$ , the solution is a composite of a rarefaction wave from  $u_l$  to  $\theta$ , followed by a non-classical shock from  $\theta$  to  $\psi(\theta)$ , followed by a classical shock from  $\psi(\theta)$  to  $u_r$ . If  $u_r \in [\psi(\theta), +\infty)$ , the solution is a rarefaction wave from  $u_l$  to  $\theta$ , followed by a non-classical shock from  $\theta$  to  $\psi(\theta)$ , then followed by a rarefaction wave from  $\psi(\theta)$  to  $u_r$ .*

*Case 2:  $u_l \in [\theta, 0]$ .*

- *If  $u_r \in (-\infty, u_l)$ , then the solution is a classical shock.*
- *If  $u_r \in [u_l, 0]$ , the solution is a single rarefaction wave.*
- *If  $u_r \in (0, \phi^{-\sharp}(u_l)]$ , the solution is a rarefaction wave from  $u_l$  to  $\phi^\sharp(u_r)$  followed by a classical shock from  $\phi^\sharp(u_r)$  to  $u_r$ .*
- *If  $u_r \in (\phi^{-\sharp}(u_l), \psi^\sharp(u_l, \psi(u_l))]$ , the solution is a single classical shock.*
- *If  $u_r \in (\psi^\sharp(u_l, \psi(u_l)), \psi(u_l))$ , the solution is a non-classical shock from  $u_l$  to  $\psi(u_l)$  followed by a classical shock from  $\psi(u_l)$  to  $u_r$ .*

- If  $u_r \in [\psi(u_l), +\infty)$ , the solution is a composite of a non-classical shock from  $u_l$  to  $\psi(u_l)$  followed by a rarefaction wave from  $\psi(u_l)$  to  $u_r$ .

Case 3:  $u_l \in (0, 1)$ . If  $u_r \in (-\infty, \phi^{-\sharp}(u_l)]$ , the solution is a classical shock.

- If  $u_r \in (\phi^{-\sharp}(u_l), 0]$ , the solution is a rarefaction wave from  $u_l$  to  $\phi^{\sharp}(u_r)$  attached by a classical shock from  $\phi^{\sharp}(u_r)$  to  $u_r$ .
- If  $u_r \in (0, u_l]$ , the solution is a single rarefaction wave.
- If  $u_r \in (u_l, \psi^{\sharp}(u_l, \psi(u_l))]$ , the solution is a single classical shock.
- If  $u_r \in (\psi^{\sharp}(u_l, \psi(u_l)), \psi(u_l))$ , the solution is a composite of a non-classical shock from  $u_l$  to  $\psi(u_l)$  followed by a classical shock from  $\psi(u_l)$  to  $u_r$ .
- If  $u_r \in [\psi(u_l), +\infty)$ , then the solution is a composite of a non-classical shock from  $u_l$  to  $\psi(u_l)$  followed by a rarefaction wave from  $\psi(u_l)$  to  $u_r$ .

Case 4:  $u_l \in (1, \psi^{-1}(0))$ .

- If  $u_r \in [u_l, +\infty)$ , the solution is a rarefaction wave.
- If  $u_r \in [\psi^{\sharp}(u_l, \psi(u_l)), u_l)$ , the solution is a single classical shock.
- If  $u_r \in (\psi(u_l), \psi^{\sharp}(u_l, \psi(u_l)))$ , the solution is a composite of two shocks: a non-classical shock from  $u_l$  to  $\psi(u_l)$  followed by a classical one from  $\psi(u_l)$  to  $u_r$ .
- If  $u_r \in [0, \psi(u_l)]$ , the solution is a non-classical shock from  $u_l$  to  $\psi(u_l)$  followed by a rarefaction wave from  $\psi(u_l)$  to  $u_r$ .
- If  $u_r \in (\phi^{-\sharp}(\psi(u_l)), 0)$ , the solution is a composite of a non-classical shock from  $u_l$  to  $\psi(u_l)$  followed by a rarefaction wave from  $\psi(u_l)$  to  $\phi^{\sharp}(u_r)$  attached by a classical shock from  $\phi^{\sharp}(u_r)$  to  $u_r$ .
- If  $u_r \in [\psi^{\sharp}(u_l, \psi(u_l)), \phi^{-\sharp}(\psi(u_l))]$ , the solution is a non-classical shock from  $u_l$  to  $\psi(u_l)$  followed by a classical shock from  $\psi(u_l)$  to  $u_r$ .
- If  $u_r \in (-\infty, \psi^{\sharp}(u_l, \psi(u_l)))$ , then the construction is classical (Section 2).

Case 5:  $u_l \in [\psi^{-1}(0), q]$ .

- If  $u_r \in [u_l, +\infty)$ , the solution is a single rarefaction wave.
- If  $u_r \in [\psi^{\sharp}(u_l, \psi(u_l)), u_l)$ , the solution is a single classical shock.
- If  $u_r \in [\phi^{-\sharp}(\psi(u_l)), \psi^{\sharp}(u_l, \psi(u_l))]$ , the solution is a composite of two shocks: a non-classical shock from  $u_l$  to  $\psi(u_l)$  followed by a classical one from  $\psi(u_l)$  to  $u_r$ .
- If  $u_r \in (0, \phi^{-\sharp}(\psi(u_l)))$ , the solution is a composite of three elementary waves: a non-classical shock from  $u_l$  to  $\psi(u_l)$  followed by a rarefaction wave from  $\psi(u_l)$  to  $\phi^{\sharp}(u_r)$ , then attached by a classical shock from  $\phi^{\sharp}(u_r)$  to  $u_r$ .
- If  $u_r \in [\psi(u_l), 0]$ , the solution is a non-classical shock from  $u_l$  to  $\psi(u_l)$  followed by a rarefaction wave from  $\psi(u_l)$  to  $u_r$ .
- If  $u_r \in [\psi^{\sharp}(u_l, \psi(u_l)), \psi(u_l))$ , then the solution is a non-classical shock from  $u_l$  to  $\psi(u_l)$  followed by a classical one from  $\psi(u_l)$  to  $u_r$ .
- If  $u_r \in (-\infty, \psi^{\sharp}(u_l, \psi(u_l)))$ , then the construction is classical.

Case 6:  $u_l \in (q, +\infty)$ , the construction is classical. The curve of discontinuity is

$$\{u_l \in (1, q], \quad u_r = \psi^{\sharp}(u_l, \psi(u_l))\} \subset \mathbb{R}^2.$$

#### 4. NON-CLASSICAL RIEMANN SOLVER USING TWO KINETIC RELATIONS

In this section, we discuss the Riemann solver to the problem (1.1) using two kinetic relations for non-classical shock-waves between two phases. It turns out that even under the condition (C), non-uniqueness appears. A stronger condition is

imposed to guarantee there is a unique choice of non-classical shocks. As expected, the unique Riemann solution does not depend continuously globally on the Riemann data.

Let us first point out several circumstances in which there are distinct choices of non-classical solutions adaptable to the condition (C). Firstly, assume that  $u_l \in (1, \psi^{-1}(0))$ , then  $\psi(u_l) \in (0, 1)$ . Therefore,

$$\phi(\psi(u_l)) \in (\phi^{-\sharp}(\psi(u_l)), \phi^{\sharp}(\psi(u_l))) \subset (\phi^{-\sharp}(\psi(u_l)), 0).$$

If we take

$$u_r = \phi(\psi(u_l)),$$

then we obtain a solution contains two nonclassical shocks: one nonclassical shock corresponding to the kinetic function  $\psi$  from  $u_l$  to  $\psi(u_l)$ , followed by one nonclassical shock corresponding to the kinetic function  $\phi$  from  $\psi(u_l)$  to  $\phi(\psi(u_l))$ . However, as derived from the construction in the subsection 3.2 that in this case, we obtained a non-classical solution containing a non-classical shock corresponding to the kinetic function  $\psi$ : one nonclassical shock from  $u_l$  to  $\psi(u_l)$  followed by a rarefaction wave from  $\psi(u_l)$  to  $\phi^{-\sharp}(u_r)$ , attached by a classical shock from  $\phi^{-\sharp}(u_r)$  to  $u_r$ . This illustrates the co-existence of two nonclassical solutions, one contains more nonclassical shocks than the other, see Figure 4.1.

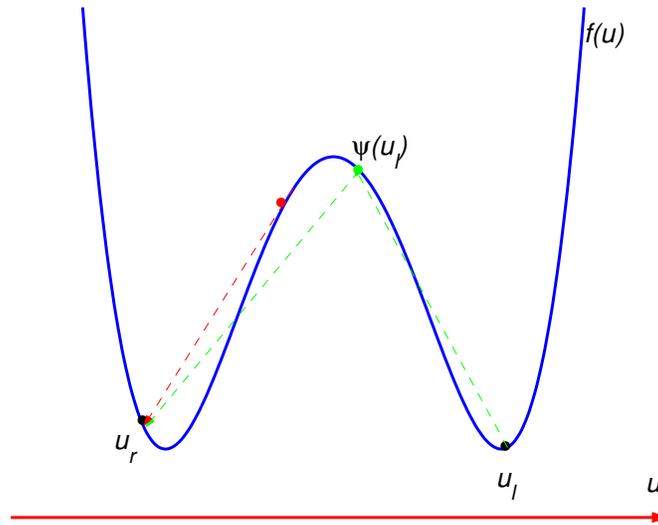


FIGURE 4. Two possible solutions: one contains one and the other contains two nonclassical shocks

Secondly, assume now

$$u_l \in [\theta, 0], \quad \text{and} \quad u_r \in (\psi^{\sharp}(u_l, \psi(u_l)), 1).$$

According to the description in Section 3, we could have two nonclassical Riemann solutions, each of them contain one nonclassical shock. Precisely, the solution would be

- either a classical shock from  $u_l$  to  $\phi^{-1}(u_r)$ , followed by a nonclassical shock corresponding to the kinetic function  $\phi$  from  $\phi^{-1}(u_r)$  to  $u_r$ ;

- or a nonclassical shock corresponding to the kinetic function  $\psi$  from  $u_l$  to  $\psi(u_l)$  followed by a classical shock from  $\psi(u_l)$  to  $u_r$ .

This is an example of the co-existence of nonclassical solutions including the same number of nonclassical shocks, see Figure 5.

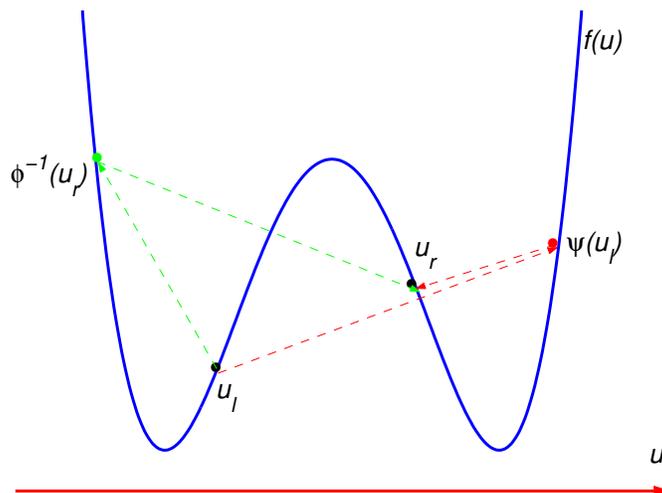


FIGURE 5. Two distinct solutions: each contains a nonclassical shock

For short, in the sequel we will name a  $0$ -shock (or a  $1$ -shock) is a nonclassical shock corresponding to the kinetic function  $\phi$  (the kinetic function  $\psi$ , resp.).

In order to select a unique solution, at least we must avoid the above circumstances. In the following, we need a more restrictive procedure than (C). That is the procedure

- (P) – A classical solution is understood to contain zero nonclassical shock.
- Nonclassical shocks are preferred whenever available in the extended sense that: If a solver  $R_1(u_l, u_r)$  contains  $m$  nonclassical shocks, and a solver  $R_2(u_l, u_r)$  contains  $n$  nonclassical shocks with  $m > n$ , then  $R_1$  excludes  $R_2$ .
  - If the left-hand state belongs to the phase  $E_I$ , then the  $0$ -shocks are preferred than the  $1$ -shocks in the sense that: if  $R_1(u_l, u_r)$  and  $R_2(u_l, u_r)$  contain the same total number of nonclassical shocks, and  $R_1(u_l, u_r)$  contain  $m$   $0$ -shocks and  $R_2(u_l, u_r)$  contain  $n$   $0$ -shocks with  $m > n$ , then  $R_1$  excludes  $R_2$ . Similarly, if the left-hand state belongs to the phase  $E_{III}$ , then  $1$ -shocks are preferred than the  $0$ -shocks.

For the construction, we first make it clear that a  $1$ -shock can not follow a  $0$ -shock.

**Proposition 4.1.** *In any Riemann solution, a  $1$ -shock can not follow a  $0$ -shock.*

*Proof.* Let the states  $u_0, u_1, u_2$  be given. Denote  $N_0(u_0, u_1)$  is the  $0$ -shock from  $u_0$  to  $u_1$  and  $N_1(u_1, u_2)$  is the  $1$ -shock from  $u_1$  to  $u_2$ . That is to say

$$u_1 = \phi(u_0), \quad \text{and} \quad u_2 = \psi(u_1).$$

In order to for  $N_1$  to follow  $N_0$  we must have the condition on shock speeds:

$$s(u_1, u_2) > s(u_0, u_1). \quad (4.1)$$

By the definition of kinetic functions, the shock speed  $s(u_1, u_2)$  has to be smaller than the slope of the tangent at  $u_1$ , which is greater than the shock speed  $s(u_0, u_1)$ . This contradicts with the condition (4.1). The proposition is proved.  $\square$

Based on the procedure (P), we proceed now to construct the Riemann solution. First, assume that  $u_l \in (-\infty, \min\{\theta, \phi^\sharp(\phi(p))\}]$ . Since  $u_l$  is out of the domain of the kinetic function  $\psi$  and, as described in the subsection 3.1, any  $u_r \in (0, +\infty)$  can be arrived at by a solution contain one 0-shock. By virtue of the procedure (P), we thus use the construction in the subsection 3.1 for this interval.

Second, let  $u_l \in (\min\{\theta, \phi^\sharp(\phi(p)), 0\})$ . The construction of the subsection 3.1 is valid for  $u_r < \phi^{-\sharp}(u_l)$ . If  $u_r \in [\phi^{-\sharp}(u_l), \phi(p)]$ , the solution can be a classical shock connecting  $u_l$  to  $u^*$  followed by a nonclassical shock from  $u^*$  to  $u_r$  provided (3.6) holds. If (3.6) fails, then the construction in the subsection 3.2 can be applied here: if  $u_l \leq \theta$  we have a rarefaction wave from  $u_l$  to  $\theta$  followed by a 1-shock from  $\theta$  to  $\psi(\theta)$ , then followed by a classical shock from  $\psi(\theta)$  to  $u_r$ , if  $u_l > \theta$ , then we have a 1-shock from  $u_l$  to  $\psi(u_l)$  followed by a classical one from  $\psi(u_l)$  to  $u_r$ . For  $u_r \in (\phi(p), +\infty)$ , if (3.7) holds then we use the construction in the subsection 3.1 to cover 0-shocks, else we use the one in the subsection 3.2 to cover 1-shocks or classical construction.

Third, let  $u_l \in [0, 1]$ . We know from Proposition 4.1 that 1-shocks can not follow 0-shocks, so we need only find the possibility of a 0-shock following a 1-shock. The interval  $[0, 1]$  can be separated by two regions

$$\begin{aligned} \mathcal{A} &:= \{u \in [0, 1] : \psi^\sharp(u, \psi(u)) < 1\} \text{ relatively open in } [0, 1], \\ \mathcal{A}^C &= [0, 1] \setminus \mathcal{A}. \end{aligned} \quad (4.2)$$

The relatively open set  $\mathcal{A}$  is thus a union of certain relatively open subintervals of the interval  $[0, 1]$ . For any  $u \in \mathcal{A}$ , there corresponds a set defined by

$$\mathcal{B} := \{v \in (a, 0) : v < \phi(\psi^\sharp(u, \psi(u))), \text{ and } v > \phi^\sharp(\psi(u), \phi^{-1}(v))\}. \quad (4.3)$$

The set  $\mathcal{B}$  is an open subset of  $\mathbb{R}$ . By definition, given any left-hand state  $u_r \in \mathcal{B}$ , the Riemann solution for the initial datum  $(u_l, u_r)$  is a three-jump wave: first a 1-shock from  $u_l$  to  $\psi(u_l)$ , followed by a classical jump from  $\psi(u_l)$  to  $\phi^{-1}(u_r)$ , then followed by a 0-shock from  $\phi^{-1}(u_r)$  to  $u_r$ . For  $u_r \in (-\infty, u_l] \setminus \mathcal{B}$ , no 1-shocks to be followed by a 0-shock, so we use the construction in the subsection 3.1. The states  $u_r \in (u_l, +\infty)$  can be reached by the construction in the subsection 3.2, as no 0-shocks are available.

Fourth, assume  $u_l \in (1, \psi^{-1}(0))$ . By the monotony, we have

$$\psi(u_l) > 0. \quad (4.4)$$

Due to (4.4) the right-hand states  $u_r \in [0, +\infty)$  should be involved with 1-shocks and the construction is the one of the subsection 3.2. If  $u_r \in (\phi(\psi(u_l)), 0)$ , then the solution is a 1-shock from  $u_l$  to  $\psi(u_l)$  followed by a rarefaction wave from  $\psi(u_l)$  to  $\phi^{-1}(u_r)$ , then followed by a 0-shock from  $\phi^{-1}(u_r)$  to  $u_r$ . If  $u_r \in (\phi^\sharp(u_l, \psi(u_l)), \phi(\psi(u_l))]$ , then  $\phi^{-1}(u_r) \in (\psi(u_l), 1)$ . The solution is a 1-shock from  $u_l$  to  $\psi(u_l)$  followed by a classical shock from  $\psi(u_l)$  to  $\phi^{-1}(u_r)$ , then followed by a 0-shock from  $\phi^{-1}(u_r)$  to  $u_r$  iff

$$s(\psi(u_l), \phi^{-1}(u_r)) < s(\phi^{-1}(u_r), u_r). \quad (4.5)$$

If (4.5) fails, then no 0-shocks are involved in the construction and we use the one in the subsection 3.2. If now  $u_r \in (-\infty, \phi^\sharp(u_l, \psi(u_l))]$ , then the classical construction is invoked.

Fifth, let  $u_l \in (\psi^{-1}(0), q]$ , then

$$\psi(u_l) < 0. \quad (4.6)$$

The right-hand states  $u_r \in [\psi^\sharp(u_l, \psi(u_l)), +\infty) \cup (-\infty, 0]$  can be arrived at as in the construction of the subsection 3.2. If  $u_r \in (0, \phi(\psi(u_l)))$ , then the solution is a 1-shock from  $u_l$  to  $\psi(u_l)$  followed by a rarefaction wave from  $\psi(u_l)$  to  $\phi^{-1}(u_r)$  by virtue of (4.6), then followed by a 0-shock from  $\phi^{-1}(u_r)$  to  $u_r$ . If  $u_r \in [\phi(\psi(u_l)), \psi^\sharp(u_l, \psi(u_l))]$ , then  $\phi^{-1}(u_r) \in (p, \psi(u_l))$ . The solution is a 1-shock from  $u_l$  to  $\psi(u_l)$  followed by a classical shock from  $\psi(u_l)$  to  $\phi^{-1}(u_r)$ , then followed by a 0-shock from  $\phi^{-1}(u_r)$  to  $u_r$  if and only if

$$s(\psi(u_l), \phi^{-1}(u_r)) < s(\phi^{-1}(u_r), u_r). \quad (4.7)$$

If (4.7) fails, then we use the one in the subsection 3.2.

Finally, if  $u_l \in (q, +\infty)$ , then the classical construction is valid. Summarizing the above arguments, we arrive at the following theorem.

**Theorem 4.2.** *Given the initial Riemann data  $(u_l, u_r)$ . Under the hypotheses (1.2), There exists a unique Riemann solution made of rarefaction waves, classical shocks and non-classical shocks satisfying the kinetic relations (3.4) and (3.11), and the selective procedure (P).*

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#### REFERENCES

- [1] R. Abeyaratne and J. K. Knowles, On the dissipative response due to discontinuous strains in bars of unstable elastic materials. *Int. J. Solids Structures*, 24:1021–1044, 1988.
- [2] R. Abeyaratne and J. K. Knowles, Kinetic relations and the propagation of phase boundaries in solids. *Arch. Rational Mech. Anal.*, 114:119–154, 1991.
- [3] B. T. Hayes and P. G. LeFloch, Non-classical shocks and kinetic relations: Scalar conservation laws. *Arch. Ration. Mech. Anal.*, 139(1):1–56, 1997.
- [4] B. T. Hayes and P. G. LeFloch, Nonclassical shocks and kinetic relations: strictly hyperbolic systems. *SIAM J. Math. Anal.*, 31:941–991, 2000.
- [5] R. D. James, The propagation of phase boundaries in elastic bars. *Arch. Rational Mech. Anal.*, 73:125–158, 1980.
- [6] P. D. Lax, Shock waves and entropy, in: E.H. Zarantonello, Ed.,. *Contributions to Nonlinear Functional Analysis*, pages 603–634, 1971.
- [7] P. G. LeFloch, Propagating phase boundaries: Formulation of the problem and existence via the Glimm method. *Arch. Rational Mech. Anal.*, 123:153–197, 1993.
- [8] P. G. LeFloch, *Hyperbolic systems of conservation laws. The theory of classical and nonclassical shock waves*. Birkhäuser, 2002.
- [9] P. G. LeFloch and M. D. Thanh, Nonclassical Riemann solvers and kinetic relations. III. A nonconvex hyperbolic model for van der Waals fluids. *Electron. J. Differential Equations*, (72):19 pp, 2000.
- [10] P. G. LeFloch and M. D. Thanh, Nonclassical Riemann solvers and kinetic relations. I. An hyperbolic model of elastodynamics. *Z.A.M.P.*, 52:597–619, 2001.
- [11] P. G. LeFloch and M. D. Thanh, Nonclassical Riemann solvers and kinetic relations. II. An hyperbolic-elliptic model of phase transition dynamics. *Proc. Roy. Soc. Edinburgh Sect.*, 132 A(1):181–219, 2002.
- [12] P. G. LeFloch and M. D. Thanh, Properties of Rankine-Hugoniot curves for Van der Waals fluid flows. *Japan J. Indus. and Appl. Math.*, 20(2):211–238, 2003.

- [13] T.-P. Liu, The Riemann problem for general  $2 \times 2$  conservation laws. *Trans. Amer. Math. Soc.*, 199:89–112, 1974.
- [14] O. A. Oleinik, Construction of a generalized solution of the cauchy problem for a quasi-linear equation of first order by the introduction of vanishing viscosity. *Amer. Math. Soc. Transl. Ser. 2*, 33:277–283, 1963.
- [15] S. Schulze and M. Shearer, Undercompressive shocks for a system of hyperbolic conservation laws with cubic nonlinearity. *J. Math. Anal. Appl.*, 229:344–362, 1999.
- [16] M. Shearer, The riemann problem for a class of conservation laws of mixed type. *J. Differential Equations*, 46:426–443, 1982.
- [17] M. Shearer and Y. Yang, The riemann problem for a system of mixed type with a cubic nonlinearity. *Proc. Royal Soc. Edinburgh*, 125A:675–699, 1995.
- [18] M. Slemrod, Admissibility criteria for propagating phase boundaries in a van der Waals fluid. *Arch. Rational Mech. Anal.*, 81:301–315, 1983.
- [19] L. Truskinovsky, *Kinks versus shocks*, in “Shock induced transitions and phase structures in general media”.
- [20] L. Truskinovsky, Dynamics of non-equilibrium phase boundaries in a heat conducting non-linear elastic medium. *J. Appl. Math. and Mech. (PMM)*, 51:777–784, 1987.

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