Electronic Journal of Differential Equations, Vol. 2006(2006), No. 19, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
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EXISTENCE AND STABILITY OF ALMOST PERIODIC SOLUTIONS FOR SHUNTING INHIBITORY CELLULAR NEURAL NETWORKS WITH CONTINUOUSLY DISTRIBUTED DELAYS

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Abstract. In this paper, we consider shunting inhibitory cellular neural networks (SICNNs) with continuously distributed delays. Sufficient conditions for the existence and local exponential stability of almost periodic solutions are established using a fixed point theorem, Lyapunov functional method, and differential inequality techniques. We illustrate our results with an example for which our conditions are satisfied, but not the conditions in [4, 6, 8].

1. Introduction

Consider the shunting inhibitory cellular neural networks (SICNNs) with continuously distributed delays

$$x'_{ij}(t) = -a_{ij}(t)x_{ij}(t) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f\Big(\int_0^\infty K_{ij}(u)x_{kl}(t-u)du\Big)x_{ij}(t) + L_{ij}(t), \tag{1.1}$$

where $i = 1,2,\dots,m$, $j = 1,2,\dots,n$, C_{ij} denote the cell at the (i,j) position of the lattice, the r-neighborhood $N_r(i,j)$ of C_{ij} is

$$N_r(i,j) = \{C_{kl} : \max(|k-i|,|l-j|) \le r, 1 \le k \le m, 1 \le l \le n\}.$$

x_{ij} is the activity of the cell C_{ij}, $L_{ij}(t)$ is the external input to C_{ij}, $a_{ij}(t) > 0$ represent the passive decay rate of the cell activity, $C_{ij}^{kl} \ge 0$ is the connection or coupling strength of postsynaptic activity of the cell transmitted to the cell C_{ij}, and the activity function f is a continuous function representing the output or firing rate of the cell C_{kl}.

Since Bouzerdout and Pinter in [1, 2, 3] described SICNNs as a new cellular neural networks(CNNs), SICNNs have been extensively applied in psychophysics, speech, perception, robotics, adaptive pattern recognition, vision, and image processing. Hence, they have been the object of intensive analysis by numerous authors

2000 *Mathematics Subject Classification.* 34C25, s4K13, 34K25.
Key words and phrases. Shunting inhibitory cellular neural networks; almost periodic solution; exponential stability; continuously distributed delays.

Submitted December 3, 2005. Published February 7, 2006.
Supported by the NNSF of China and by grant 05JJ40009 from Hunan Provincial Natural Science Foundation of China.

in recent years. In particular, there have been extensive results on the problem of the existence and stability of periodic and almost periodic solutions of SICNNs with constant time delays and time-varying delays in the literature. We refer the reader to [4, 6, 8] and the references cited therein. Moreover, in the above-mentioned literature, we observe that the assumption

(T0) there exists a nonnegative constant M_f such that $M_f = \sup_{x\in\mathbb{R}} |f(x)|$

has been considered as fundamental for the considered existence and stability of periodic and almost periodic solutions of SICNNS. However, to the best of our knowledge, few authors have considered SICNNS without the assumptions (T0). Thus, it is worth while to continue to investigate the existence and stability of almost periodic solutions of SICNNS.

The main purpose of this paper is to obtain some sufficient conditions for the existence and stability and local exponential stability of the almost periodic solutions for system (1.1). By applying fixed point theorem, Lyapunov functional method and differential inequality techniques, we derive some new sufficient conditions ensuring the existence and local exponential stability of the almost periodic solution of system (1.1), which are new and they complement previously known results. In particular, we do not need the assumption (T0). Moreover, an example is also provided to illustrate the effectiveness of the new results.

Throughout this paper, we set

$$\{x_{ij}(t)\} = (x_{11}(t), \dots, x_{1n}(t), \dots, x_{i1}(t), \dots, x_{in}(t), \dots, x_{m1}(t), \dots, x_{mn}(t)).$$

For all $x = \{x_{ij}(t)\} \in \mathbb{R}^{m\times n}$, we define the norm $\|x\| = \max_{(i,j)}\{|x_{ij}(t)|\}$. Set

$$\begin{aligned} B = \big\{\varphi : \varphi = \{\varphi_{ij}(t)\} = (&\varphi_{11}(t), \dots, \varphi_{1n}(t), \dots, \\ &\varphi_{i1}(t), \dots, \varphi_{in}(t), \dots, \varphi_{m1}(t), \dots, \varphi_{mn}(t))\big\}, \end{aligned}$$

where φ is an almost periodic function on $\mathbb{R}$. For all $\varphi \in B$, we define induced module $\|\varphi\|_B = \sup_{t\in\mathbb{R}} \|\varphi(t)\|$, then B is a Banach space.

The initial conditions associated with system (1.1) are

$$x_{ij}(s) = \varphi_{ij}(s), \quad s \in (-\infty, 0],\ i = 1, 2, \dots, m,\ j = 1, 2, \dots, n, \tag{1.2}$$

where $\varphi_{ij}(\cdot)$ denotes real-valued bounded continuous function defined on $(-\infty, 0]$.

We also assume that the following conditions

(T1) For $i \in \{1, 2, \dots, m\}$, $j \in \{1, 2, \dots, n\}$, the delay kernels $K_{ij} : [0, \infty) \to \mathbb{R}$ are continuous, integrable and there exist nonnegative constants k_{ij} such that
$$\int_0^\infty |K_{ij}(s)|ds \le k_{ij}.$$

(T2) For each $i \in \{1, 2, \dots, m\}$, $j \in \{1, 2, \dots, n\}$, $L_{ij}(t)$ and $a_{ij}(t)$ are almost periodic functions on R, let $L^+_{ij} = \sup_{t\in\mathbb{R}} |L_{ij}(t)|$, $0 < a_{ij} = \inf_{t\in\mathbb{R}} a_{ij}(t)$.

(T3) $f(0) = 0$, $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz with Lipschitz constant μ, i.e.,
$$|f(u) - f(v)| \le \mu|u - v|, \quad \text{for all } u, v \in \mathbb{R}.$$

(T4) there exist nonnegative constants L, q and δ such that

$$L = \max_{(i,j)}\{\frac{L_{ij}^+}{a_{ij}}\}, \quad \delta = \max_{(i,j)}\Big\{\frac{\mu k_{ij}\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}}{a_{ij}}\Big\} < 1,$$

$$\frac{L}{(1-\delta)} \le 1, \quad q = 2\delta\frac{L}{(1-\delta)} < 1.$$

(T5) For $i \in \{1,2,\dots,m\}$, $j \in \{1,2,\dots,n\}$, there exists a constant $\lambda_0 > 0$ such that

$$\int_0^\infty |K_{ij}(s)|e^{\lambda_0 s}ds < +\infty.$$

Definition. (see [5, 6]) Let $u(t) : \mathbb{R} \to \mathbb{R}^n$ be continuous in t. $u(t)$ is said to be almost periodic on $\mathbb{R}$ if, for each $\varepsilon > 0$, the set $T(u,\varepsilon) = \{\delta : |u(t+\delta) - u(t)| < \varepsilon,\ \forall t \in \mathbb{R}\}$ is relatively dense; i.e., for all $\varepsilon > 0$, it is possible to find a real number $l = l(\varepsilon) > 0$, so that for any interval of length $l(\varepsilon)$, there exists a number $\delta = \delta(\varepsilon)$ in this interval such that $|u(t+\delta) - u(t)| < \varepsilon$, for all $t \in \mathbb{R}$.

The remaining part of this paper is organized as follows. In Section 2, we shall derive new sufficient conditions for checking the existence of almost periodic solutions. In Section 3, we present some new sufficient conditions for the local exponential stability of the almost periodic solution of (1.1). In Section 4, we shall give an example to illustrate our results obtained in previous sections.

2. Existence of Almost Periodic Solutions

Theorem 2.1. *Under conditions* (T1)–(T4)*there exists a unique almost periodic solution of* (1.1) *in the region* $B^* = \{\varphi : \varphi \in B, \|\varphi - \varphi_0\|_B \le \frac{\delta L}{1-\delta}\}$*, where*

$$\begin{aligned}\varphi_0(t) &= \Big\{\int_{-\infty}^t e^{-\int_s^t a_{ij}(u)du})L_{ij}(s)ds\Big\}\\ &= \Big(\int_{-\infty}^t e^{-\int_s^t a_{1j}(u)du}L_{11}(s)ds, \dots, \int_{-\infty}^t e^{-\int_s^t a_{ij}(u)du}L_{ij}(s)ds,\\ &\quad \dots, \int_{-\infty}^t e^{-\int_s^t a_{mn}(u)du}L_{mn}(s)ds\Big).\end{aligned}$$

Proof. For $\varphi \in B$, we consider the almost periodic solution $x_\varphi(t)$ of nonlinear almost periodic differential equation

$$\frac{dx_{ij}}{dt} = -a_{ij}(t)x_{ij}(t) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(\int_0^\infty K_{ij}(u)\varphi_{kl}(t-u)du)\varphi_{ij}(t) + L_{ij}(t), \tag{2.1}$$

Because $\varphi_{ij}(t), L_{ij}(t)$, $i = 1,2,\dots,m$, $j = 1,2,\dots,n$, are almost periodic functions. By [7, P. 90-120], (2.1) has a unique almost periodic solution

$$\begin{aligned}x_\varphi(t) = \Big\{\int_{-\infty}^t e^{-\int_s^t a_{ij}(u)du}\Big[-\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}\\ \times f\big(\int_0^\infty K_{ij}(u)\varphi_{kl}(s-u)du\big)\varphi_{ij}(s) + L_{ij}(s)\Big]ds\Big\}.\end{aligned} \tag{2.2}$$

Now, we define a mapping $T : B \to B$ by setting

$$T(\varphi)(t) = x_\varphi(t), \quad \forall \varphi \in B.$$

Since $B^* = \{\varphi : \varphi \in B, \|\varphi - \varphi_0\|_B \le \frac{\delta L}{1-\delta}\}$, it is easy to see that B^* is a closed convex subset of B. According to the definition of the norm of Banach space B, we have

$$\begin{aligned}\|\varphi_0\|_B &= \sup_{t\in\mathbb{R}} \max_{(i,j)} \Big\{ \int_{-\infty}^{t} L_{ij}(s) e^{-\int_s^t a_{ij}(u)du} ds \Big\} \\ &\le \sup_{t\in\mathbb{R}} \max_{(i,j)} \big\{ \frac{L_{ij}^+}{a_{ij}} \big\} = \max_{(i,j)} \big\{ \frac{L_{ij}^+}{a_{ij}} \big\} = L.\end{aligned} \tag{2.3}$$

Therefore, for all $\varphi \in B^*$, we have

$$\|\varphi\|_B \le \|\varphi - \varphi_0\|_B + \|\varphi_0\|_B \le \frac{\delta L}{1-\delta} + L = \frac{L}{1-\delta}. \tag{2.4}$$

In view of (T3), we have

$$|f(u)| = |f(u) - f(0)| \le \mu |u|, \quad \forall u \in \mathbb{R}. \tag{2.5}$$

Now, we prove that the mapping T is a self-mapping from B^* to B^*. In fact, for all $\varphi \in B^*$, together with (2.4), (2.5) and $\frac{L}{1-\delta} \le 1$, we obtain

$$\begin{aligned}&\|T\varphi - \varphi_0\|_B \\ &= \sup_{t\in\mathbb{R}} \max_{(i,j)} \Big\{ \Big| \int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)du} \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f\Big(\int_0^\infty K_{ij}(u)\varphi_{kl}(s-u)du\Big)\varphi_{ij}(s) ds \Big| \Big\} \\ &\le \sup_{t\in\mathbb{R}} \max_{(i,j)} \{ \int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)du} \mu \|\varphi\|_B \int_0^\infty |K_{ij}(u)| du \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} |\varphi_{ij}(s)| ds \} \\ &\le \sup_{t\in\mathbb{R}} \max_{(i,j)} \Big\{ \mu k_{ij} \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} \int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)du} ds \|\varphi\|_B^2 \Big\} \\ &\le \max_{(i,j)} \big\{ \frac{\mu k_{ij} \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}}{a_{ij}} \big\} \|\varphi\|_B^2 \\ &= \delta \|\varphi\|_B^2 \le \delta \Big(\frac{L}{1-\delta}\Big)^2 \le \delta \frac{L}{1-\delta},\end{aligned}$$

which implies $T(\varphi)(t) \in B^*$. So, the mapping T is a mapping from B^* to B^*. Next, we prove that the mapping T is a contraction mapping of the B^*. In fact, for all

$\varphi, \psi \in B^*$, we have

$$
\begin{aligned}
&\|T(\varphi) - T(\psi)\|_B \\
&= \sup_{t\in\mathbb{R}} \|T(\varphi)(t) - T(\psi)(t)\| \\
&= \sup_{t\in\mathbb{R}} \max_{(i,j)}\{|\int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)du} \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(f(\int_0^\infty K_{ij}(u)\varphi_{kl}(s-u)du) \\
&\quad \times \varphi_{ij}(s) - f(\int_0^\infty K_{ij}(u)\psi_{kl}(s-u)du)\psi_{ij}(s))ds|\} \\
&\le \sup_{t\in\mathbb{R}} \max_{(i,j)}\{\int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)du} \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}|f(\int_0^\infty K_{ij}(u)\varphi_{kl}(s-u)du)\varphi_{ij}(s) \\
&\quad - f(\int_0^\infty K_{ij}(u)\psi_{kl}(s-u)du)\varphi_{ij}(s) + f(\int_0^\infty K_{ij}(u)\psi_{kl}(s-u)du)\varphi_{ij}(s) \\
&\quad - f(\int_0^\infty K_{ij}(u)\psi_{kl}(s-u)du)\psi_{ij}(s)|ds\}\,.
\end{aligned}
$$

In view of condition (T3), (2.4), (2.5) and the above inequality, we have

$$
\begin{aligned}
&\|T(\varphi) - T(\psi)\|_B \\
&\le \sup_{t\in\mathbb{R}} \max_{(i,j)}\{\int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)du} \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}[|f(\int_0^\infty K_{ij}(u)\varphi_{kl}(s-u)du) \\
&\quad - f(\int_0^\infty K_{ij}(u)\psi_{kl}(s-u)du)||\varphi_{ij}(s)| \\
&\quad + |f(\int_0^\infty K_{ij}(u)\psi_{kl}(s-u)du)||\varphi_{ij}(s) - \psi_{ij}(s)|]ds\} \\
&\le \sup_{t\in\mathbb{R}} \max_{(i,j)}\{\int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)du} \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}[\int_0^\infty |K_{ij}(u)|\mu|\varphi_{kl}(s-u) \\
&\quad - \psi_{kl}(s-u)|du|\varphi_{ij}(s)| + \int_0^\infty |K_{ij}(u)|\mu|\psi_{kl}(s-u)|du|\varphi_{ij}(s) - \psi_{ij}(s)|]ds\} \\
&\le \sup_{t\in\mathbb{R}} \max_{(i,j)}\{\int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)du} \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} k_{ij}\mu[\|\varphi\|_B + \|\psi\|_B]\|\varphi - \psi\|_B ds\} \\
&\le \max_{(i,j)}\{\frac{\mu k_{ij}\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}}{a_{ij}} \frac{2L}{(1-\delta)}\}\|\varphi - \psi\|_B \\
&= 2\delta\frac{L}{(1-\delta)}\|\varphi - \psi\|_B,
\end{aligned}
$$

i.e.

$$\|T(\varphi) - T(\psi)\|_B \le q\|\varphi - \psi\|_B.$$

Note that $q = 2\delta\frac{L}{(1-\delta)} < 1$, it is clear that the mapping T is a contraction. Therefore the mapping T possesses a unique fixed point $\varphi^* \in B^*$, $T\varphi^* = \varphi^*$. By (2.1), φ^* satisfies (1.1). So φ^* is an almost periodic solution of system (1.1) in B^*. The proof is complete. $\square$

3. Stability of the almost periodic solution

In this section, we establish some results for the uniqueness and local exponential stability of the almost periodic solution of system (1.1) in the region B^*.

Theorem 3.1. *Let $\delta(1+2\frac{L}{1-\delta})<1$ and suppose that conditions (T1)–(T5) hold. Then (1.1) has exactly one almost periodic solution $\varphi^*(t)=\{x^*_{ij}(t)\}=\{\varphi^*_{ij}(t)\}$ in the region B^*. Moreover, $\varphi^*(t)$ is locally exponentially stable, and the domain of attraction of $\varphi^*(t)$ is the set*

$$G_1(\varphi^*)=\{\varphi:\varphi\in C((-\infty,\,0];\;R^{mn}),\ \|\varphi-\varphi^*\|=\sup_{-\infty\le s\le 0}\max_{(i,j)}|\varphi_{ij}(s)-\varphi^*_{ij}(s)|<1\};$$

namely, there exist constants $\lambda>0$ and $M>1$ such that for every solution $Z(t)=\{x_{ij}(t)\}$ of system (1.1) with any initial value $\varphi=\{\varphi_{ij}(t)\}\in G_1(\varphi^*)$,

$$|x_{ij}(t)-x^*_{ij}(t)|\le M\|\varphi-\varphi^*\|e^{-\lambda t},$$

for all $t>0$, $i=1,2,\dots,m$, $j=1,2,\dots,\ n$.

Proof. From Theorem 2.1, system (1.1) has exactly one almost periodic solution $\varphi^*(t)=\{x^*_{ij}(t)\}=\{\varphi^*_{ij}(t)\}$ in the region B^*. Let $Z(t)=\{x_{ij}(t)\}$ be an arbitrary solution of system (1.1) with initial value $\varphi=\{\varphi_{ij}(t)\}\in G_1(\varphi^*)$. Set $y(t)=\{y_{ij}(t)\}=\{x_{ij}(t)-x^*_{ij}(t)\}=Z(t)-\varphi^*(t)$. Then

$$\begin{aligned} y'_{ij}(t)=-a_{ij}(t)y_{ij}(t)-\sum_{C_{kl}\in N_r(i,j)}C^{kl}_{ij}[f(\int_0^\infty K_{ij}(u)x_{kl}(t-u)du)x_{ij}(t)\\ -f(\int_0^\infty K_{ij}(u)x^*_{kl}(t-u)du)x^*_{ij}(t)], \end{aligned}\tag{3.1}$$

for $i=1,2,\dots,m$, $j=1,2,\dots,n$. Since $\delta<1$, $\delta(1+2\frac{L}{1-\delta})<1$, we can easily obtain

$$\begin{aligned} a_{ij}>\mu k_{ij}\sum_{C_{kl}\in N_r(i,j)}C^{kl}_{ij}+\mu k_{ij}\sum_{C_{kl}\in N_r(i,j)}C^{kl}_{ij}\frac{L}{1-\delta}\\ +\mu k_{ij}\sum_{C_{kl}\in N_r(i,j)}C^{kl}_{ij}\frac{L}{1-\delta}, \end{aligned}$$

where $i=1,2,\dots,m$, $j=1,2,\dots,n$. Set

$$\begin{aligned} \Gamma_{ij}(\omega)=\omega-a_{ij}+\sum_{C_{kl}\in N_r(i,j)}C^{kl}_{ij}[\mu\int_0^\infty|K_{ij}(u)|e^{\omega u}du+\mu k_{ij}\frac{L}{1-\delta}\\ +\mu\int_0^\infty|K_{ij}(u)|e^{\omega u}du\frac{L}{1-\delta}], \end{aligned}$$

where $i=1,2,\dots,m$, $j=1,2,\dots,n$. Clearly, $\Gamma_{ij}(\omega)$, $i=1,2,\dots,m$, $j=1,2,\dots,n$, are continuous functions on $[0,\lambda_0]$. Since

$$\begin{aligned}\Gamma_{ij}(0) &= -a_{ij} + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}[\mu\int_0^\infty |K_{ij}(u)|du + \mu k_{ij}\frac{L}{1-\delta} \\ &\quad + \mu\int_0^\infty |K_{ij}(u)|du\frac{L}{1-\delta}] \\ &\le -a_{ij} + \mu k_{ij}\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} + \mu k_{ij}\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}\frac{L}{1-\delta} \\ &\quad + \mu k_{ij}\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}\frac{L}{1-\delta} < 0,\end{aligned}$$

where $i=1,2,\dots,m, j=1,2,\dots,n$. It follows that we can choose a positive constant $\lambda\in[0,\lambda_0]$ such that

$$\begin{aligned}\Gamma_{ij}(\lambda) &= (\lambda-a_{ij}) + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}[\mu\int_0^\infty |K_{ij}(u)|e^{\lambda u}du + \mu k_{ij}\frac{L}{1-\delta} \\ &\quad + \mu\int_0^\infty |K_{ij}(u)|e^{\lambda u}du\frac{L}{1-\delta}] < 0,\end{aligned} \tag{3.2}$$

where $i=1,2,\dots,m$, $j=1,2,\dots,n$. We consider the Lyapunov functional

$$V_{ij}(t) = |y_{ij}(t)|e^{\lambda t}, \quad i=1,2,\dots,m,\ j=1,2,\dots,n. \tag{3.3}$$

Calculating the upper right derivative of $V_{ij}(t)$ along the solution $y(t)=\{y_{ij}(t)\}$ of system (3.1) with the initial value $\bar\varphi=\varphi-\varphi^*$, we have

$$\begin{aligned}&D^+(V_{ij}(t)) \\ &\le -a_{ij}|y_{ij}(t)|e^{\lambda t} + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}|f(\int_0^\infty K_{ij}(u)x_{kl}(t-u)du)x_{ij}(t) \\ &\quad - f(\int_0^\infty K_{ij}(u)x_{kl}^*(t-u)du)x_{ij}^*(t)|e^{\lambda t} + \lambda|y_{ij}(t)|e^{\lambda t} \\ &= (\lambda-a_{ij})|y_{ij}(t)|e^{\lambda t} + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}|f(\int_0^\infty K_{ij}(u)x_{kl}(t-u)du)y_{ij}(t) \\ &\quad + [f(\int_0^\infty K_{ij}(u)x_{kl}(t-u)du) - f(\int_0^\infty K_{ij}(u)x_{kl}^*(t-u)du)]x_{ij}^*(t)|e^{\lambda t},\end{aligned} \tag{3.4}$$

where $i=1,2,\dots,m$, $j=1,2,\dots,n$. Let

$$\|\varphi-\varphi^*\| = \sup_{-\infty\le s\le 0}\max_{(i,j)} |\varphi_{ij}(s)-\varphi_{ij}^*(s)| > 0.$$

Since $\|\varphi-\varphi^*\|<1$, we can choose a positive constant $M>1$ such that

$$M\|\varphi-\varphi^*\| < 1, \quad (M\|\varphi-\varphi^*\|)^2 < M\|\varphi-\varphi^*\|. \tag{3.5}$$

It follows from (3.3) that

$$V_{ij}(t) = |y_{ij}(t)|e^{\lambda t} < M\|\varphi-\varphi^*\|,$$

for all $t\in(-\infty,0]$, $i=1,2,\dots,m$, $j=1,2,\dots,n$. We claim that

$$V_{ij}(t) = |y_{ij}(t)|e^{\lambda t} < M\|\varphi-\varphi^*\|, \tag{3.6}$$

for all $t > 0$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$. Contrarily, there must exist $ij \in \{11, 12, \dots, 1n, \dots, m1, m2, \dots, mn\}$ and $t_{ij} > 0$ such that

$$V_{ij}(t_{ij}) = M\|\varphi - \varphi^*\|, \quad V_{\overline{ij}}(t) < M\|\varphi - \varphi^*\|, \forall t \in (-\infty,\ t_{ij}), \tag{3.7}$$

where $\overline{ij} \in \{11, 12, \dots, 1n, \dots, m1, m2, \dots, mn\}$. It follows from (3.7) that

$$V_{ij}(t_{ij}) - M\|\varphi - \varphi^*\| = 0, \quad V_{\overline{ij}}(t) - M\|\varphi - \varphi^*\| < 0, \tag{3.8}$$

for all $t \in (-\infty, t_{ij})$, where $\overline{ij} \in \{11, 12, \dots, 1n, \dots, m1, m2, \dots, mn\}$. From (2.4), (2.5), (3.4) , (3.5) and (3.8), we obtain

$$
\begin{aligned}
&0 \le D^+(V_{ij}(t_{ij}) - M\|\varphi - \varphi^*\|)\\
&= D^+(V_{ij}(t_{ij}))\\
&\le (\lambda - a_{ij})|y_{ij}(t_{ij})|e^{\lambda t_{ij}} + \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} |f(\int_0^\infty K_{ij}(u) x_{kl}(t_{ij} - u)du) y_{ij}(t_{ij})\\
&\quad + [f(\int_0^\infty K_{ij}(u) x_{kl}(t_{ij} - u)du) - f(\int_0^\infty K_{ij}(u) x_{kl}^*(t_{ij} - u)du)] x_{ij}^*(t_{ij})|e^{\lambda t_{ij}}\\
&\le (\lambda - a_{ij}) M\|\varphi - \varphi^*\| + \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} [\mu \int_0^\infty |K_{ij}(u)||x_{kl}(t_{ij} - u)|du|y_{ij}(t_{ij})|e^{\lambda t_{ij}}\\
&\quad + \int_0^\infty |K_{ij}(u)|e^{\lambda u} \mu |y_{kl}(t_{ij} - u)|e^{\lambda(t_{ij} - u)} du |x_{ij}^*(t_{ij})|]\\
&\le (\lambda - a_{ij}) M\|\varphi - \varphi^*\| + \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} [\mu \int_0^\infty |K_{ij}(u)|(|x_{kl}(t_{ij} - u) - x_{ij}^*(t_{ij} - u)|\\
&\quad + |x_{ij}^*(t_{ij} - u)|)du|y_{ij}(t_{ij})|e^{\lambda t_{ij}}\\
&\quad + \int_0^\infty |K_{ij}(u)|e^{\lambda u} \mu |y_{kl}(t_{ij} - u)|e^{\lambda(t_{ij} - u)} du |x_{ij}^*(t_{ij})|]\\
&\le (\lambda - a_{ij}) M\|\varphi - \varphi^*\| + \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} [\mu \int_0^\infty |K_{ij}(u)|e^{\lambda u} |y_{kl}(t_{ij} - u)|e^{\lambda(t_{ij} - u)} du\\
&\quad \times |y_{ij}(t_{ij})|e^{\lambda t_{ij}} e^{-\lambda t_{ij}} + \mu \int_0^\infty |K_{ij}(u)||x_{ij}^*(t_{ij} - u)|du|y_{ij}(t_{ij})|e^{\lambda t_{ij}}\\
&\quad + \int_0^\infty |K_{ij}(u)|e^{\lambda u} \mu |y_{kl}(t_{ij} - u)|e^{\lambda(t_{ij} - u)} du |x_{ij}^*(t_{ij})|]\\
&\quad (\lambda - a_{ij}) M\|\varphi - \varphi^*\| + \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} [\mu \int_0^\infty |K_{ij}(u)|e^{\lambda u} du (M\|\varphi - \varphi^*\|)^2 e^{-\lambda t_{ij}}\\
&\quad + \mu \int_0^\infty |K_{ij}(u)|du \frac{L}{1 - \delta} M\|\varphi - \varphi^*\| + \mu \int_0^\infty |K_{ij}(u)|e^{\lambda u} du \frac{L}{1 - \delta} M\|\varphi - \varphi^*\|]\\
&\quad \{(\lambda - a_{ij}) + \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} [\mu \int_0^\infty |K_{ij}(u)|e^{\lambda u} du + \mu k_{ij} \frac{L}{1 - \delta}\\
&\quad + \mu \int_0^\infty |K_{ij}(u)|e^{\lambda u} du \frac{L}{1 - \delta}]\} M\|\varphi - \varphi^*\|.
\end{aligned}
$$

Therefore,

$$\begin{aligned}0 \le (\lambda - a_{ij}) + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}[\mu \int_0^\infty |K_{ij}(u)|e^{\lambda u}du + \mu k_{ij}\frac{L}{1-\delta} \\ + \mu \int_0^\infty |K_{ij}(u)|e^{\lambda u}du \frac{L}{1-\delta}],\end{aligned}$$

which contradicts (3.2). Hence, (3.6) holds. It follows that

$$|x_{ij}(t) - x_{ij}^*(t)| = |y_{ij}(t)| < M\|\varphi - \varphi^*\|e^{-\lambda t},$$

for $t > 0$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$. This completes the proof. □

4. Example

To illustrate the results obtained in previous sections we present the following example. Consider the shunting inhibitory cellular neural network with delays

$$\frac{dx_{ij}}{dt} = -a_{ij}(t)x_{ij} - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(\int_0^\infty K_{ij}(u)x_{kl}(t-u)du)x_{ij} + L_{ij}(t),$$

where $i = 1, 2, 3$, $j = 1, 2, 3$,

$$\begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) \end{bmatrix} = \begin{bmatrix} 1+|\sin t| & 1+|\sin t| & 3+|\sin t| \\ 3+|\sin t| & 1+|\sin t| & 3+|\sin t| \\ 3+|\sin t| & 1+|\sin t| & 3+|\sin t| \end{bmatrix},$$

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 & 0.1 \\ 0.2 & 0 & 0.2 \\ 0.1 & 0.2 & 0.1 \end{bmatrix},$$

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} = \begin{bmatrix} 0.5\sin t & 0.5\cos t & 0.2\sin t \\ 0.4\cos t & 0.2\sin t & 0.3\sin t \\ 0.4\cos t & 0.6\sin t & 0.2\cos t \end{bmatrix}.$$

Set $r = 1$, $K_{ij}(u) = (\sin u)e^{-u}$, $i = 1, 2, 3$, $j = 1, 2, 3$, and $f(x) = \frac{1}{10}x$. Clearly $\mu = 0.1$, $\sum_{C_{kl}\in N_1(1,1)} C_{11}^{kl} = 0.5$,

$$\begin{gathered}\sum_{C_{kl}\in N_1(1,2)} C_{12}^{kl} = 0.8\,, \quad \sum_{C_{kl}\in N_1(1,3)} C_{13}^{kl} = 0.5\,, \\ \sum_{C_{kl}\in N_1(2,1)} C_{21}^{kl} = 0.8\,, \quad \sum_{C_{kl}\in N_1(2,2)} C_{22}^{kl} = 1.2\,, \\ \sum_{C_{kl}\in N_1(2,3)} C_{23}^{kl} = 0.8\,, \quad \sum_{C_{kl}\in N_1(3,1)} C_{31}^{kl} = 0.5\,, \\ \sum_{C_{kl}\in N_1(3,2)} C_{32}^{kl} = 0.8\,, \quad \sum_{C_{kl}\in N_1(3,3)} C_{33}^{kl} = 0.5\,, \\ \sum_{(i,j)} \sum_{C_{kl}\in N_1(i,j)} C_{ij}^{kl} = 6.4\,,\end{gathered}$$

$k_{ij} = 1,\ i = 1,2,3,\ j = 1,2,3,$

$$\delta = \max_{(i,j)} \Big\{ \frac{\mu \sum_{C_{kl} \in N_1(i,j)} C_{ij}^{kl}}{a_{ij}} \Big\} = 0.12 < 1,$$

$$L = \max_{(i,j)} \Big\{ \frac{L_{ij}^+}{a_{ij}} \Big\} = 0.6, \quad \frac{L}{(1-\delta)} = \frac{0.6}{1-0.12} < 1,$$

$$q = 2\delta \frac{L}{(1-\delta)} = 2 \times 0.12 \times \frac{0.6}{1-0.12} < 1,$$

$$\delta(1 + 2\frac{L}{1-\delta}) = 0.12(1 + 2 \times 0.12 \times \frac{0.6}{1-0.12}) < 1\,.$$

By theorem 3.1, the system (4) has a unique almost periodic solution $\varphi^*(t)$ in the region $\|\varphi - \varphi_0\|_B \le 0.08128$. Moreover, $\varphi^*(t)$ is locally exponentially stable, the domain of attraction of $\varphi^*(t)$ is the set $G_1(\varphi^*)$.

We remark that System (4) is a very simple form of SICNNs, and that it does not not satisfy the condition (T0). Therefore, the results in [4, 6, 8] can not be applied to this system. This implies that the results of this paper are essentially new.

Conclusion. The shunting inhibitory cellular neural networks with continuously distributed delays have been studied. Some sufficient conditions for the existence and local exponential stability of almost periodic solutions have been established. The obtained results are new and complement previously known results. Moreover, an example is given to illustrate our results.

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