Electronic Journal of Differential Equations, Vol. 2006(2006), No. 156, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

ELASTO-PLASTIC TORSION PROBLEM AS AN INFINITY LAPLACE'S EQUATION

AHMED ADDOU, ABDELUAAB LIDOUH, BELKASSEM SEDDOUG

Abstract. In this paper, we study a perturbed infinity Laplace's equation, the perturbation corresponds to an Leray-Lions operator with no coercivity assumption. We consider the case where data are distributions or L^1 elements. We show that this problem has an unique solution which is the solution to the variational inequality arising in the elasto-plastic torsion problem, associated with and operator A.

1. Introduction

Given a bounded open subset Ω of $\mathbb{R}^N$, $N \geq 1$, we consider the Dirichlet Problem

$$\begin{gathered} Au - \Delta_\infty u = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Delta_\infty u = u_{x_i} u_{x_j} u_{x_i x_j}$ (see [3]), f in $L^1(\Omega)$ or $W^{-1,p'}(\Omega)$ and A is a Leray-Lions operator with no coercivity assumption, i.e.

$$Av = -\operatorname{div}(a(x, \nabla v(x)))$$

where $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Caratheodory function satisfying the following assumptions:

For almost every $x \in \Omega$ and for all $\xi, \eta \in \mathbb{R}^N$, $(\xi \neq \eta)$, one has:

$$a(x, \xi)\xi \geq 0, \tag{1.2}$$

$$|a(x, \xi)| \leq \beta\big[h(x) + |\xi|^{p-1}\big], \tag{1.3}$$

$$\big[a(x, \xi) - a(x, \eta)\big](\xi - \eta) > 0 \tag{1.4}$$

with $1 < p < +\infty$, $\beta > 0$, $h \in L^{p'}(\Omega)$ (p' denotes the conjugate exponent of p, i.e: $\frac{1}{p} + \frac{1}{p'} = 1$).

By a solution to 1.1 we will mean a variational solution in the sense which extends that given in ([3]) and ([9]), that is, a function u which is the limit of the sequence (u_n) of solutions to the Dirichlet problems

$$\begin{gathered} Au_n - \Delta_n u_n = f \quad \text{in } \Omega, \\ u_n = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

2000 *Mathematics Subject Classification.* 35J70, 35J85, 74C05.
Key words and phrases. Infinity Laplace equation; elasto-plastic torsion problem; variational inequality.

Submitted October 11, 2006. Published December 18, 2006.

as $n \to \infty$, where Δ_n is the n-Laplacian operator ($\Delta_n v = \operatorname{div}(|\nabla v|^{n-2}\nabla v)$.

We show that in the variational case ($f \in W^{-1,p'}(\Omega)$), the sequence (u_n) converges to the unique solution to the variational inequality

$$\begin{gathered}\langle Au, v-u\rangle \geq \langle f, v-u\rangle, \text{ for all } v \in \mathcal{K},\\ u \in \mathcal{K}.\end{gathered}$$

Where $\mathcal{K}$ is the bounded convex cone of $W_0^{1,p}(\Omega)$ defined as:

$$\mathcal{K} = \{v \in W_0^{1,p}(\Omega) : |\nabla v(x)| \leq 1 \text{ a.e. in } \Omega\},$$

and in the case $f \in L^1(\Omega)$, the sequence (u_n) converges to the unique solution to the problem

$$\begin{gathered}\langle Au, T_k(v-u)\rangle \geq \int_\Omega f T_k(v-u)dx, \quad \text{for all } v \in \mathcal{K},\\ u \in \mathcal{K}, \quad \text{for all } k > 0.\end{gathered}$$

Where $T_k : \mathbb{R} \to \mathbb{R}$ is the cut function defined as

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k\\ k\operatorname{sign}(s) & \text{if } |s| > k.\end{cases}$$

here $\langle .,. \rangle$ denotes the duality pairing between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$.

Our approach is also inscribed among the techniques of "the increase of power", first introduced by Boccardo and Murat in [4], where they approached the problem

$$\begin{gathered}\langle Au, v-u\rangle \geq \langle f, v-u\rangle, \quad \text{for all } v \in \mathcal{K}_0,\\ u \in \mathcal{K}_0 = \{v \in W_0^{1,p}(\Omega) : |v(x)| \leq 1 \text{ a.e. in } \Omega\},\end{gathered}$$

by the sequence of the Dirichlet equations

$$\begin{gathered}Au_n - |u_n|^{n-1}u_n = f \quad \text{in } D'(\Omega),\\ u_n \in W_0^{1,p}(\Omega) \cap L^n(\Omega),\end{gathered}$$

where $f \in W^{-1,p'}(\Omega)$ and A is modelled on the p-Laplacian.

Then in [5], Dall'Aglio and Orsina generalized this result by considering increasing powers depending of a certain Caratheodory function satisfying the sign condition and an integrability assumption.

Then finally in [2] the authors extended this result to the case where increasing powers are multiplied by a quantity depending on the gradient and verifying adequate conditions, they examine the two cases, f in $L^1(\Omega)$ and in $W^{-1,p'}(\Omega)$.

In this paper we examine the case where the increasing powers carry on the gradients and not on quantities independent of the gradient.

2. The variational case

Let $f \in W^{-1,p'}(\Omega)$, $1 < p < +\infty$. For all integer $n \geq p$, we consider the Dirichlet problem

$$\begin{gathered}Au_n - \Delta_n u_n = f \quad \text{in } \Omega,\\ u_n \in W_0^{1,n}(\Omega).\end{gathered} \tag{2.1}$$

It is known [7, 8] that, under assumptions (1.2)–(1.4), the problem (2.1) has an unique solution u_n, in the following sense:

$$\forall v \in W_0^{1,n}(\Omega) : \int_\Omega [a(x,\nabla u_n)\nabla v + |\nabla u_n|^{n-2}\nabla u_n \nabla v]dx = \langle f, v\rangle. \tag{2.2}$$

In the sequel $W_0^{1,p}(\Omega)$ is equipped with its usual norm

$$\|v\|_{W_0^{1,p}(\Omega)} = \Big[\int_\Omega |\nabla v|^p dx\Big]^{1/p}$$

Let us now, state our first main result.

Theorem 2.1. *Let $f \in W^{-1,p'}(\Omega)$, $1 < p < +\infty$. Under assumptions (1.2)–(1.4), if u_n designates the solution to the problem (2.1), then the sequence (u_n) converges strongly in $W_0^{1,p}(\Omega)$, to the unique solution u to the problem*

$$\begin{gathered} \langle Au, v-u\rangle \geq \langle f, v-u\rangle, \quad \text{for all } v \in \mathcal{K}, \\ u \in \mathcal{K}. \end{gathered} \tag{2.3}$$

Proof of Theorem 2.1.

A priori estimate. With u_n as a test function in (2.2), we get

$$\int_\Omega a(x,\nabla u_n)\nabla u_n dx + \int_\Omega |\nabla u_n|^n dx = \langle f, u_n\rangle \leq \|f\|_{-1,p'}\|u_n\|_{1,p}$$

hence

$$\int_\Omega |\nabla u_n|^n dx \leq c\|u_n\|_{1,p} \quad \text{for all } n \geq p\,. \tag{2.4}$$

In the sequel $c, c_1, c_2 \dots$. designate arbitrary constants.

From (2.4), and by splitting $\int_\Omega |\nabla u_n|^p dx$ as

$$\int_\Omega |\nabla u_n|^p dx = \int_{[|\nabla u_n|\leq 1]} |\nabla u_n|^p dx + \int_{[|\nabla u_n|>1]} |\nabla u_n|^p dx,$$

one deduces that

$$\int_\Omega |\nabla u_n|^p dx \leq |\Omega| + c[\int_\Omega |\nabla u_n|^p dx]^{\frac{1}{p}} \quad \text{for all } n \geq p$$

and so

$$\int_\Omega |\nabla u_n|^p dx \leq c \quad \text{for all } n \geq p\,. \tag{2.5}$$

Thereafter,

$$\int_\Omega |\nabla u_n|^n dx \leq c \quad \forall n \quad \text{and} \quad \int_\Omega |\nabla u_n|^q dx \leq c \quad \forall q,\ \forall n \geq q. \tag{2.6}$$

Therefore, one can construct a subsequence, still denoted by $(u_n)_n$, such that

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,q}(\Omega) \text{ and uniformly in } \bar{\Omega}, \tag{2.7}$$

for some $u \in W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$, for all $q > 1$. More precisely, we have

$$u \in W_0^{1,\infty}(\Omega) \quad \text{and} \quad \|\nabla u\|_\infty \leq 1. \tag{2.8}$$

Indeed, from (2.6) and (2.7), one has

$$\|\nabla u\|_\infty = \lim_{q\to\infty} \|\nabla u\|_q \leq \lim_{q\to\infty}\big(\liminf_{n\to\infty} \|\nabla u_n\|_q\big) \leq \lim_{q\to\infty} c^{\frac{1}{q}} = 1.$$

Almost everywhere convergence of gradients. With $v = u_n - u$, as a test function in (2.2), and using the fact that

$$\nabla u_n(\nabla u_n - \nabla u) \geq 0$$

in the set $\{|\nabla u_n| \geq |\nabla u|\}$, one has

$$\langle Au_n, u_n - u\rangle + \int_{\{|\nabla u_n|<|\nabla u|\}} |\nabla u_n|^{n-2}\nabla u_n(\nabla u_n - \nabla u)dx \leq \varepsilon_n, \tag{2.9}$$

We will denote by ε_n any quantity which converges to zero as n tends to infinity.

Let $\varepsilon > 0$, for the second term on the left in (2.9), one puts

$$A_1 = \{|\nabla u_n| < |\nabla u| \text{ and } |\nabla u_n| \leq 1-\varepsilon\}, \quad A_2 = \{1-\varepsilon < |\nabla u_n| < |\nabla u|\}$$

and so we have

$$\int_{A_1} |\nabla u_n|^{n-2}\nabla u_n(\nabla u_n - \nabla u)dx = \sigma_{n,\varepsilon}, \tag{2.10}$$

where $\sigma_{n,\varepsilon}$ denotes a quantity depending on n and ε, such that, for any fixed $\varepsilon > 0$, $\sigma_{n,\varepsilon} \to 0$, as $n \to \infty$, and which may change from line to line. Also

$$\begin{aligned}
&\int_{A_2} |\nabla u_n|^{n-2}\nabla u_n(\nabla u_n - \nabla u)dx \\
&= \int_{A_2} |\nabla u_n|^{n-2}(|\nabla u_n|^2 - |\nabla u|^2)dx + \int_{A_2} |\nabla u_n|^{n-2}\nabla u(\nabla u - \nabla u_n)dx \\
&= q_n + I_n,
\end{aligned} \tag{2.11}$$

where the quantity I_n is nonnegative, and $q_n \in [-2\varepsilon|\Omega|, 0]$. Combining (2.9), (2.10) and (2.11), one gets

$$\langle Au_n, u_n - u\rangle \leq \sigma_{n,\varepsilon} + 2\varepsilon|\Omega|, \forall\varepsilon > 0$$

On the other hand, $\langle Au, u_n - u\rangle \to 0$, as $n \to \infty$, so that

$$0 \leq \langle Au_n - Au, u_n - u\rangle \leq \sigma_{n,\varepsilon} + 2\varepsilon|\Omega|, \forall\varepsilon > 0.$$

Passing to the limit as $n \to \infty$, for any fixed ε, one has

$$0 \leq \liminf_{n\to\infty}\langle Au_n - Au, u_n - u\rangle \leq \limsup_{n\to\infty}\langle Au_n - Au, u_n - u\rangle \leq 2\varepsilon|\Omega| \quad \forall\varepsilon > 0.$$

By the arbitrariness of ε (and since $\langle Au_n - Au, u_n - u\rangle$ does not depend on ε) it follows that

$$\langle Au_n - Au, u_n - u\rangle \to 0 \quad \text{as } n \to \infty. \tag{2.12}$$

Which implies, thanks to (1.4), that (for a subsequence),

$$(a(x,\nabla u_n) - a(x,\nabla u))(\nabla u_n - \nabla u) \to 0 \text{ a.e. in } \Omega.$$

For a fixed $k > 1$, we put

$$X = \bigcap_{q\in\mathbb{N}}\bigcup_{n\geq q}\{|\nabla u_n| \geq k\}, \text{ and its complement } Y = \bigcup_{q\in\mathbb{N}}\bigcap_{n\geq q}\{|\nabla u_n| < k\},$$

for all $x \in Y$, the sequence $(\nabla u_n(x))$ is bounded in $\mathbb{R}^N$, so

$$\nabla u_n(x) \to \xi$$

for a subsequence and some $\xi \in \mathbb{R}^N$, while (1.4) and the continuity of $a(x,.)$, implies that $\xi = \nabla u(x)$, we can then conclude that

$$\nabla u_n(x) \to \nabla u(x) \quad \text{for all } x \in Y.$$

To show the almost everywhere convergence of (∇u_n), it suffices to prove that $\operatorname{meas}(X)=0$. In deed, from (2.6), one has

$$\operatorname{meas}\{|\nabla u_n|\geq k\}=\int_{\{|\nabla u_n|\geq k\}}1dx\leq \frac{c}{k^n}\,. \tag{2.13}$$

Since $X\subset \bigcup_{n\geq q}\{|\nabla u_n|\geq k\}$, for all q, one deduces that

$$\operatorname{meas}(X)\leq \sum_{n\geq q}\operatorname{meas}\{|\nabla u_n|\geq k\}\to 0 \quad \text{as } q\to\infty.$$

Strong convergence in $W_0^{1,p}(\Omega)$. Thanks to Vitali's theorem, it suffices to show the equi-integrability of $(|\nabla u_n|^p)$ in $L^1(\Omega)$, what follows from (2.6) with $q=p+1$.

Indeed for a measurable subset E of Ω, one has

$$\int_E|\nabla u_n|^p dx\leq \Big(\int_E|\nabla u_n|^{p+1}dx\Big)^{\frac{p}{p+1}}\Big(\int_E 1\,dx\Big)^{\frac{1}{p+1}}\leq c\big(\operatorname{meas}(E)\big)^{\frac{1}{p+1}}.$$

The function u is solution to problem (2.3). Let $v\in\mathcal{K}$ and $0<\theta<1$, taking $z=u_n-\theta T_k(v)$ as a test function in (2.2), one gets

$$\langle Au_n,z\rangle+\int_\Omega|\nabla u_n|^{n-2}\nabla u_n\nabla z dx=\langle f,z\rangle$$

While noticing that

$$\int_{\{|\nabla u_n|\geq\theta|\nabla T_k(v)|\}}|\nabla u_n|^{n-2}\nabla u_n(\nabla u_n-\theta\nabla T_k(v))dx\geq 0$$

one has

$$\langle Au_n,z\rangle+\int_{\{|\nabla u_n|<\theta|\nabla T_k(v)|\}}|\nabla u_n|^{n-2}\nabla u_n\nabla z dx\leq\langle f,z\rangle$$

Passing to the limit as $n\to\infty$, and using standard result about Caratheodory functions satisfying (1.3), one gets

$$\langle Au,u-\theta T_k(v)\rangle\leq\langle f,u-\theta T_k(v)\rangle$$

The result is then obtained while passing to the limit as $\theta\to 1$ and $k\to\infty$.

3. The case $f\in L^1(\Omega)$

In this section, we suppose that $f\in L^1(\Omega)$, as in the previous section. Now we prove our second main result.

Theorem 3.1. *Let $f\in L^1(\Omega)$, $1<p<+\infty$. Under assumptions (1.2)–(1.4), if u_n $(n>N)$ designates the solution to the problem (2.1), then the sequence (u_n) converges strongly in $W_0^{1,p}(\Omega)$, to the unique solution u to the problem*

$$\begin{gathered}\langle Au,T_k(v-u)\rangle\geq\int_\Omega fT_k(v-u)dx \quad \text{for all } v\in\mathcal{K},\\ u\in\mathcal{K},\quad \text{for all } k>0.\end{gathered}\tag{3.1}$$

Proof of Theorem 3.1. According to the previous section, it is clear that the estimate (2.6) permits to show that the sequence (u_n) converges in $W_0^{1,p}(\Omega)$ and uniformly in $\bar\Omega$ (for a subsequence) to u satisfying (2.8).

We are going to prove (2.6) and the fact that u is the solution to (3.1).

A priori estimate. With u_n ($n > N$) as a test function in (2.2), we get

$$\int_\Omega a(x, \nabla u_n)\nabla u_n dx + \int_\Omega |\nabla u_n|^n dx = \int_\Omega f u_n dx \le \|f\|_1 \|u_n\|_\infty$$

Let $q > N$ (fixed), by splitting $\int_\Omega |\nabla u_n|^q dx$ as

$$\int_\Omega |\nabla u_n|^q dx = \int_{\{|\nabla u_n|<1\}} |\nabla u_n|^q dx + \int_{\{|\nabla u_n|\ge 1\}} |\nabla u_n|^q dx$$

and using Sobolev's inequality [1], one has

$$\int_\Omega |\nabla u_n|^q dx \le c \quad \forall n \ge q; \tag{3.2}$$

therefore,

$$\int_\Omega |\nabla u_n|^n dx \le c \quad \forall n > N\,.$$

It follows that the estimate (3.2) holds for all $q > 1$, what leads to the estimate (2.6).

The function u is solution to problem (3.1). Let $v \in \mathcal{K}$ and $0 < \theta < 1$, taking $z = T_k(u_n - \theta v)$ as a test function in (2.2), one gets

$$\langle Au_n, z\rangle + \int_\Omega |\nabla u_n|^{n-2}\nabla u_n \nabla z dx = \int_\Omega f z dx$$

While noticing that

$$\int_{\{|\nabla u_n|\ge\theta|\nabla v|\}} |\nabla u_n|^{n-2}\nabla u_n \nabla T_k(u_n - \theta v) dx \ge 0$$

one has

$$\langle Au_n, z\rangle + \int_{\{|\nabla u_n|<\theta|\nabla v|\}} |\nabla u_n|^{n-2}\nabla u_n \nabla z dx \le \int_\Omega f z\, dx$$

Passing to the limit as $n \to \infty$, one gets

$$\langle Au, T_k(u - \theta v)\rangle \le \int_\Omega f T_k(u - \theta v)\, dx$$

The result is obtained when passing to the limit as $\theta \to 1$.

Remark 3.2. Since $u \in W_0^{1,\infty}(\Omega)$, the problem can be formulated in this space by choosing $\mathcal{K} = \{v \in W_0^{1,\infty}(\Omega) : \|\nabla v(x)\|_\infty \le 1\}$, what permits to write the problem (3.1) without truncation operator, and simplify the proof of the step *The function u is solution to the problem* (3.1). But traditionally (see for example [6]), the elasto-plastic torsion problem is written with $\mathcal{K} = \{v \in W_0^{1,p}(\Omega) : |\nabla v(x)| \le 1 \text{ a.e. in } \Omega\}$, it's why we have done this choice.

Acknowledgement. The authors would like to thank the anonymous referee for his/her interesting remarks.

References

[1] R. Adams. *Sobolev spaces*. Academic Press, New York, 1975.

[2] A. Benkirane, A. EL Mahi, D. Meskine. *On the limit of some penalized problems involving increasing powers*. Asymptotic Analysis, Volume 36, Numbers 3–4 (2003), pp. 303–317.

[3] T. Bhattacharya, E. DiBenedetto, J. Manfredi. *Limits as $p \to \infty$ of $\Delta_p(u) = f$ and related extremal problems*. Rend. Sem. Mat. Univ. Pol. Torino, Fascicolo Speciale Nonlinear PDE's, (1989), 15–68.

[4] L. Boccardo, F. Murat. *Increase of power leads to bilateral problems*. Composite Media and Homogenization Theory, G. Dal Maso and G. F. Dell' Antonio. eds., World Scientific, Singapore, 1995, pp. 113–123.

[5] A. Dall'Aglio, L. Orsina. *On the limit of some nonlinear elliptic equations involving increasing powers*. Asympt. Anal., **14** (1997), 49–71.

[6] D. Kinderlehrer, G. Stampacchia. *An introduction to variational inequalities and their applications*. Academic Press, New York, 1980.

[7] L. Leray, J. L. Lions. *Quelques résultats de Visik sur les problèmes non linéaires par les méthodes de Minty-Browder*. Bull. Soc. Math. France, **93** (1965), 97–107.

[8] J. L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod et Gauthiers-Villars (1969).

[9] E. Rosset. *Symmetry and convexity of level sets of solutions to the infinity Laplace's equation*. Electron. J. Differential Equations, vol. 1998, (1998) No. 34, pp. 1–12.

Ahmed Addou
Université Mohammed premier, Faculté des sciences, Oujda, Maroc
E-mail address: addou@sciences.univ-oujda.ac.ma

Abdeluaab Lidouh
Université Mohammed premier, Faculté des sciences, Oujda, Maroc
E-mail address: lidouh@sciences.univ-oujda.ac.ma

Belkassem Seddoug
Université Mohammed premier, Faculté des sciences, Oujda, Maroc
E-mail address: seddougbelkassem@yahoo.fr